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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 10

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OBITUARY NOTICE

S. H. BURBURY.

SAMUEL HAWKSLEY BURBURY was born at Kenilworth, in 1831, and was educated at Shrewsbury School, whence, like many Salopians of that and later times, he proceeded to St. John's College, Cambridge. His university career was one of great distinction in classical literature; he obtained the Porson Prize for Greek verse, and was Craven Scholar in 1853; and he gained the second place in the Classical Tripos in 1854 and the second Chancellor's Medal. To this he added in the same year the position of fifteenth wrangler in the Mathematical Tripos, thus constituting one of the most brilliant double degrees on record. In due course he became a Fellow of St. John's College and was called to the Bar. After an interval of more than twenty years he revived his mathematical studies under the influence of his friend Dr. H. W. Watson, afterwards F.R.S., who had come under the spell of Clerk Maxwell, and, through his advice and direction, wrote in 1876 (second edition 1893) the earliest systematic dynamical treatise on the Theory of Gases, expounding, in thorough fashion, mainly the work of Maxwell himself and his successor Boltzmann. In the preface Watson mentions his difficulties with Boltzmann's deduction of the second law of thermodynamics, and states his obligation to his friend Burbury for an argument which he considered to be unexceptionable and applicable to all cases. This led to further co-operation; and under their joint names there appeared in 1883–5 a *Treatise on the Mathematical Theory of Electricity*, in two volumes, again concerned mainly with the exposition of Maxwell's electrical ideas, then still on their trial, and not yet fully accepted on the Continent of Europe. This treatise, a very valuable and original one for the time, contained extensive exposition and development of the difficult parts of Maxwell's own classical treatise of 1872, and at the same time incorporated much new mathematical material that had accumulated in the intervening years, with original developments of their own. Mr. Burbury's main original work lay, however, in the Dynamical Theory of Gases, in which at first he communicated memoirs to the *Philosophical Transactions* and the *Philosophical Magazine*, defending Maxwell's law of equable partition of energy among the degrees of freedom of the molecules against the attacks then current, including those of Lord Kelvin, and elucidating it in various directions. In course of time he became convinced that the simple Maxwellian law of distribution e^{-hQ}, where $Q = \Sigma m(u^2+v^2+w^2)$, did not adequately include the interactions of a crowded set of molecules, and that Q should be replaced by $Q' = \Sigma m(u^2+v^2+w^2) + \Sigma\Sigma b(uu'+vv'+ww')$; and he worked out a new gas-theory on this basis in various memoirs which he condensed into *A Treatise on the Kinetic Theory of Gases*, published by the Cambridge University Press in 1899. He remarks that for gases under ordinary conditions the effect of the coefficient b is probably very small. But in denser gases it leads to local streaming effects, of course on a molecular scale of magnitude, while the equable partition of energy ceases to hold: and he criticizes the more recent developments of Boltzmann's work in that regard. The difficulty of the subject has hitherto prevented many physicists from following him in this very acute piece of work: but signs are not wanting that in the future his train of ideas may be needed in connexion with the modern developments relating to Brownian movements and the physical theory of emulsions. Whether in his own constructive work, or in his trenchant criticism of the work of others, Mr. Burbury's literary gifts always stood revealed in the conciseness and elegance of his modes of expression. For many years he was afflicted by deafness, which limited considerably his sphere of activity; but he retained his vigour, both mental and bodily, and was to the last a very regular attendant at the weekly meetings of the Royal Society, and was closely interested in all branches of physical science. He was elected Fellow of the Royal Society in 1890. He died at the ripe age of 79 years.

J. L.

NOTES AND CORRECTIONS.

Mr. G. T. Bennett sends the following correction of his paper on "Deformable Octahedra":—

P. 342, last line but three, *for* β_1, *read* β_1'.

Dr. W. H. Young writes:—

Vol. 9, p. 459.—The argument taken from p. 467 of the companion paper does not apply. When the intervals concerned are finite, but then only, the behaviour of a periodic function of bounded variation can be inferred from that of a monotone function.

Miss H. P. Hudson sends the following corrections to her paper in Vol. 9:—

Page 21, last line, *for* UzW^2 *read* Uzw^2.
,, 22, line 7 from bottom, *for* I1 *read* I7.
,, 33, I4 diagram, *transpose* 3, 4.
,, 36, L′1 diagram, *dele* 3.
,, 38, $P3_2$, last equation, *for* $z(Fx+My)$ *read* $Z(Fx+My)$; $P4_1$ diagram, the vertex 4 should lie on the lowest line.
,, 39, Q_1 diagram, *insert* 4 at lower left-hand corner.
,, 41, $T4_1$ left-hand diagram, *transpose* 1, 2; *insert* 4 at lower right hand corner.
,, 42, T7 right-hand diagram, *transpose* 1, 2; T9 diagram, *transpose* 1, 2, *also* 3, 4.
,, 43, $U2_3$ diagram, *for* 3 *read* 4; $U3_2$ diagram, *for* 2, 3, 4, *read* 1, 4, 3.
,, 44, $U4_2$, right-hand side, *for* Q *read* Q_1; V_1, left-hand diagram, *for* 3 *read* 4; right-hand diagram should have three adjacent lines I4.
,, 45, $W1_2$, right-hand side, *for* V_2 *read* V_1.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE FUNDAMENTAL THEOREM IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

By W. H. YOUNG, Sc.D., F.R.S.

[Received December 24th, 1910.—Read January 12th, 1911.]

1. The fundamental theorem in the Theory of Functions of a Complex Variable states, as is well known, that, if dw/dz exists at all points in the interior of a circle of radius c, the function w of the complex variable z is analytic, so that, in particular, all the higher differential coefficients of w exist. There is no difficulty in translating this theorem into one which employs the language of the real variable, and the question has thus been raised as to whether it is not possible to prove the theorem without the use of the complex variable. I understand, indeed, that this very matter was the subject of conversation at a meeting of the Society held early in the present year.

The progress of the Theory of Functions of a Real Variable has been recently very rapid; but, regarded as a science, in the modern sense of the term, it is but a younger brother of the Theory of Functions of a Complex Variable. This has been largely owing to the great simplicity brought about in the latter theory by precisely the theorem of which it is the question. On the other hand, from the point of view of the Theory of Functions of a Real Variable, it is certainly unsatisfactory that a theorem so important, and in any but its most recent formulations long well known, should not yet have been proved except by the use of the complex variable.

I hope, therefore, that the present contribution, though not a complete solution of the problem in all its generality, will be regarded as of interest, more especially in view of the fact that the only restriction introduced is that of the boundedness of dw/dz. In point of generality, therefore, the result arrived at comes between those of Goursat's former and latter papers on the subject. I may add that the nature of the reasoning employed admits obviously of a slightly more general statement than that given, and it is not impossible that further knowledge of the properties of functions of a real variable may admit of the argument used below applying *mutatis mutandis* to the most general case.

As to the importance of the method exposed, two further remarks may be made. The one is that the actual expression of the function is obtained, so that in point of length the present method compares favourably with the classical one, as completed by Goursat. Further, just as even mathematicians of the calibre of Kronecker endeavoured to minimise or even banish the employment of irrational numbers in mathematics, so there have always been, and it may be expected there always will be, mathematicians who regard a theorem as only then completely established when the mode of proof employed avoids the use of the complex variable. For such mathematicians the complex variable is only a powerful method of investigation. In the present case, the theorem with which we are concerned is that which is fundamental in the whole Theory of Functions of a Complex Variable; it consequently opens the door to the use of chains of reasoning by which the real results of that theory are established from first to last by the methods of the real function theory alone.

It may be added that Goursat's extremely elegant and skilful proof is one which, besides involving the complex variable, employs considerations which offer numerous difficulties to the reader. Only a select few can be expected to be able to judge, in the present state of mathematical teaching, of the validity of that proof.

2. It is known that the necessary and sufficient conditions that $w = u+iv$ should possess a differential coefficient dw/dz with respect to the complex variable $z = x+iy$ are that u and v should possess first differentials and that

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}.$$

Let, then, $u(x, y)$ and $v(x, y)$ be functions of (x, y) which have the following properties:—

(i) u_x, u_y, v_x, v_y all exist, and u and v possess differentials at all points of a certain circle of radius c and centre the origin.

(ii)
$$u_x = v_y, \quad \text{and} \quad u_y = -v_x. \tag{1}$$

(iii) u_x, u_y, v_x, and v_y are bounded functions of (x, y) in the circle considered.

Then, since $u(x, y)$ possesses a differential,

$$\frac{du}{dr} = u_x \frac{dx}{dr} + u_y \frac{dy}{dr} \quad \text{and} \quad \frac{du}{d\theta} = u_x \frac{dx}{d\theta} + u_y \frac{dy}{d\theta};$$

therefore

$$\left.\begin{aligned} \frac{du}{dr} &= u_x \cos\theta + u_y \sin\theta = \frac{1}{r}\frac{dv}{d\theta} \\ \text{and} \quad \frac{1}{r}\frac{du}{d\theta} &= -u_x \sin\theta + u_y \cos\theta = -\frac{dv}{dr} \end{aligned}\right\}. \tag{2}$$

We may evidently suppose, without loss of generality, that $u = 0$ and $v = 0$ at the origin. It follows also, from (2), that u_r, v_r and u_θ/r, v_θ/r are all bounded functions of (r, θ).

Hence, by the simplest case of Lebesgue's theorem,*

$$u - u_0 = u(x, y) - u(r, 0) = \int_0^\theta u_\theta \, d\theta,$$

and we may integrate both sides of this equation with respect to r after dividing by r.

Again, since $1/r . du/d\theta$ is a bounded function of (r, θ),

$$\int_0^r dr \int_0^\theta \frac{1}{r} u_\theta \, d\theta = \int_0^\theta d\theta \int_0^r \frac{1}{r} u_\theta \, dr,$$

whence
$$\int_0^r \frac{1}{r}(u - u_0)\, dr = -\int_0^\theta v \, d\theta. \tag{3}$$

Thus the left-hand side is an integral when regarded as a function of θ, so that we may expand it in a Fourier series.

Similarly
$$\int_0^r \frac{1}{r}(v - v_0)\, dr = \int_0^\theta u \, d\theta. \tag{4}$$

We have then

$$\int_0^r \frac{1}{r}(u - u_0)\, dr = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta). \tag{5}$$

Also, since u_θ is bounded, u is an integral with respect to θ, so that

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

---

[*] For a proof of this theorem without the use of Cantor's numbers, see W. H. Young and Grace Chisholm Young, "On the Existence of a Differential Coefficient," Appendix II, 1910, *Proc. London Math. Soc.* (to appear).

where the constant term has been omitted, since, by (4), the integral of u with respect to θ is periodic. Here

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \left(\cos n\theta \int_0^r \frac{(u-u_0)}{r}\, dr\right), \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u \cos n\theta \, d\theta.$$

Now for all modes of variation of (r, θ), as we approach the origin, it follows from the Theorem of the Mean for functions of two variables,* bearing in mind that u is zero at the origin, that

$$\frac{u}{r} = \frac{x}{r}\, u_{x'} + \frac{y}{r}\, u_{y'},$$

where (x', y') is a certain point between $(0, 0)$ and (x, y), whence it follows that u/r is a bounded function of (r, θ) in the whole circle. The same is, of course, true of u_0/r, since it is the result of putting $\theta = 0$ in u/r.

Hence in the expression for A_n we may change the order of integration, which gives us at once

$$A_n = \int_0^r \frac{a_n}{r}\, dr.$$

Similarly
$$B_n = \int_0^r \frac{b_n}{r}\, dr. \tag{6}$$

We remark also that a_n/r, and similarly b_n/r, since it is the result of a single integration with respect to θ of a bounded function of (r, θ), is certainly a bounded function of r up to and including the point $r = 0$.

Now v is an integral with respect to θ. Hence we may write

$$v = \sum_{n=1}^{\infty} (a_n' \cos n\theta + b_n' \sin n\theta),$$

where, as before, the constant term has been omitted, since, by (3), the integral of v with respect to θ is periodic. Hence, by the theory of Fourier series, we have

$$\int_0^{\theta} v\, d\theta = \tfrac{1}{2}A_0' + \sum_{n=1}^{\infty} (a_n' \sin n\theta - b_n' \cos n\theta)/n,$$

whence, by equations (3), (5), and (6), we see that

$$\int_0^r \frac{a_n}{r}\, dr = \frac{b_n'}{n} \quad \text{and} \quad \int_0^r \frac{b_n}{r}\, dr = -\frac{a_n'}{n}. \tag{7}$$

---

* See, for instance, § 22 of my tract, *On the Fundamental Theorems of the Differential Calculus*, Cambridge University Press, 1910, where Case (i) of the Theorem of the Mean, § 15, is used. Using Case (ii), the result is in the form here required.

Similarly, using (4) instead of (3), we see that

$$\int_0^r \frac{a'_n}{r}\,dr = -\frac{b_n}{n} \quad \text{and} \quad \int_0^r \frac{b'_n}{r}\,dr = \frac{a_n}{n}. \tag{8}$$

From equations (7) and (8) we see that a_n, b_n, a'_n, and b'_n are all continuous functions of r, and therefore, except possibly for the value zero of r, this remains the case when they are divided by r. Hence the integrands of the integrals in (7) and (8), except possibly for the value $r = 0$, are the differential coefficients with respect to r of their integrals. From (8) it therefore follows that

$$a'_n = -r\frac{d}{dr}\left(\frac{b_n}{n}\right). \tag{9}$$

Similarly

$$b_n = -r\frac{d}{dr}\left(\frac{a'_n}{n}\right). \tag{10}$$

From (9) and (10) we have $n^2 b_n = r\frac{b}{dr}\,r\frac{d}{dr}\,b_n,$

which has been proved to hold for all values of r except zero. Hence, writing

$$t = \log r,$$

we see that, for all values of t corresponding to positive values of $r < c$, that is, for all finite values of t less than $\log c$,

$$n^2 b_n = \frac{d^2}{dt^2}\,b_n,$$

which gives

$$b_n = K_n r^n + L_n r^{-n}.$$

Here we may at once put $L_n = 0$, since b_n/r is bounded up to and including $r = 0$. Thus

$$b_n = K_n r^n.$$

Hence, by (7),

$$a'_n = -K_n r^n.$$

In precisely the same way we see that b'_n and a_n are equal with the same sign, and have a value of the same form as b_n and a'_n.

We may, therefore, with a convenient change in the notation, write

$$\left.\begin{aligned} u &= \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta), \\ v &= \sum_{n=1}^{\infty} r^n(-b_n\cos n\theta + a_n\sin n\theta), \end{aligned}\right\} \tag{I}$$

these equations certainly holding for all points in the interior of the circle of radius c, the origin being obviously included.

Now, regarded as functions of r, u and v are here expressed as power series; hence all their differential coefficients with respect to r exist. Hence it also follows that all their differential coefficients with respect to θ exist.

To see this, let and

$$\left.\begin{aligned}&\sum_{n=1}^{\infty} r^n(A_n\cos n\theta+B_n\sin n\theta)\\ &\sum_{n=1}^{\infty} r^n(-B_n\cos n\theta+A_n\sin n\theta)\end{aligned}\right\} \qquad \text{(II)}$$

be the series got by performing the operation $(r\,d/dr)^p$ term-by-term on the above series (I) representing u and v respectively. Then, since both these series converge for $\theta = 0$, the series $A_n r^n$ and $B_n r^n$ both converge, and therefore, by the theory of power series, the series $|A_n|r^n$ and $|B_n|r^n$ both converge, and therefore the series $(|A_n|+|B_n|)r^n$ also converges, for all positive values of r less than c. Hence, by Weierstrass's test, it follows that the series (II) converge uniformly for all values of the ensemble (r, θ) inside any circle of radius less than c.

But, apart from sign, these series (II) are the same as the series got by differentiating the series (I) p times with respect to θ. Hence, by the theory of term-by-term differentiation of series, these latter series, being uniformly convergent series of functions of θ, represent the p-th differential coefficients of the two series (I).

Thus all the differential coefficients of u and v with respect to r and θ, and therefore also with respect to x and y, exist. Moreover, we have obtained analytical expressions for u and v which can be differentiated term-by-term as often as we please, namely, the series (I).

Finally, $u+iv$ is, of course, a power series in $x+iy$. Hence, translating into the language of the complex variable, we have the theorem stated in the first article, with the restriction there noted.

THE TRANSFORMATION OF A PARTICULAR TYPE OF ELECTROMAGNETIC FIELD

By H. BATEMAN.

[Received November 7th, 1910.—Read November 10th, 1910.]

1. In a former paper* a short statement was made of some results connected with a transformation of the type

$$f^2(dx^2+dy^2+dz^2-dt^2)+(s_x dx+s_y dy+s_z dz-s\,dt)^2$$
$$= dx'^2+dy'^2+dz'^2-dt'^2, \quad (1)$$

where s_x, s_y, s_z, s are functions of x, y, z, t connected by the equation†

$$s_x^2+s_y^2+s_z^2 = s^2. \quad (2)$$

I have recently come across an interesting class of transformations of this type, and have thought it worth while to give some account of the analysis. The transformation was suggested to me by a remarkable solution of the equation of wave motion which was discovered some years ago by Forsyth.‡ This solution is obtained as follows.

If $\alpha(u)$, $\beta(u)$, $\gamma(u)$, $\sigma(u)$ are four functions satisfying the equation

$$\sigma^2(u) = \alpha^2(u)+\beta^2(u)+\gamma^2(u), \quad (3)$$

and if u be defined in terms of x, y, z, t by means of the equation

$$F(u) = x\alpha(u)+y\beta(u)+z\gamma(u)-t\sigma(u), \quad (4)$$

then $\phi(u)$ is a solution of the differential equations

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial t}\right)^2,$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2},$$

---

* These *Proceedings*, Vol. 8 (1910), p. 469.

† The units of length and time have been chosen so that the velocity of radiation is represented by unity.

‡ *Messenger of Mathematics*, Vol. 27, p. 114 (1898); *Phil. Trans.*, A (1898).

and the function $\dfrac{\chi(u)}{F'(u)-x\alpha'(u)-y\beta'(u)-z\gamma'(u)+t\sigma'(u)}$

is another solution of these equations.

2. The transformation is obtained by putting

$$\left.\begin{aligned} x' &= x+X(u), & y' &= y+Y(u) \\ z' &= z+Z(u), & t' &= t+T(u) \end{aligned}\right\}. \tag{5}$$

These equations give

$$\begin{aligned} dx'^2+dy'^2+dz'^2-dt'^2 \\ = dx^2+dy^2+dz^2-dt^2+2du\left[X'(u)dx+Y'(u)dy+Z'(u)dz-T'(u)dt\right] \\ +du^2\left[X'^2(u)+Y'^2(u)+Z'^2(u)-T'^2(u)\right]. \end{aligned} \tag{6}$$

Now, if the variable u is defined by means of the equation

$$F(u) = x\alpha(u)+y\beta(u)+z\gamma(u)-t\sigma(u),$$

we have $$\frac{\partial u}{\partial x} = \frac{\alpha(u)}{F'(u)-x\alpha'(u)-y\beta'(u)-z\gamma'(u)+t\sigma'(u)} = \frac{\alpha(u)}{M}, \tag{7}$$

and three similar equations. Accordingly, if we put

$$X'(u) = \alpha(u), \quad Y'(u) = \beta(u), \quad Z'(u) = \gamma(u), \quad T'(u) = \sigma(u),$$

we shall have

$$\begin{aligned} X'(u)dx+Y'(u)dy+Z'(u)dz-T'(u)dt \\ = \left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz+\frac{\partial u}{\partial t}dt\right)M = M\,du, \end{aligned}$$

and $$X'^2(u)+Y'^2(u)+Z'^2(u)-T'^2(u) = 0. \tag{8}$$

Equation (6) thus reduces to the form

$$dx'^2+dy'^2+dz'^2-dt'^2 = dx^2+dy^2+dz^2-dt^2+2M\,du^2. \tag{9}$$

Putting $$s_x = \sqrt{2M}\frac{\partial u}{\partial x}, \quad s_y = \sqrt{2M}\frac{\partial u}{\partial y},$$

$$s_z = \sqrt{2M}\frac{\partial u}{\partial z}, \quad s = \sqrt{2M}\frac{\partial u}{\partial t},$$

we see that the equation $s_x^2+s_y^2+s_z^2 = s^2$

is satisfied in virtue of the relation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial t}\right)^2, \tag{10}$$

and so the transformation is of the type (I).

3. The results stated on pp. 473 and 474 of my former paper indicate that if V is a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial t^2}$$

and
$$\frac{\partial u}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial V}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial V}{\partial z}=\frac{\partial u}{\partial t}\frac{\partial V}{\partial t},$$

then V is also a solution of

$$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=\frac{\partial^2 V}{\partial t'^2}.$$

Secondly, if U is a solution of

$$\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2=\left(\frac{\partial U}{\partial t}\right)^2$$

and
$$\frac{\partial U}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial u}{\partial z}=\frac{\partial U}{\partial t}\frac{\partial u}{\partial t},$$

then
$$\left(\frac{\partial U}{\partial x'}\right)^2+\left(\frac{\partial U}{\partial y'}\right)^2+\left(\frac{\partial U}{\partial z'}\right)^2=\left(\frac{\partial U}{\partial t'}\right)^2.$$

Again, if the components (E_x, E_y, E_z), (H_x, H_y, H_z) in an electromagnetic field are connected with u by the relations

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial t}E_x-\frac{\partial u}{\partial y}H_z+\frac{\partial u}{\partial z}H_y\right)&=\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}E_x+\frac{\partial u}{\partial y}E_y+\frac{\partial u}{\partial z}E_z\right)\\
\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial t}E_y-\frac{\partial u}{\partial z}H_x+\frac{\partial u}{\partial x}H_z\right)&=\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}E_x+\frac{\partial u}{\partial y}E_y+\frac{\partial u}{\partial z}E_z\right)\\
\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial t}E_z-\frac{\partial u}{\partial x}H_y+\frac{\partial u}{\partial y}H_x\right)&=\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial x}E_x+\frac{\partial u}{\partial y}E_y+\frac{\partial u}{\partial z}E_z\right)
\end{aligned}\right\}, \tag{11}$$

and the quantities (E'_x, E'_y, E'_z), (H'_x, H'_y, H'_z) are defined by means of the equations

$$\left.\begin{aligned}
E_x&=E'_x\frac{\partial(y', z')}{\partial(y, z)}+E'_y\frac{\partial(z', x')}{\partial(y, z)}+E'_z\frac{\partial(x', y')}{\partial(y, z)}\\
&\qquad-H'_x\frac{\partial(x', t')}{\partial(y, z)}-H_y\frac{\partial(y', t')}{\partial(y, z)}-H'_z\frac{\partial(z', t')}{\partial(y, z)}\\
-H_x&=E'_x\frac{\partial(y', z')}{\partial(x, t)}+E'_y\frac{\partial(z', x')}{\partial(x, t)}+E'_z\frac{\partial(x', y')}{\partial(x, t)}\\
&\qquad-H'_x\frac{\partial(x', t')}{\partial(x, t)}-H'_y\frac{\partial(y', t')}{\partial(x, t)}-H'_z\frac{\partial(z', t')}{\partial(x, t)}, \;\ldots
\end{aligned}\right\}, \tag{12}$$

then (E'_x, E'_y, E'_z), (H'_x, H'_y, H'_z) are the components of the electric and magnetic forces of a corresponding electromagnetic field in the (x', y', z', t') system of coordinates.

To prove this we shall first show that the equations

$$\left.\begin{aligned} H_x &= H'_x\frac{\partial(y', z')}{\partial(y, z)}+H'_y\frac{\partial(z', x')}{\partial(y, z)}+H'_z\frac{\partial(x', y')}{\partial(y, z)} \\ &\qquad +E'_x\frac{\partial(x', t')}{\partial(y, z)}+E'_y\frac{\partial(y', t')}{\partial(y, z)}+E'_z\frac{\partial(z', t')}{\partial(y, z)} \\ E_x &= H'_x\frac{\partial(y', z')}{\partial(x, t)}+H'_y\frac{\partial(z', x')}{\partial(x, t)}+H'_z\frac{\partial(x', y')}{\partial(x, t)} \\ &\qquad +E'_x\frac{\partial(x', t')}{\partial(x, t)}+E'_y\frac{\partial(y', t')}{\partial(x, t)}+E'_z\frac{\partial(z', t')}{\partial(x, t)} \end{aligned}\right\} \tag{13}$$

may be deduced from the equations given above.

By differentiating (5), we get

$$\frac{\partial(y', z')}{\partial(y, z)} = 1+\beta\frac{\partial u}{\partial y}+\gamma\frac{\partial u}{\partial z}, \qquad \frac{\partial(z', x')}{\partial(y, z)} = -\alpha\frac{\partial u}{\partial y}, \qquad \frac{\partial(x', y')}{\partial(y, z)} = -\alpha\frac{\partial u}{\partial z},$$

$$\frac{\partial(x', t')}{\partial(y, z)} = 0, \qquad \frac{\partial(y', t')}{\partial(y, z)} = \sigma\frac{\partial u}{\partial z}, \qquad \frac{\partial(z', t')}{\partial(y, z)} = -\sigma\frac{\partial u}{\partial y},$$

$$\frac{\partial(x', t')}{\partial(x, t)} = 1+\alpha\frac{\partial u}{\partial x}+\sigma\frac{\partial u}{\partial t}, \qquad \frac{\partial(y', t')}{\partial(x, t)} = \beta\frac{\partial u}{\partial x}, \qquad \frac{\partial(z', t')}{\partial(x, t)} = \gamma\frac{\partial u}{\partial x},$$

$$\frac{\partial(y', z')}{\partial(x, t)} = 0, \qquad \frac{\partial(z', x')}{\partial(x, t)} = -\gamma\frac{\partial u}{\partial t}, \qquad \frac{\partial(x', y')}{\partial(x, t)} = \beta\frac{\partial u}{\partial t}.$$

The equations (12) thus give

$$E_x = E'_x\left(1+\beta\frac{\partial u}{\partial y}+\gamma\frac{\partial u}{\partial z}\right)-\alpha\frac{\partial u}{\partial y}E'_y-\alpha\frac{\partial u}{\partial z}E'_z-\sigma\frac{\partial u}{\partial z}H'_y+\sigma\frac{\partial u}{\partial y}H'_z,$$

$$H_x = H'_x\left(1+\alpha\frac{\partial u}{\partial x}+\sigma\frac{\partial u}{\partial t}\right)+\beta\frac{\partial u}{\partial x}H'_y+\gamma\frac{\partial u}{\partial x}H'_z+\gamma\frac{\partial u}{\partial t}E'_y-\beta\frac{\partial u}{\partial t}E'_z,$$

and in the same way the equations (13) give

$$E_x = E'_x\left(1+\alpha\frac{\partial u}{\partial x}+\sigma\frac{\partial u}{\partial t}\right)+\beta\frac{\partial u}{\partial x}E'_y+\gamma\frac{\partial u}{\partial x}E'_z-\gamma\frac{\partial u}{\partial t}H'_y+\beta\frac{\partial u}{\partial t}H'_z,$$

$$H_x = H'_x\left(1+\beta\frac{\partial u}{\partial y}+\gamma\frac{\partial u}{\partial z}\right)-\alpha\frac{\partial u}{\partial y}H'_y-\alpha\frac{\partial u}{\partial z}H'_z+\sigma\frac{\partial u}{\partial z}E'_y-\sigma\frac{\partial u}{\partial z}E'_z.$$

Now since $$\alpha = M\frac{\partial u}{\partial x},\quad \beta = M\frac{\partial u}{\partial y},\quad \gamma = M\frac{\partial u}{\partial z},$$

$$\sigma = -M\frac{\partial u}{\partial t},\quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial t}\right)^2,$$

it is easy to see that the two sets of equations are equivalent if

$$2\sigma E'_x\frac{\partial u}{\partial t} + 2\alpha\left(\frac{\partial u}{\partial x}E'_x + \frac{\partial u}{\partial y}E'_y + \frac{\partial u}{\partial z}E'_z\right) = 2\gamma\frac{\partial u}{\partial t}H'_y - 2\beta\frac{\partial u}{\partial t}H'_z,$$

i.e., if

$$\frac{\partial u}{\partial t}\left[\frac{\partial u}{\partial t}E'_x - \frac{\partial u}{\partial y}H'_z + \frac{\partial u}{\partial z}H'_y\right] = \frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial x}E'_x + \frac{\partial u}{\partial y}E'_y + \frac{\partial u}{\partial z}E'_z\right],$$

and two similar equations are satisfied identically.

Now the first set of equations gives

$$\frac{\partial u}{\partial x}E_x + \frac{\partial u}{\partial y}E_y + \frac{\partial u}{\partial z}E_z = \frac{\partial u}{\partial x}E'_x + \frac{\partial u}{\partial y}E'_y + \frac{\partial u}{\partial z}E'_z,$$

$$\frac{\partial u}{\partial t}E_x - \frac{\partial u}{\partial y}H_z + \frac{\partial u}{\partial z}H_y = \frac{\partial u}{\partial t}E'_x - \frac{\partial u}{\partial y}H'_z + \frac{\partial u}{\partial z}H'_y;$$

hence the equations which ought to be satisfied are

$$\frac{\partial u}{\partial t}\left[\frac{\partial u}{\partial t}E_x - \frac{\partial u}{\partial y}H_z + \frac{\partial u}{\partial z}H_y\right] = \frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial x}E_x + \frac{\partial u}{\partial y}E_y + \frac{\partial u}{\partial z}E_z\right],$$

and these are satisfied in virtue of the conditions imposed upon the vectors E and H.

4. The fact that (E'_x, E'_y, E'_z), (H'_x, H'_y, H'_z) are the components of the electric and magnetic forces in an electromagnetic field may be deduced at once from the equation (12) and (13) for we have

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial E_x}{\partial t}$$

$$= \frac{\partial(y', z', t')}{\partial(y, z, t)}\left[\frac{\partial H'_z}{\partial y'} - \frac{\partial H'_y}{\partial z'} - \frac{\partial E'_x}{\partial t'}\right] + \frac{\partial(z', x', t')}{\partial(y, z, t)}\left[\frac{\partial H'_x}{\partial z'} - \frac{\partial H'_z}{\partial x'} - \frac{\partial E'_y}{\partial t'}\right]$$

$$+ \frac{\partial(x', y', t')}{\partial(y, z, t)}\left[\frac{\partial H'_y}{\partial x'} - \frac{\partial H'_x}{\partial y'} - \frac{\partial E'_z}{\partial t'}\right] - \frac{\partial(x', y', z')}{\partial(y, z, t)}\left[\frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'}\right],$$

and similarly

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial H_x}{\partial t}$$

$$= \frac{\partial(y', z', t')}{\partial(y, z, t)}\left[\frac{\partial E'_z}{\partial y'}-\frac{\partial E'_y}{\partial z'}+\frac{\partial H'_x}{\partial t'}\right]+\frac{\partial(z', x', t')}{\partial(y, z, t)}\left[\frac{\partial E'_x}{\partial z'}-\frac{\partial E'_z}{\partial x'}+\frac{\partial H'_y}{\partial t'}\right]$$

$$+\frac{\partial(x', y', t')}{\partial(y, z, t)}\left[\frac{\partial E'_y}{\partial x'}-\frac{\partial E'_x}{\partial y'}+\frac{\partial H'_z}{\partial t'}\right]-\frac{\partial(x', y', z')}{\partial(y, z, t)}\left[\frac{\partial H'_x}{\partial x'}+\frac{\partial H'_y}{\partial y'}+\frac{\partial H'_z}{\partial z'}\right].$$

There are clearly four equations of each type.

Hence the equations

$$\operatorname{curl}' H' = \frac{\partial E'}{\partial t'}+\rho' w', \qquad \operatorname{curl}' E' = -\frac{\partial H'}{\partial t'},$$

$$\operatorname{div}' E' = \rho', \qquad \operatorname{div} H' = 0,$$

are a consequence of

$$\operatorname{curl} H = \frac{\partial E}{\partial t}+\rho w, \qquad \operatorname{curl} E = -\frac{\partial H}{\partial t},$$

$$\operatorname{div} E = \rho, \qquad \operatorname{div} H = 0,$$

if ρ', w'_x, w'_y, w'_z are defined in terms of ρ, w_x, w_y, w_z by means of the equations

$$\rho w_x = \rho' w'_x \frac{\partial(y', z', t')}{\partial(y, z, t)}+\rho' w'_y \frac{\partial(z', x', t')}{\partial(y, z, t)}$$

$$+\rho' w'_z \frac{\partial(x', y', t')}{\partial(y, z, t)}-\rho' \frac{\partial(x', y', z')}{\partial(x, y, z)}$$

$$-\rho = \rho' w'_x \frac{\partial(y', z', t')}{\partial(x, y, z)}+\rho' w'_y \frac{\partial(z', x', t')}{\partial(x, y, z)}$$

$$+\rho' w'_z \frac{\partial(x', y', t')}{\partial(x, y, z)}-\rho' \frac{\partial(x', y', z')}{\partial(x, y, z)},$$

just as in my paper on "The Transformation of the Electrodynamical Equations."*

It may now be shown by the methods used in this former paper that the integral forms

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* These *Proceedings*, Ser. 2, Vol. 8, p. 223.

$$H_x\,dy\,dz+H_y\,dz\,dx+H_z\,dx\,dy+E_x\,dx\,dt+E_y\,dy\,dt+E_z\,dz\,dt,$$

$$E_x\,dy\,dz+E_y\,dz\,dx+E_z\,dx\,dy-H_x\,dx\,dt-H_y\,dy\,dt-H_z\,dz\,dt,$$

$$\rho w_x\,dy\,dz\,dt+\rho w_y\,dz\,dx\,dt+\rho w_z\,dx\,dy\,dt-\rho\,dx\,dy\,dz,$$

$$dx\,dy\,dz\,dt,$$

$$(E_x^2+E_y^2+E_z^2-H_x^2-H_y^2-H_z^2)\,dx\,dy\,dz\,dt,$$

$$(E_xH_x+E_yH_y+E_zH_z)\,dx\,dy\,dz\,dt,$$

are invariants for the transformation. The invariant

$$V(s_x\,dx+s_y\,dy+s_x\,dz-s\,dt),$$

which is mentioned on p. 474 of my former paper, is in this case simply du or

$$\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy+\frac{\partial u}{\partial z}\,dz+\frac{\partial u}{\partial t}\,dt.$$

[*May* 19*th*, 1911.—The following is an example of an electromagnetic field which is transformed into another electromagnetic field by the above transformation. Let

$$A_x=\frac{\alpha'(u)}{M},\quad A_y=\frac{\beta'(u)}{M},\quad A_z=\frac{\gamma'(u)}{M},\quad \Phi=\frac{\sigma'(u)}{M}.$$

These potentials have the same values at two corresponding points (x, y, z, t), (x', y', z', t'), provided the function $\bar{F}(u)$ corresponding to $F(u)$ is given by the equation

$$\bar{F}(u)=F(u)+\alpha(u)\,X(u)+\beta(u)\,Y(u)+\gamma(u)\,Z(u)-\sigma(u)\,T(u).$$

The values of u and M are unaltered by the transformation.]

5. It should be mentioned that the transformation which has been considered in this paper may be generalized.

Let us put

$$x'=xf(u)+X(u),\qquad t'=tf(u)+T(u),$$

$$y'=yf(u)+Y(u),\qquad z'=zf(u)+Z(u),$$

where

$$f'(u)\,[x^2+y^2+z^2-t^2]+2X'(u)\,x+2Y'(u)\,y+2Z'(u)z-2T'(u)\,t+2F(u)=0.$$

Then it is easy to see that

$$\frac{\partial u}{\partial x} = -\frac{2[xf'(u)+X'(u)]}{(x^2+y^2+z^2-t^2)f''(u)+2xX''(u)+2yY''(u)+2zZ''(u)-2tT''(u)+2F''(u)}$$

or

$$xf'(u)+X'(u) = M\frac{\partial u}{\partial x}.$$

Calculating $dx'^2+dy'^2+dz'^2-dt'^2$, we find that

$$dx'^2+dy'^2+dz'^2-dt'^2$$
$$= f^2(u)\,[dx^2+dy^2+dz^2-dt^2]$$
$$+2Mdu^2+[X'^2(u)+Y'^2(u)+Z'^2(u)-T'^2(u)]\,du^2.$$

The transformation is thus of the general form (8) considered on p. 474 of my former paper. The transformation reduces to the type in which

$$s_x^2+s_y^2+s_z^2 = s^2,$$

if

$$X'^2(u)+Y'^2(u)+Z'^2(u)-T'^2(u) = 2F(u)f'(u),$$

and in this case the equation for u is

$$[xf'(u)+X'(u)]^2+[yf'(u)+Y'(u)]^2+[zf'(u)+Z'(u)]^2 = [tf'(u)+T'(u)]^2.$$

When u is kept constant this equation may be regarded as defining a system of spherical waves.

The function u is again a solution of the equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial t}\right)^2,$$

and also satisfies the equation

$$\frac{\partial}{\partial x}\left(\frac{1}{M^2}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{M^2}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{M^2}\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial t}\left(\frac{1}{M^2}\frac{\partial u}{\partial t}\right).$$

The formulæ for the transformation of the electric and magnetic forces are the same as before, but now the invariant $dx\,dy\,dz\,dt$ must be replaced by an invariant of the form

$$\lambda\,dx\,dy\,dz\,dt,$$

where λ is some function of u.

ON THE 3–3 BIRATIONAL TRANSFORMATION IN THREE DIMENSIONS[*]

(SECOND PAPER.[†])

By HILDA P. HUDSON, M.A.

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CONTENTS.

1. General considerations on lines of contact.
2. Investigation of the possible cases.
3. Discussion of a nodal line.
4. Formulæ and diagrams of the transformations discovered.

1.

In continuation of the discussion of Cayley's bilinear space-transformation, those cases are here considered in which the first fundamental sextic degenerates into six straight lines, some of which coincide. The geometrical language of actual and apparent double points ceases to have an obvious meaning, and more help is derived from the analytical discussion of the numbers of conditions and of independent surfaces, although it is possible to trace the geometrical meaning in some cases by reference to an adjacent position in which the lines do not coincide. If this adjacent non-coincident pair intersect, there is a distinction between lines in the plane of the pair, which meet both, and other lines which may meet one or neither; so, for the coincident pair which is the limit of this, we expect to find a special plane tangent to all the surfaces at all points of the line, and any line in this plane meets the surfaces in two points on the line. If the adjacent non-coincident pair do not intersect, there is still the distinction between lines meeting both and lines meeting one; the former are the sets of generators of all the quadrics containing the pair of lines. So in the limiting case we expect to find a special family of quadrics touching the surfaces all along the line. But the cubic surfaces can touch each other all along the line without so touching any quadric or plane; in this case the adjacent non-coincident pair, if it existed, would have to be treated as intersecting in -1 points.

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[*] I am indebted to Professor C. A. Scott for kindly reading the manuscript of this paper and making some valuable suggestions.

[†] The first paper appeared in these *Proceedings*, Vol. 9, p. 51.

The fact that certain cases cannot be derived from cases of non-coincident lines becomes apparent on comparing the diagrams of the second fundamental sextic, given at the end of this paper, with those in the former paper. Some of these involve non-degenerate quartics or pairs of conics, which cannot be derived from any of the former set, which all consisted partly of three straight lines.

If the intersection of two surfaces consists partly of two or more coincident straight lines, the surfaces have contact of first or higher order at every point of that line; analytically, in the general equation of the family, the independent terms or groups of terms, *i.e.*, those by which any two members of the family can differ, are small of the second or higher order in the neighbourhood of the line.

Let the equations of the straight line be $0 = z = w$; the terms independent of z, w are absent from the equation of the general cubic surface containing the line, and the remaining terms fall into three groups according to their degree in z, w, which is in general the same as their order of smallness; these are

linear terms of first order $z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2)$,

quadratic terms of second order $z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$,

cubic terms of third order $Sz^3+Tz^2w+Uzw^2+Vw^3$.

Two cases are excluded from present consideration, in which

(i) the general surface degenerates; for then the transformation, if it exists, is of lower degree;

(ii) the linear group disappears entirely; then the line is a common nodal line on all the surfaces, and the transformation is of a different type. This is treated separately in § 3.

If no special relations hold, then if two cubic surfaces touch all along the line, their linear terms must be the same; if they have contact of second order, their linear and quadratic terms must be the same; and they cannot have higher contact without coinciding. Although in general the order of smallness of any group of terms is equal to the lowest degree in z, w of any one term of the group, yet the group of linear terms as a whole, though apparently of first order, is really of second order, for it can be equated to quadratic and cubic terms; and the group of linear and quadratic terms together, though apparently of first order, is really of third order. Any group containing either of these groups as a factor is also of higher order of smallness than appears from considering its separate terms; but such a group is more than cubic in x, y, z, w, and cannot enter into the equation. But if the linear or linear + quadratic group falls into factors, one is necessarily finite (since, if both factors were small, there could be no linear terms in the product), and the other contains terms linear in z, w, so that it is apparently of first order, but its order is really that of the group. If this special factor ϕ is quadratic in x, y, z, w, then the cubic functions $x\phi$, $y\phi$ are of the same order as ϕ, and $z\phi$, $w\phi$ are of higher order. If the special factor θ is linear, the same holds of multiples of θ and of θ^2. Again, if the linear + quadratic group has a factor ϕ and also the linear group a factor θ, the product $\theta\phi$ must be considered; again, if θ or θ^2 is also a factor of the cubic group (but not of the quadratic group, for if so the general surface degenerates), then the cubic group is of fourth or higher order, and ϕ is of the same order.

In all these ways certain groups may occur, which are of higher order of smallness than appears from considering their separate terms, and so increase the number of groups that are independent for a given order of contact all along the line, *i.e.*, for a given number of lines of intersection coinciding in $0 = z = w$. Table I (p. 18) gives the terms which may occur in the various forms which the cubic equation can assume from this point of view. In the first column are the functions θ, ϕ, which in each type have orders of smallness higher than appears, reduced to their simplest forms by choice of a frame of reference. The order of smallness is found in the second column. The rest of the table gives the terms or groups that can then occur, arranged in ascending order of smallness. If one of these is not cubic, it will appear in the equation multiplied by an arbitrary function of x, y only, of proper degree. The frames of reference are so chosen that w is always small of first order, θ occurs in the form z, and ϕ in one of the forms ϕ_1, ϕ_2.

To obtain the types of cubic families which have k straight lines of common intersection coinciding in $0 = z = w$, and have therefore contact of order $k-1$ all along this line, we must consider terms or groups of order of smallness k or higher to be independent; such a family

TABLE I.

		Terms or groups that may occur, of order of smallness:									Type
		1	2	3	4	5	6	7	8	9	
		z, w	z^2, zw, w^2	z^3, z^2w, zw^2, w^3							i
ϕ_1	2		z^2, zw, w^2 ϕ_1	z^3, z^2w, zw, w^3 $z\phi_1, w\phi_1$							ii
ϕ_1	3			z^3, z^2w, zw^2, w^3 ϕ_1	$z\phi_1, w\phi_1$						iii
θ	2		w^2 z	w^3 wz	w^2z z^2	wz^2	z^3				iv
θ	3			w^3 z	wz	w^2z	z^2	wz^2		z^3	v
θ ϕ_2	2 3			w^3 wz, ϕ_2	w^2z $z^2, w\phi_2$	wz^2 $z\phi_2$	z^3				vi
θ ϕ_2	2 4				w^2z z^2, ϕ_2	wz^2 $w\phi_2$	z^3 $z\phi_2$				vii
θ ϕ_2	2 5	$\theta = z$ $\phi_1 = zx + wy$ $\phi_2 = zx + w^2$				wz^2 ϕ_2	z^3 $w\phi$	$z\phi_2$			viii
θ ϕ_2	2 6						z^3 ϕ_2	$w\phi_2$	$z\phi_2$		ix

TABLE II.

Number of coincident lines	2	3	4	5	6	7	8	9	Type
Number of independent groups	11	5							i
	12*	7							ii
		8	3						iii
	13	9	6	3	2				iv
		11	8	6	5	3		2	v
		10*	7†	4	2				vi
			8†	5	3				vii
				6	4	2			viii
					5	3	2		ix

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* Here the groups $z\phi$, $w\phi$ are counted under their separate terms, and do not give additional groups.

† Here $z\phi$ is counted under its separate terms, but $w\phi$ gives an additional group.

will not arise in a type in which there are no terms of order as high as k, nor in a type in which there are no terms of order as low as k, for then there is necessarily contact of higher order.

Table II (p. 18) gives the numbers of independent surfaces which have any given number of coincident lines in each type.

Since the surfaces touch all along the line, the tangent plane is assigned at each of its points. Now the equation of the tangent plane is given by the linear group, $(x, y, 0, 0)$ being the point of contact; and to assign this plane for all values of $x : y$ is, in general, to assign all the coefficients of the group (type i). But if the group has a factor ψ involving x, y only, this does not affect the position of the plane, and the coefficients contained in the factor remain independent. If ψ is linear (type ii), the surfaces all touch the quadric $\phi_1 = 0$, or any one of the family of quadrics $\phi_1 + \alpha z^2 + \beta zw + \gamma w^2 = 0$, all along the line; in type iii the surfaces all osculate the special quadric $\phi_1 = 0$ of this family. If ψ is quadratic, the surfaces touch the plane $\theta = 0$ all along the line (type iv); in type v they osculate the plane $\theta = 0$; in type vi they osculate any one of the family of cones $\phi_2 + \theta(\alpha z + \beta w) = 0$; in type vii they have contact of third order with the more restricted family of cones $\phi_2 + \alpha\theta^2 = 0$; in types viii and ix there is contact of fourth and fifth orders with the special cone $\phi_2 = 0$.

Using the notation of the former paper, a line b meeting the coincident pair a will not in general lie in the given tangent plane at the point ab. If it does not, a surface which also contains b must have a conical point; therefore, in calculating the reduction in the number of conditions imposed by b, due to its intersection with a, we must consider separately those lines which do or do not lie in the given tangent planes at their points of intersection with a. In type iii a further particularity attaches to generators of the special quadric; in types vi and vii, to lines through the vertex of the family of cones; and in types viii and ix, to generators of the special cone.

The same idea appears in analytical form as follows. In any one type a certain number of the coefficients, though arbitrary, are not independent, *i.e.*, they are the same for all the members of the family; these are the coefficients that occur in the group of all terms that are not independent in the case considered, and also those contained in the special functions θ, ϕ; such coefficients will be distinguished by circumflexes. Now, if a line b imposes a condition involving these alone, this condition is not to be counted for the purpose of this investigation; for it does not reduce the number of independent surfaces, but merely restricts the generality of the assigned set of tangent planes, causing it to have some particular relation to b.

The 127 transformations investigated are classified as follows, where, *e.g.*, the symbol $1^4.2$ denotes four distinct lines and one coincident pair. The first set are those given in the former paper.

Reference...	A to H	I to M	N to Q	R to S	T	U	V	W	X	Y	Z
Symbol ...	1^6	$1^4.2$	$1^2.2^2$	2^3	$1^3.3$	$1.2.3$	3^2	$1^2.4$	2.4	1.5	6
Number ...	28	26	19	3	14	14	3	8	5	4	3

We have seen that the nature of the sextic curve is not enough to determine the transformation. The fact that a certain number of the six straight lines coincide is capable of interpretation in several ways; we need to know something of the set of tangent planes, osculating quadrics, &c., given all along the coincident lines. In the set of formulæ and diagrams in § 4, the classification according to the form of the sextic has been retained; but in the investigation of § 2 it has been found more convenient to arrange the transformations, to a certain extent, under the different types to which they belong.

In this second paper the interest lies in coincidence of lines and its interpretation, rather than in coincidence of intersections. No separate investigation is given of the cases in which coincident intersections can be deduced from distinct intersections by obvious changes. The references to the deduced transformations are given, and their equations and diagrams will be found in their places in § 4. In the latter part of § 2 certain cases of coincidence of lines are dismissed in the same way. When three or more lines coincide (T to Z), coincident intersections lose their interest still further, since one arm of the triple point must be of higher multiplicity than any of the others; these varieties do not receive separate letters of reference. When distinct transformations arise from the same form of sextic, they are grouped under the same letter and number.

The method of enumerating the possible transformations is as follows. Considering in turn the cases in which 2, 3, 4, 5, or 6 straight lines coincide, the general cubic equation is written down for each of the possible types; the rest of the six lines have then to be arranged so as to reduce the number of independent cubics to four. Every such arrangement gives rise to a transformation.

It is clearly useless to consider a case in which there are fewer than four independent cubics; it is also useless to consider a case in which there are more than four. In Cayley's language, the condition of equivalence implies the condition of postulation. For, in general, let F_1, F_2, F_3, F_4, F_5 be five members of a linear family satisfying the condition of equivalence, *i.e.*, such that any three independent surfaces of the form

$$\sum_1^5 (A_n F_n) = 0$$

meet in one and only one point not common to all the surfaces of the family. Take the general points P', P'', and let F' be the value of F at P', &c. Then the three equations contained in

$$\left\| \begin{matrix} F_1 & F_2 & F_3 & F_4 & F_5 \\ F_1' & F_2' & F_3' & F_4' & F_5' \\ F_1'' & F_2'' & F_3'' & F_4'' & F_5'' \end{matrix} \right\| = 0$$

represent surfaces which intersect in the two points P', P'' not common to the family. They are therefore not independent, but connected by a linear relation, which is a linear relation connecting F_1, F_2, F_3, F_4, F_5; therefore the family does not contain more than four linearly independent members.

With regard to the connection between the first and second fundamental sextics of any of these transformations, it may be remarked that if one has an ordinary triple point, the other contains a real or degenerate plane cubic, and *vice versa.*

It may also be remarked that the transformations B4 and B5, and those deducible from them, may be obtained as squares of the quadri-quadric transformation; A3 and W as squares of the quadri-cubic transformation; and D as the product of a quadri-quadric and a quadri-cubic transformation.

2.

I to M. *Two coincident and four distinct straight lines.*

A reference to the second table shows that types i, ii, and iv are the only ones to be considered, and that the four distinct lines must give exactly 7, 8, and 9 conditions in the respective types.

Type i.

$$\begin{aligned} & z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+w(\hat{H}x^2+\hat{J}xy+\hat{K}y^2) \\ & +z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry) \\ & +Sz^3+Tz^2w+UzW^2+Vw^3 \qquad = 0. \end{aligned}$$

A line b meeting a gives two conditions. (Here, as in all that follows, "condition" is used in the sense of "condition which reduces the number of independent surfaces." Of the four conditions imposed by b, one is satisfied identically, one causes the given tangent plane at the point ab to contain b, and two are effective for our purpose.)

If $bcde$ all meet a, then two of them meet, say, bc; this may be specialized by allowing intersections to coincide [I1 ; J1 ; J′1 ; K1 ; L1 ; L′1 ; M. Since x^3 is absent, and the terms in x^2 are not independent, each line through the vertex (1, 0, 0, 0) gives only two conditions, and a quadruple point does not offer the peculiarity noticed in the former paper (F). An ordinary triple point acd gives a degenerate system unless another arm passes through the point.]

If only bcd meet a, then
either e meets bcd [I2 ; J2 ; J3 ; K2];
or e meets bc, and d meets b [I3; J4 ; J5 ; J′2 ; L′2].

If only bc meet a, the system imposes too many conditions.

Type ii.

$$(zx+wy)(Ex+Fy)+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line b meeting a and lying in the given tangent plane at the point ab gives two conditions. A line meeting a and not lying in the given tangent plane gives a relation between E, F and reduces this to a particular case of type i; therefore we need not consider a triangle nor an ordinary triple point on a, for in these cases one line could not lie in its tangent plane. A pair of lines meeting on a and both lying in the same tangent plane also gives a relation between E, F; therefore we need not consider a special triple point on a.

Then $bcde$ may all meet a [I4].
If only bcd meet a, then e meets two of them [I5].
If only bc meet a, then d meets both, and
either e meets bc [I6 ; J6];
or e meets bd [I1 ; J′3].
If only b meets a, there are too many conditions.

Type iv.

$$z(Ex^2+Fxy+Gy^2)+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line lying in the tangent plane $z=0$ gives two conditions. A line meeting a and not lying in the tangent plane gives three conditions.

There are too many conditions unless there is a line b in the tangent plane; there cannot be more than one.

If only b meets a, there are too many conditions.

If only bc meet a, there must be four more intersections; the configurations are the same as in the corresponding cases of type ii. Taking c to be $0 = w = x$, then $G = 0$, and these are particular cases of those of type ii, obtained by omitting the term wy from the quadratic factor.

If only bcd meet a, we have in the same way particular cases of type i with two relations between E, F, G; except in the case there excluded, in which acd form an ordinary triple point; then

either e meets bc [J7; $\text{L}2_1$; bce cannot concur, for if so, c lies in the tangent plane];

or e meets cd; then b must meet one side of the triangle cde, and therefore passes through the vertex acd [$\text{L}2_2$].

If $bcde$ all meet a, there must be two more intersections, and b can meet no other line; the system degenerates unless there is a triple point, and gives a particular case of type i unless there is a quadruple point $acde$ [L3]. A quintuple point at (1, 0, 0, 0) causes x to disappear entirely.

N to S. *Two coincident pairs and two distinct straight lines.*

First let the coincident pairs not intersect. Then let them be

$$(a)\quad 0 = z = w, \qquad (b)\quad 0 = x = y.$$

The cubic containing these can be rearranged in the form

$$\begin{aligned} z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2) & \\ +x(Lz^2+Nzw+Qw^2) & \\ +y(Mz^2+Pzw+Rw^2) &= 0. \end{aligned}$$

Each of a, b may belong to any one of the types i, ii, or iv.

Type a i, b i gives only two independent surfaces.

Type a ii, b i gives only three.

Type a iv, b i gives four, all containing the line $(c)\ 0 = z = \hat{Q}x+\hat{R}y$, but no other line.

Type a ii, b ii gives four, the general form being

$$(zx+wy)(Ex+Fy)+[x(\hat{N}z+\hat{P}w)+y(\hat{Q}z+\hat{R}w)](Lz+Mw) = 0,$$

all containing c, d the two remaining lines of intersection of the two quadrics; cd both meet ab [N1].

Then *cd* may coincide, type ii [R].

Type *a* iv, *b* ii gives five independent surfaces, the general form being

$$z(Ex^2+Fxy+Gy^2)+(zx+wy)(Lz+Mw)=0,$$

all containing (*c*) $0=y=z$, which meets *ab*. Now *d* must give only one condition; but it cannot meet *abc*, for if so *abcd* would be in a plane; it cannot lie in the tangent plane along *a*, for if so the system would degenerate; therefore *d* meets *b* and lies in the given tangent-plane at the point *bd* and also meets another line, which cannot be *c*, but must be *a*, giving a particular case of N1, in which the first quadric degenerates.

Type *a* iv, *b* iv gives six independent surfaces, the general form being

$$z(Ex^2+Fxy+Gy^2)+x(Lz^2+Nzw+Qw^2)=0,$$

all containing (*c*) $0=x=z$, which is the line of intersection of the tangent planes. Then *d* must give two conditions; but it cannot lie in either tangent plane, and therefore cannot meet *ac*; therefore *d* meets *ab*, giving again a particular case of N1.

[The transformation R is the only one with three pairs of coincident lines which do not concur; this arrangement need not be sought again.]

Next let the coincident pairs intersect. Then let them be

$$(a)\quad 0=z=w,\qquad (b)\quad 0=y=w.$$

Type *a* i, *b* i.

$$zy(\hat{F}x+\hat{G}y+\hat{M}z)+w(\hat{H}x^2+\hat{J}xy+\hat{K}y^2)+wz(\hat{N}x+\hat{T}z)$$
$$+Pzwy+w^2(Qx+Ry+Uz+Vw)=0,$$

giving six independent surfaces, unless $\hat{F}=\hat{H}=0$, in which case there are seven, for then the terms linear in y, w are independent of those linear in z, w.

If this is not so, the surfaces all contain (*c*) $0=w=\hat{F}x+\hat{G}y+\hat{M}z$. Then *d* must give two conditions and therefore meet *a* only [N2; P1; P2; P′1; Q′1. If *cd* coincide, there are too few conditions.]

But if $\hat{F}=\hat{H}=0$, we must have three conditions. A line in the plane *ab* reduces this to the last case.

If *cd* meet *a*, they meet each other; then if the plane of the triangle is $z = 0$, one side is $0 = z = \hat{J}y + Qw$ forming a triple point with ab [$P3_1$; $Q'2$; S_1 (*cd* coincide, type iv, the tangent plane containing *a*)].

If only c meets a, then d meets bc; the case immediately following includes this [N3].

Type *a* ii, *b* i.

$$(zy+wx)(\hat{E}x+Fy)+\hat{M}z^2y+zw(\hat{N}x+\hat{T}z)$$
$$+Pzwy+w^2(Qx+Ry+Uz+Vw) = 0.$$

We have seen that any line meeting a lies in its given tangent plane, and that there are no triangles nor triple points on a. A line meeting either a or b gives two conditions.

If cd both meet a, they cannot meet each other, and there are too many conditions.

If only c meets a, then d meets bc [N3].

If no line meets a, then bcd form a triangle [N4 ; $P'2$].

Type *a* iv, *b* i.—First let (b) $0 = y = w$ not lie in $z = 0$, the tangent plane along a

$$zy(\hat{F}x+Gy+\hat{M}z)+wz(\hat{N}x+\hat{T}z)+Pwzy+w^2(Qx+Ry+Uz+Vw) = 0.$$

If G is not independent, a is type i; therefore any line meeting a either lies in the plane $z = 0$ (but there can only be one such), or forms a triple point with ab, giving in either case two conditions; therefore

either c meets a, then d meets bc (a particular case of N3);

or bcd form a triangle; then if its plane is $y = 0$, one side is $0 = y = \hat{N}z + Qw$, forming a triple point with ab [$P3_2$];

or there is an ordinary quadruple point [Q_1 ; S_2 (*cd* coincide, type iv)].

Next let b lie in the tangent plane $w = 0$ along a.

$$w(\hat{H}x^2+\hat{N}xz+\hat{T}z^2)+\hat{M}yz^2$$
$$+wy(Jx+Ky+Pz)+w^2(Qx+Ry+Uz+Vw) = 0.$$

No other line can lie in $w = 0$. A line meeting a gives three conditions and a line meeting b gives two. Therefore

either c meets a, and d meets bc (a particular case of N3);

or cd both meet b [$N5_1$; if *bcd* concur, the system degenerates ; $P4_1$; Q_2 ; if *ca* coincide, we have S_1 with *a*, *c* interchanged].

Type *a* ii, *b* ii.

$$(zy+wx)(Ex+Fy+Mz)+Pzwy+w^2(Qx+Ry+Uz+Vw)=0.$$

There are no triangles nor triple points. Any line meeting *a* or *b* gives two conditions. Therefore

either *cd* both meet *a* [N5_2];
or *c* meets *a*, and *d* meets *b* [N6].

Type *a* iv, *b* ii cannot occur, since it requires more terms linear in y, w than are present in either form of type *a* iv, *b* i.

Type *a* iv, *b* iv. First let (*b*) $0=y=w$ not lie in $z=0$, the tangent plane along *a*, nor *a* in $y=0$, the tangent plane along *b*.

$$zy(Fx+Gy+Mz+Pw)+w^2(Qx+Ry+Uz+Vw)=0.$$

A line in the plane $z=0$ gives two conditions, but two such lines would make the system degenerate. A line *c* meeting *a*, but not lying in the tangent plane, gives a relation between F, G and reduces *a* to type i, except in the case in which *abc* form a triple point; then *c* gives two conditions, but two such lines would cause x to disappear entirely.

Therefore *abc* form a triple point, and *d* lies in one of the tangent planes [P4_2].

Next let *b* lie in $w=0$, the tangent plane along *a*. Then in the second form of type *a* iv, *b* i,

either the tangent plane along *b* is $y=0$ not containing *a*; then
$\hat{H}=\hat{N}=\hat{T}=0$, and *b* is type i;
or the tangent plane along *b* is $w=0$ containing *a*; then
$\hat{M}=0$, and the system degenerates.

T to V. *Three coincident and three distinct straight lines.*

Type i gives five independent cubics, and the three distinct lines give too many conditions.

Type ii.

$$(zx+wy)(\hat{E}x+\hat{F}y+Lz+Rw)+\hat{M}z^2y+zw(\hat{N}x+\hat{P}y)+\hat{Q}w^2x$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

A line (*b*) $0=y=w$, not lying in the tangent plane at the point *ab*, gives two conditions $\hat{E} = L = S = 0$, and then all the surfaces contain (*c*) $0 = w = \hat{F}x + \hat{M}z$, lying in the tangent plane at the point *ac*, which is the plane *ab*. A line *b* in its tangent plane gives one condition. Therefore

either *bcd* all meet *a*, each in its tangent plane [$\mathrm{T1}_1$; T7; $\mathrm{U2}_1$ (*bc* coincide, type ii); $\mathrm{U2}_2$ (*bc* coincide, type iv); V_1 (*bcd* coincide, type iii); V_2 (*bcd* coincide, type iv)];

or *bc* meet *a*, each in its tangent plane and *d* meets *ab* [T2; $\mathrm{T4}_1$; $\mathrm{U1}_1$ and $\mathrm{U5}_1$ (*bc* coincide, type i)].

Type iii.

$$(zx+wy)(Ex+Fy+Lz+Rw)+Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

The surfaces osculate the quadric $zx+wy=0$; therefore if they contain any other point on the quadric, the generator through that point which meets *a* meets the surfaces in four points, and therefore lies on them all.

Therefore a line *b* not meeting *a* fixes two generators *cd* and imposes four conditions [$\mathrm{T3}_1$; $\mathrm{U3}_1$ (*cd* coincide, type ii)].

A line *b* meeting *a*, not a generator, fixes one generator, but reduces *a* to type ii.

Three generators *bcd* give too few conditions unless *bc* coincide, type i, [$\mathrm{U2}_3$], or *bcd* all coincide, type ii [see above; if *bcd* coincide, type iv, *a* becomes type ii].

Type iv.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+w^2(\hat{Q}x+\hat{R}y)+z^2(Lx+My)+zw(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line in the tangent plane gives one condition; there can only be one such line. A line meeting *a* and not lying in the tangent plane gives two conditions.

If *b* lies in the tangent plane,

either *cd* meet *a* [$\mathrm{T1}_2$; $\mathrm{T5}_1$ (*acd* concur); $\mathrm{T5}_2$ (*abc* concur); T8; $\mathrm{U2}_4$ and $\mathrm{U4}_1$ (*bc* coincide, type i); $\mathrm{U2}_5$ and $\mathrm{U4}_2$ (*cd* coincide, type iv); if *bcd* all coincide, *a* is type iii];

or *c* meets *a*, and *d* meets *bc* [$\mathrm{T3}_2$; $\mathrm{T6}_1$; $\mathrm{U3}_2$ (*bc* coincide, type i)].

If no line lies in the tangent plane, *bcd* all meet *a*, and *c* meets *d*; but this causes the linear terms to vanish entirely unless there is a triple point abc [$T4_2$; $T9$; $U1_2$ and $U5_2$ (bc coincide, type i); if cd coincide, type i, we have a particular case of $U4_2$; V_3 (bcd coincide, type v)].

Type v.

$$z(Ex^2+Fxy+Gy^2)+z^2(Lx+My)+zw(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

The plane $z=0$ meets the surfaces in a thrice, and therefore every other line meets a and does not lie in $z=0$, giving three conditions. There would have to be two intersections between bcd, but then $abcd$ would all lie in a plane.

Type vi.

$$(zx+w^2)(Ex+Fy)+z^2(Lx+My)+wz(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

If E, F are related, this becomes type iv; therefore the only lines to be considered are those which do not meet a, giving four conditions, and those through the vertex of the special cone and not lying in the tangent plane, giving two conditions.

If bcd all pass through this point, y disappears entirely unless bc coincide, type i, but then a is type iii unless further bcd coincide, type iv (see above).

If only bc pass through the vertex, the triple point cannot be special and d meets bc [$T6_2$; $U4_3$ (bc coincide, type i)].

W, X. *Four coincident and two distinct straight lines.*

Type iv.—

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+zw(\hat{N}x+\hat{P}y)+w^2(\hat{Q}x+\hat{R}y+\hat{V}w)$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0,$$

all containing (b) $0=z=\hat{Q}x+\hat{R}y+\hat{V}w$.

Then c must give two conditions, and therefore meet a [$W1_1$; $W3_1$; if bc coincide, there are too few conditions].

Type v.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+\hat{V}w^3+z^2(Lx+My)+zw(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2=0.$$

No other line can lie in the tangent plane: bc must both meet a [$W1_2$; if abc concur, we have a particular case of type vi].

If bc coincide, let them be $0=y=w$; therefore $\hat{E}=L=S=0$, and the terms linear in y, w are $yz(\hat{F}x+Mz)+wz(Nx+Tz)$. Then

either $\hat{F}=0$, so that this group is independent of the terms linear in z, w, and b is type i [X_1];

or $\hat{F}\neq 0$; then b must be type iv, the tangent plane along b containing a, giving a particular case of type vi.

Type vi.

$$(zx+w^2)(\hat{E}x+\hat{F}y+Rw)+wz(\hat{N}x+\hat{P}y)+\hat{V}w^3$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

A line in the tangent plane would make R not independent and reduce a to type iv. A line meeting a and not lying in the tangent plane gives two conditions; but two such lines cause the linear terms to disappear entirely, unless one passes through the vertex of the cone; a second line through the vertex gives only one condition; therefore

either abc form a triangle with one side through the vertex [W2; W4; X_2; X_3 (bc coincide, type iv, the tangent plane containing a)],

or abc form an ordinary triple point at the vertex of the cone [W3_2].

Type vii.

$$(zx+w^2)(Ex+Fy+Rw)+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

If E, F are not independent, this reduces to type vi; therefore bc coincide, type i, and pass through the vertex of the cone, but do not lie in the tangent plane [X_4].

Y. *Five coincident straight lines and one distinct.*

Type v.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+zw(\hat{N}x+\hat{P}y)+\hat{V}w^3$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

No other line can lie in the tangent plane; b must meet a, giving the two conditions required [Y_1].

Type vi.

$$(zx+w^2)(\hat{E}x+\hat{F}y+Qz)+z^2(\hat{L}x+\hat{M}y)+zw(\hat{N}x+\hat{P}y)+\hat{U}w^2z+\hat{V}w^3$$
$$+Sz^3+Tz^2w=0,$$

all containing (*b*) $0 = z = \hat{E}x + \hat{F}y + \hat{V}w$, and giving four independent surfaces $[Y_2]$.

Type vii.

$$(zx + w^2)(\hat{E}x + \hat{F}y + Qz + Rw) + z^2(\hat{L}x + \hat{M}y) + \hat{U}zw^2 + Sz^3 + Tz^2w = 0.$$

If b lies in the tangent plane, R is not independent, giving type vi. There are too many conditions unless b passes through the vertex of the special cone $[Y_3]$.

Type viii.

$$(zx + w^2)(Ex + Fy + Qz + Rw) + Sz^3 + Tz^2w = 0.$$

A line in the tangent plane reduces this to type vi. A generator of the special cone gives too few conditions; an arbitrary line through the vertex causes y to disappear entirely. Any other line gives too many conditions.

Z. *Six coincident straight lines.*

Type viii is the only one giving exactly four independent surfaces $[Z]$.

3.

It remains to consider the case of a common nodal line, reserved (p. 16) for separate treatment. Cayley's investigation excludes this, but the extended formulæ, given without proof at the end of his paper,* show that, besides one nodal and two distinct straight lines, the fundamental system must contain two additional points. The birational transformation which results is not in general bilinear. It will be shown that if it is so, the fundamental points must both lie on the nodal line, *i.e.*, at each of two fixed points of this line one set of sheets of the cubic surfaces touch a fixed plane.

Only a particular case of Cayley's formula is required, the proof of which may be outlined as follows:—

Lemma.—The two distinct lines bc of the sextic must meet the nodal line a, and cannot meet each other unless abc form a triple point.

For the terms linear in z, w are absent, and if b does not meet a, we may take it to be $0 = x = y$, and the terms cubic in z, w vanish. Then $XYZW$ are all formed from terms quadratic in z, w and linear in x, y, and their ratios are functions of the two quantities $z : w$ and $x : y$ only; therefore b and c must meet a. Now any plane through a meets the surface in only one other line, therefore bc cannot lie in a plane through a and so cannot meet each other except on a.

We may therefore assume (*b*) $0 = y = w$, (*c*) $0 = x = z$. Then $XYZW$ are formed from the terms z^2y, w^2x, zwx, zwy, z^2w, zw^2, and may be chosen so that the first two of these terms do not enter into Z, W, which are therefore zwp_1, zwp_2, where p_1, p_2 are linear in x, y, z, w.

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* Cayley, *Proc. London Math. Soc.*, Vol. III, p. 179.

Now three of the cubic surfaces can meet in only one point not belonging to the fundamental system; but $Z=0$, $W=0$ meet the general surface in three points lying on the straight line $0=p_1=p_2$; therefore two of these points are fundamental, and either are isolated fundamental points or lie on the fundamental lines. In the former case we can proceed to construct the transformation; the latter case can be obtained from the former by letting the straight line $0=p_1=p_2$ move from its general position to intersect two or coincide with one of the fundamental lines. The surfaces will then be found to have contact at the limiting positions of the isolated fundamental points which have come to lie on the fundamental lines.

If the isolated fundamental points are $(\alpha\beta\gamma\delta)$ and $(\xi\eta\zeta\theta)$, the equations of transformation are

$$\begin{aligned} X:Y:Z:W = {} & w^2x\gamma\zeta(\delta\zeta-\gamma\theta)+zw\left[z\delta\theta(\alpha\zeta-\gamma\xi)-w(\alpha\delta\zeta^2-\gamma^2\xi\theta)\right] \\ : {} & z^2y\delta\theta(\delta\zeta-\gamma\theta)+zw\left[z(\beta\gamma\theta^2-\delta^2\eta\zeta)-w\gamma\zeta(\beta\theta-\delta\eta)\right] \\ : {} & zw\left[x(\delta\zeta-\gamma\theta)+z(\alpha\theta-\delta\xi)-w(\alpha\zeta-\gamma\xi)\right] \\ : {} & zw\left[y(\delta\zeta-\gamma\theta)+z(\beta\theta-\delta\eta)-w(\beta\zeta-\gamma\eta)\right], \end{aligned}$$

which will be found to be birational and symmetrical, but not in general bilinear.

Now, if three bilinear relations exist, and the matrix of coefficients of $XYZW$ is

$$\left\|\begin{matrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{matrix}\right\|,$$

the surfaces $X=0$, $Y=0$ meet in the fundamental sextic and a cubic curve given by

$$\left\|\begin{matrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{matrix}\right\| = 0.$$

If $P(1, 0, 0, 0)$ is a fundamental point, it lies on all these cubic curves, and its coordinates annihilate all the determinants of order two contained in the first matrix. This implies that the terms in x in any column are proportional to those in any other column, so that the terms in x^3, x^2 vanish in all the four determinants X, Y, Z, W.

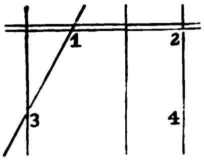
Therefore P is a conical point on all the surfaces, and lies on the common nodal line; for if not the surfaces degenerate, containing the plane aP.

Therefore both the fundamental points lie on the nodal line.

This gives the transformation $W1$. Then bc may meet on a $[W3]$, or coincide, type i $[X]$. Or b may coincide with a; then one sheet of every surface touches a fixed plane all along the line $[Y]$. Or, finally, bc may both coincide with a; then either both sheets of every surface touch fixed planes all along the line $[Z_1]$, or one sheet osculates a fixed plane $[Z_2]$.

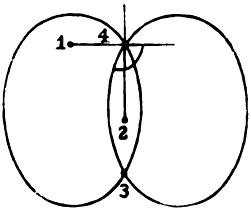
4.

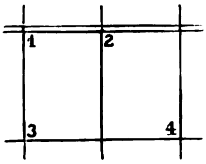
This section contains the actual bilinear equations of all the transformations discovered in the preceding investigation. In each case the frame of reference has been chosen so as to give these equations in their most simple and symmetrical form. All superfluous constants have been absorbed; of those which remain, Greek letters denote constants depending on the position of the straight lines, and Roman letters those depending on the position of the given tangent planes. Coincident lines are represented by parallel lines drawn close together and the conventions explained in the former paper are observed.



I 1

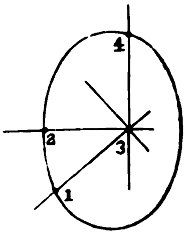
$$\begin{aligned} Xz &= Zx \\ Y(w+z) &= Z(x+y) \\ Ww &= (X-Y)(\alpha x+\beta y)+\gamma Xy \end{aligned}$$

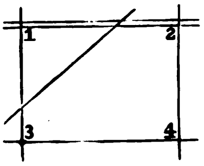




I 2

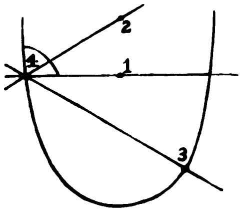
$$\begin{aligned} (X+Y)w &= (W-X)z \\ Zz &= X[H(x+y)-Fy]+Y[J(x+y)+Fx] \\ Zw &= W(Hx+Jy)-X[H(x+y)-Fy] \end{aligned}$$

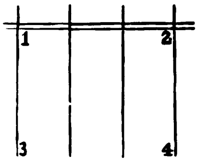




I 3

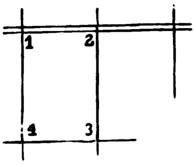
$$\begin{aligned} Xy &= Yx \\ Xw &= Zz \\ Ww &= Y(x+y+z)+Z(Hx+Jy) \end{aligned}$$





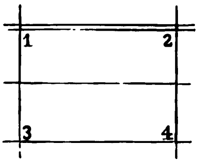
I 4

$$\begin{aligned} \alpha(Xw - Wx) &= \beta[Yz - Z(y+z)] \\ &= \gamma(Xz - Wy) \\ &= \delta(Zx - Yw) \end{aligned} \tag{I 4}$$



I 5

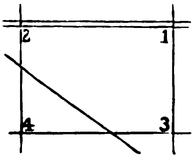
$$\begin{aligned} Xy &= Yx \\ X(z+w) &= x(Z+W) - Ww \\ Y(z+w) &= y(Z+W) + Zz \end{aligned} \tag{I 5}$$



I 6

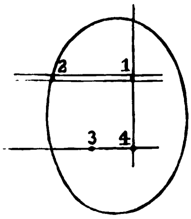
$$\begin{aligned} Zw &= Wz \\ \alpha Xy - \beta Yx &= Zy + Wx \\ &= Yz + Xw \end{aligned} \tag{I 6}$$

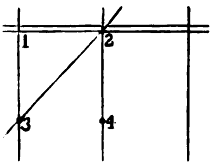
De Paolis, *Rendic. R. Acc. Linc.* (4), I, p. 740, 2°; $\mu = \nu_1 = \nu_2 = 1$.



I 7

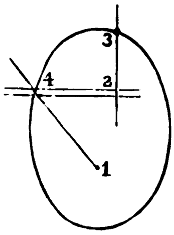
$$\begin{aligned} Zw &= Wz \\ Xx &= Y(y+z+w) \\ Yw &= Zy + Wx \end{aligned}$$

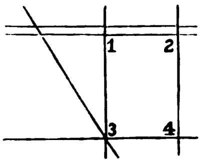




J 1

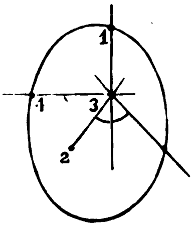
$$\begin{aligned} Xz &= Zx \\ Y(z+w) &= Z(x+y) \\ Ww &= X(x+Hy) - Yx \end{aligned}$$



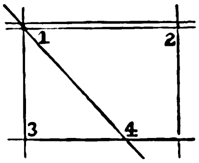


J 2

$$\begin{aligned} Xy &= Yx \\ Xw &= Wz \\ Zw &= Y(x+y) + W(Hx + Jy) \end{aligned}$$

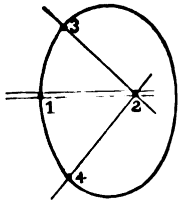


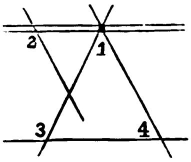
D



J 3

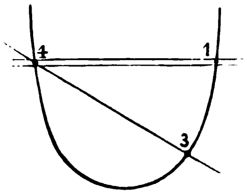
$$Zw = Wz$$
$$Yz = Z(Fx + Gy) + (J - F)Xy$$
$$Yw = W(Jx + Gy) - (J - F)Xy$$

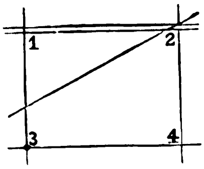




J 4

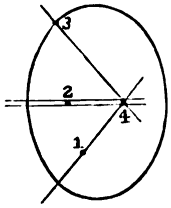
$$Xy = Yx$$
$$Yw = Zz$$
$$Ww = Y(x + z) + Z(Jx + Ky)$$

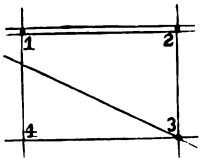




J 5

$$Xy = Yx$$
$$Xw = Zz$$
$$Ww = Y(x + z) + Z(Hx + Jy)$$

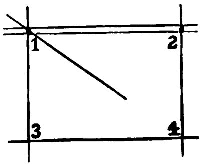




J 6

$$Zw = Wz$$
$$Zx = Xy - Yx$$
$$= Yw - Wy$$

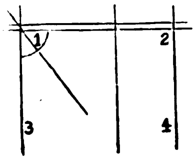
I 6



J 7

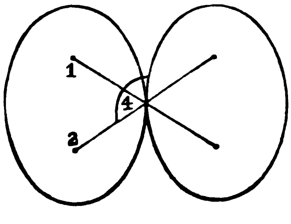
$$Xy = Yx$$
$$Y(z + w) = Z(y + w)$$
$$Zx = X(z + w) + Wz$$

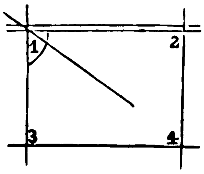
N 3



J'1

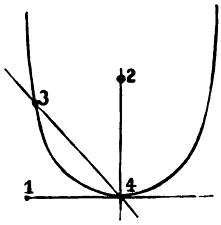
$$Xz = Zx$$
$$Y(z + w) = Z(x + y)$$
$$Ww = HXy + (X - Y)(Hx + y + z)$$

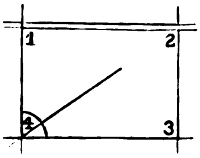




J'2

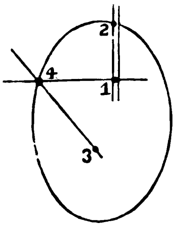
$$Xy = Yx$$
$$Xw = Zz$$
$$Ww = Y(y+z) + Z(Hx+Jy)$$

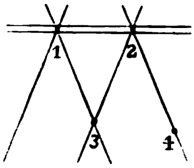




J'3

$$Zw = Wz$$
$$Xy = Y(x+z)$$
$$Yw = Wy + Zx$$

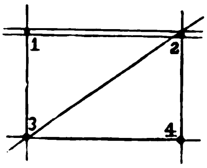




K 1

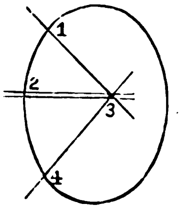
$$Xz = Zx$$
$$Zy = Y(z+w)$$
$$Ww = JXy + Yx$$

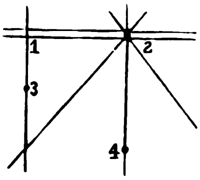
N 3



K 2

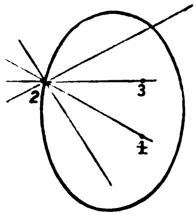
$$Xy = Yx$$
$$= Zw + W(Hx+Jy)$$
$$Xw = Wz$$

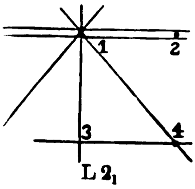




L 1

$$Xw = Wz$$
$$X(z+w) = Zx$$
$$Yw = y(X+Z) + W(Hx+Jy)$$





L 2_1

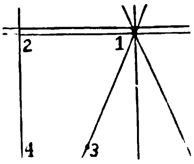
$$Xy = Yx$$
$$Zw = Wz$$
$$Y(z+w) = Z(y+w)$$

P 3_2

L 2_2

$$Xy = Yx$$
$$Zw = W(z + \delta w)$$
$$Y(z + w) = y(Z + W) + Wz$$

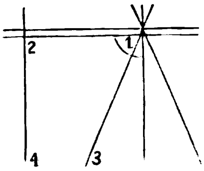
L 2_2



L 3

$$Xz = Zx$$
$$Y(\gamma z + w) = W(y + w)$$
$$Y(z + w) = y(Z + W)$$

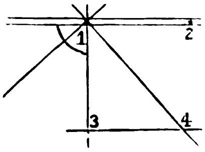
L 3



L′ 1

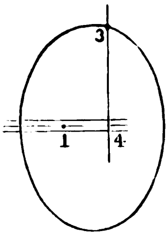
$$Xz = Zx$$
$$Zy = Y(z + w)$$
$$Ww = J(Xy - Yx) + (\beta y + \gamma z)(Y - Z)$$

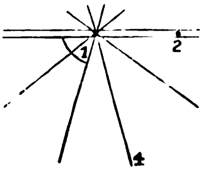
T 1



L′ 2

$$Xy = Yx$$
$$Yw = Zz$$
$$Ww = Y(y + z) + Z(Jx + Ky)$$

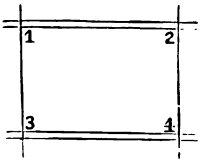




M

$$Yy = Zz$$
$$Zw = W(z + w)$$
$$Xw = Wx + (Y - Z)(\beta y + \gamma z)$$

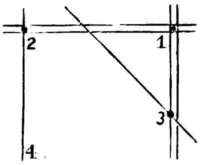
M



N 1

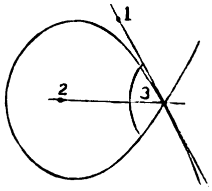
$$Xy = Yx$$
$$Zw = Wz$$
$$Xw + Zy = JWx + NYz$$

N 1

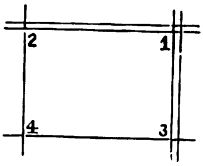


N 2

$$Xz = Wx$$
$$Yw = Wy$$
$$Zw = Y(x + y) + X(Hx + Jy) + W(Nx + Tz)$$

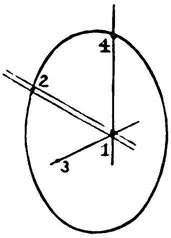


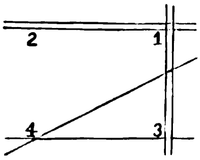
(One line is a tangent at the node of the quartic.)



N 3

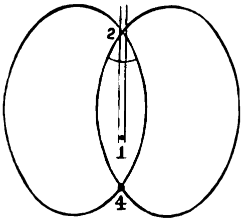
$$Yw = Wy$$
$$Xy = Y(Ex + Mz) - (M - N)\,Zx$$
$$Xw = W(Ex + Nz) + MZz$$

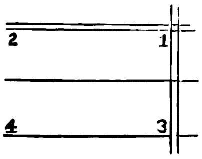




N 4

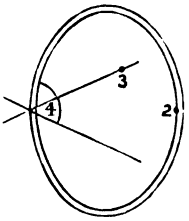
$$Zw = Wz$$
$$Zy = Xw - Wx$$
$$Yy = M(Xz - Zx) + x(X + Z)$$





N 5_1

$$Xy = Yx$$
$$Zy = Yz + w(X + Z)$$
$$Ww = Zz + M(Xz - Zx)$$

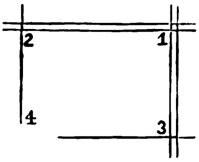


(Two coincident conics.)

N 5_2

$$Xy = Yx$$
$$Zw = W(z + Nw)$$
$$Xw = W(x + Nw) + Zy - Yz$$

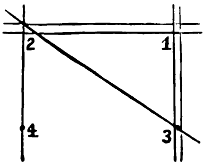
N 5_2



N 6

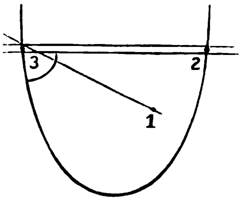
$$Xz = Zx$$
$$Yz = Zy + Ww$$
$$Y(x - w) = y(X - W)$$

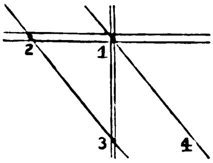
N 6



P 1

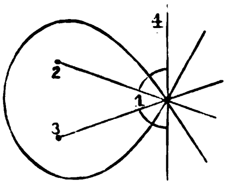
$$Yw = Wy$$
$$Xz = Wx$$
$$Zw = FYx + X(Hx + Jy) + W(Nx + Tz)$$



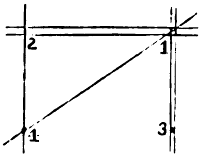


P 2

$$Yy = Zz$$
$$= Ww$$
$$Xw = Wx + Y(Nx + Tz) + Z(Jx + Ky)$$

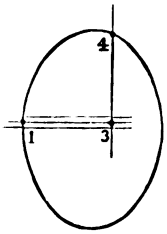


(Plane nodal cubic.)



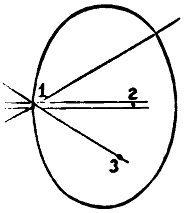
P 3_1

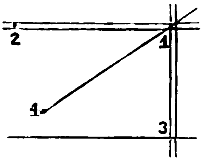
$$Yw = Wy$$
$$Zy = MXz - Y(Jx + Tz)$$
$$Zw = GYz + W(Nx + Tz)$$



P 3_2

$$Yy = Ww$$
$$Zw = Wz$$
$$Xz = x(Y + W) + z(Fx + My)$$





P 4_1

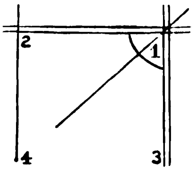
$$Xy = Yx$$
$$Y(z + w) = Z(y + w)$$
$$Ww = Z(z - x) + X(z + w)$$

U 2_1

P 4_2

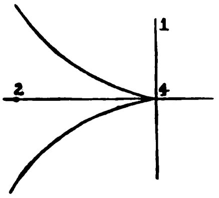
$$Xy = Yx$$
$$Zw = W(z + w)$$
$$Z(y - w) = z(Y - W)$$

P 4_2

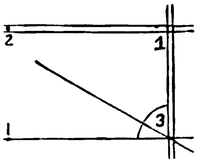


P′ 1

$$Xz = Zx$$
$$Yw = Zy$$
$$Ww = Y(y + z) + X(Hx + Jy) + Z(Nx + Tz)$$



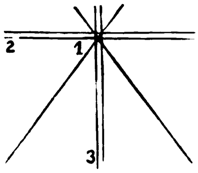
(One line is a tangent at the cusp of the quartic.)



P′2

$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Yw &= Zy + W(x+w) \end{aligned}$$

N 5_1



Q_1

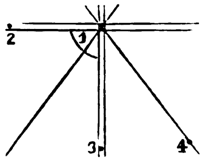
$$\begin{aligned} Yw &= Wy \\ Xy &= Yx + MZz \\ Xw &= Wx - MZz + (z+w)(JZ + MW) \end{aligned}$$

Q_1

Q_2

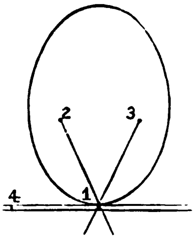
$$\begin{aligned} Xw &= Wx + Zz \\ Yw &= W(y-w) \\ Yz &= Z(y+z) \end{aligned}$$

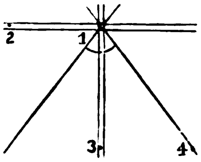
Q_2



Q′1

$$\begin{aligned} Yy &= Zz \\ &= Ww \\ Xw &= W(y+z) + Y(Nx+Tz) + Z(Jx+Ky) \end{aligned}$$

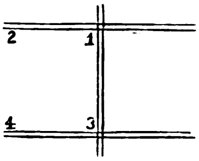




Q′2

$$\begin{aligned} Zw &= Wz \\ Xw &= Wx + Zy \\ Xz &= Yy + Z(x - z - w) \end{aligned}$$

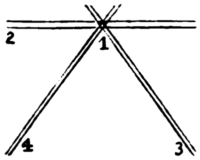
W 3_2



R

$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xw + Y(z + Nw) &= Wx + y\,(Z + FW \end{aligned}$$

R



S_1

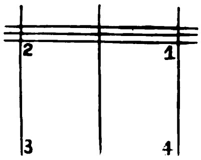
$$\begin{aligned} Zw &= Wz \\ Xw &= Wx + GZy \\ Xz &= Zx + M(Yy + Zz) \end{aligned}$$

X_2

S_2

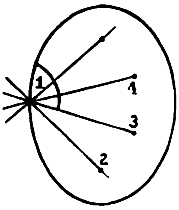
$$Yw = Wy$$
$$Xy = Yx + TZz$$
$$Xw = W(x+Tz) + Z(Fz+Jw)$$

S_2



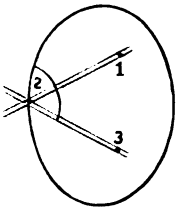
T 1_1

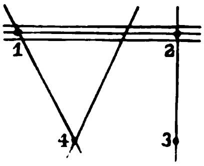
$$Zw = Wz$$
$$Xz = Yy + Z(Ex+Fy+w)$$
$$Xw = Yx + W(Ex+Fy+z)$$



T 1_2

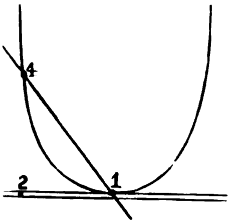
$$Xw = Wx$$
$$Z(z+w) = W(x+y)$$
$$Yz = X(x+y) - Zx + RWy$$

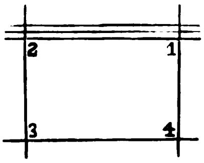




T 2

$$Zw = Wz$$
$$Yw = Wy + Zx$$
$$Xz = Y(x+y) + Z(Nx+Py)$$





T 3_1

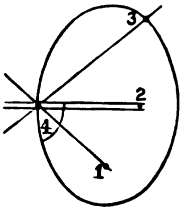
$$Xy = Yx$$
$$Zw = Wz$$
$$Xz = Zx + Yw - W(y-z)$$

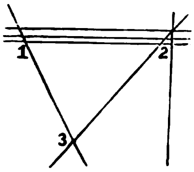
T 3_1

De Paolis, *l.c.*, p. 741, 1°; p. 754, 1°; $\mu = 1$, $\nu = 2$.

T 3_2

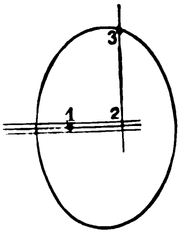
$$Xy = Yx$$
$$Xz = Zw$$
$$Wz = RYw + Z(Ex+Fy)$$





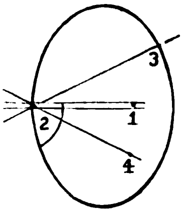
T 4_1

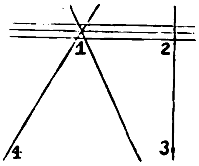
$$Zw = Wz$$
$$Xz = Zx + Wy$$
$$Xy = Yw + W(Nx + Py)$$



T 4_2

$$Zw = Wz$$
$$X(w + z) = Zx$$
$$X(y + Qw) = Yw + W(Qx + Ry)$$





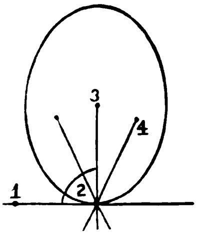
T 5_1

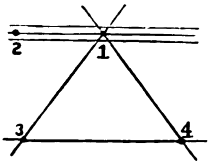
$$Xw = Wx$$
$$Y(z + w) = Wy$$
$$Yw = Zz + F(Xy - Yx)$$

U 2_3

T 5_2

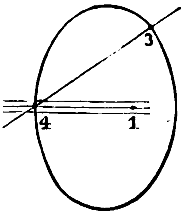
$$Xz = Zw$$
$$X(z + w) = Wx$$
$$Yz = Z(Ex + Fy) + Wy$$





T 6_1

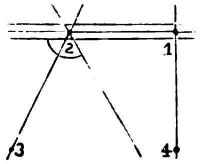
$$Xy = Yx$$
$$Yz = Zw$$
$$Yw = Wz + Z(Fx + Gy)$$



T 6_2

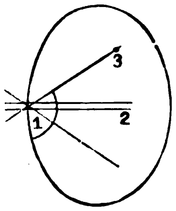
$$Xy = Yx$$
$$Zw = Wz$$
$$Xz = Zx + (W + Z)w$$

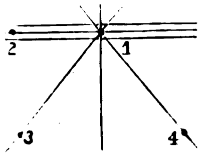
T 6_2



T 7

$$Zw = Wz$$
$$Xz = Zx + Wy$$
$$Yw = EX(x+z) + W(Nx+Py)$$

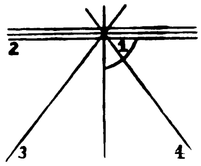




T 8

$$Zw = Wz$$
$$Y(y+w) = W(z+w)$$
$$Xz = Zx + Yy$$

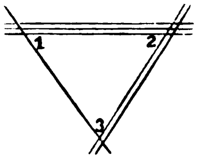
M



T 9

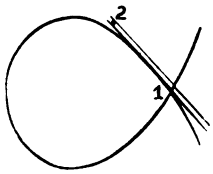
$$Xz = Zx$$
$$Zw = W(z-w)$$
$$X(x+z) = Yw - Wy + Q(Xw - Wx)$$

T 9



U 1_1

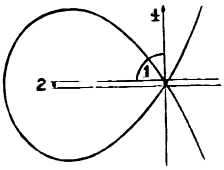
$$Zw = Wz$$
$$Yw = Wy + Zx$$
$$Xw = Yy + QWx + Z(Nx + Py + Tz)$$



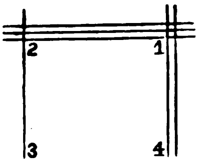
(Two lines coincide in a tangent at the node of the quartic.)

U 1_2

$$Zw = Wz$$
$$Yw = Zy$$
$$Xw = Yx + TZz + W(Qx + Ry)$$

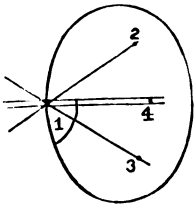


(Plane nodal cubic.)



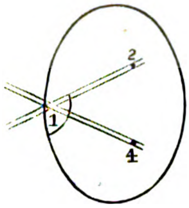
U 2_1

$$Zw = Wz$$
$$Xz = PYy + Z(Ex + Fy)$$
$$Xw = PYx + W(Ex + Fy + Pz)$$



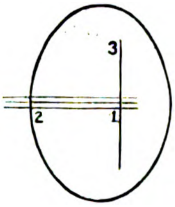
U 2_2

$$\begin{aligned} Zw &= Wz \\ Yw &= Zx + Wy \\ Xz &= Yy + Z(Nx + Py) \end{aligned}$$



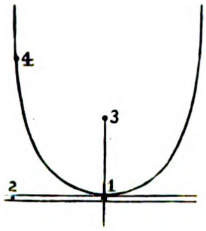
U 2_3

$$\begin{aligned} Yz &= Zy \\ Xy &= Y(Ex + Rw) + Zw + Wx \\ Xz &= Z(Ex + Rw) + Ww \end{aligned}$$



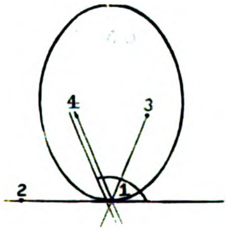
U 2_4

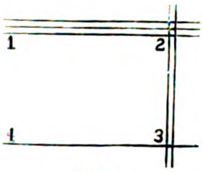
$$\begin{aligned} Yz &= Wy \\ Zw &= Wx \\ Xz &= Yw + Z(Ex + Fy + Nw) \end{aligned}$$



U 2_5

$$\begin{aligned} Yz &= Zy \\ Yw &= Wz \\ Xw &= Z(x + Tw) + W(Jx + Ky) \end{aligned}$$





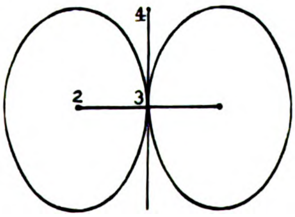
U 3_1

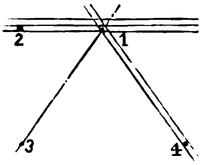
$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xz &= Yw + Zx - W(y - w) \end{aligned}$$

U 3_1

U 3_2

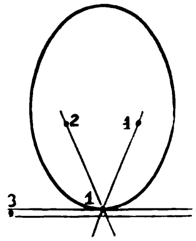
$$\begin{aligned} Xy &= Yx \\ Yz &= Zw \\ Zz &= Ww + X(Hx + Nz) + Y(Jx + Ky) \end{aligned}$$





U 4_1

$$\begin{aligned} Yw &= Wy \\ &= Zz \\ Xz &= Zw + Y(Fx+Gy) + W(Nx+Uw) \end{aligned}$$



Q

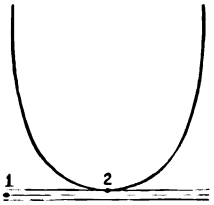
U 4_2

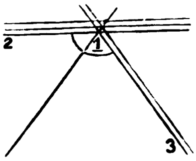
$$\begin{aligned} Xz &= Zx + RYw \\ Zw &= z(FY+QW) \\ Xw &= Y(Fx+Rw) + W(Qx-Ry) \end{aligned}$$

U 4_3

$$\begin{aligned} Yw &= Wy \\ Xw &= Wx + TZz \\ Yx &= Xy + T(Z-W)(z+w) \end{aligned}$$

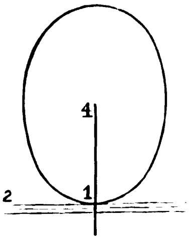
U 4_3





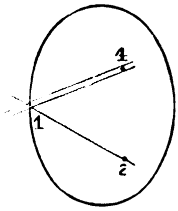
U 5_1

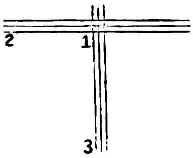
$$\begin{aligned} Zw &= Wz \\ Xw &= Wx + Zy \\ Yw &= GX(y+z) + RWy + Z(Nx+Py+Tz) \end{aligned}$$



U 5_2

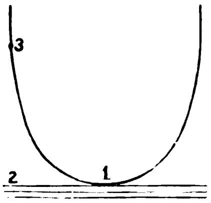
$$\begin{aligned} Zw &= Wz \\ Yw &= Zy \\ Xw &= GY(y+z) + Z(Nx+Tz) + W(Qx+Ry) \end{aligned}$$





V_1

$$\begin{aligned} Zw &= Wz \\ Xz &= MYy + Z(Ex+Fy) \\ Xw &= M(Zz-Yx) + W(Ex+Fy) \end{aligned}$$



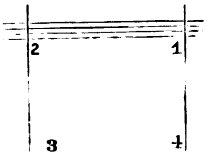
V_2

$$\begin{aligned} Zw &= Wz \\ Yz &= Zx + Wy \\ Xz &= Yy + W(Nx+Uw) \end{aligned}$$

V_3

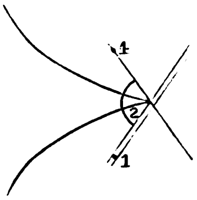
$$\begin{aligned} Yw &= Wy \\ Xy &= Yx + UZw \\ Xw &= Wx + Uz\,(Z - W) \end{aligned}$$

$U\,4_3$



$W\,1_1$

$$\begin{aligned} Zw &= Wz \\ Xw &= Zx \\ Yz &= x\,(EX + NZ) + y\,(FX + PZ + RW) \end{aligned}$$

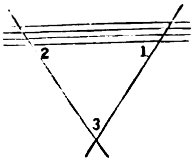


(Plane cuspidal cubic.)

$W\,1_2$

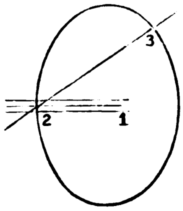
$$\begin{aligned} Yw &= Wy \\ X\,(z + w) &= Wx \\ Zz &= Ww + F\,(Xy - Yx) \end{aligned}$$

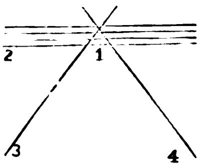
V_2



$W\,2$

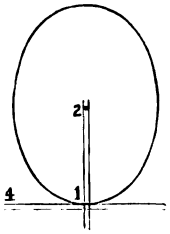
$$\begin{aligned} Zw &= Wz \\ Xz &= Zx + Ww \\ Xy &= Yw + V\,(Xw - Wx) \end{aligned}$$



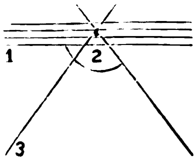


$W\,3_1$

$$\begin{aligned} Zw &= Wz \\ Yw &= Zy \\ Xz &= Wy + Y\,(Fx + Gy) + Z\,(Nx + Py) \end{aligned}$$



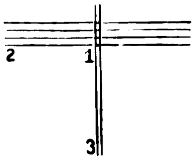
$W\,3_2$ (see Q′2)



$W\,4$

$$\begin{aligned} Zw &= Wz \\ Xz &= Zx + Ww \\ Yw &= Wy + x\,(X + Z) \end{aligned}$$

$W\,4$



X_1

$$\begin{aligned} Yw &= Wy \\ Xw &= Wx + Yz \\ Zz &= Ww + G\,(Xy - Yx) \end{aligned}$$

Z

X_2 (see S_1)

X_3

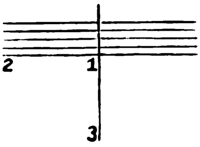
$$Zw = Wz$$
$$Yz = Zy + Ww$$
$$Xw = Wx + Yy$$

X_3

X_4

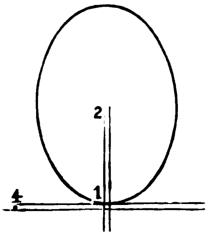
$$Yw = Wy$$
$$Xw = Wx + TZz$$
$$Xy = Yx + T(Zw - Wz)$$

X_4



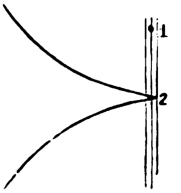
Y_1

$$Zw = Wz$$
$$Yw = Zy$$
$$Xz = Ww + Y(Fx + Gy) + Z(Nx + Py)$$



Y_2

$$Zw = Wz$$
$$Xw = Zz + Wx$$
$$Xy = Yw + Z(Nx + Py) + W(Qx + Ry)$$

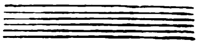


(Plane cuspidal cubic.)

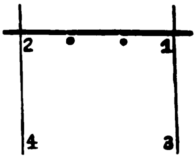
Y_3

$$Zw = Wz$$
$$Xz = Zx + Yw$$
$$Xw = Y(y + Pw) + W(x + z)$$

U 4_2



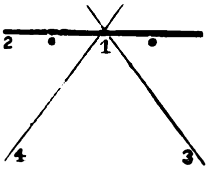
Z (see X_1)



*W*1

$$Zw = Wz$$
$$Xz = Wx + Z(Ax + By)$$
$$Yw = Zy + W(Cx + Dy)$$

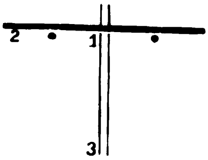
*W*1



*W*2

$$Zw = Wz$$
$$Xz = Wy + Z(Ax + By)$$
$$Yw = Zy + W(Cx + Dy)$$

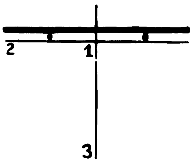
*W*1



X

$$Zw = Wz$$
$$Xw = Yz + Z(Ax + By) + W(Ex + Fy)$$
$$Yw = Zy + W(Cx + Dy)$$

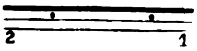
W 1



Y

$$Zw = Wz$$
$$Xz = Ww + Z(Ax + By)$$
$$Yw = Zy + W(Cx + Dy)$$

Y



Z_1

$$Zw = Wz$$
$$Xz = Zz + Ww$$
$$Yw = Wy + Zz$$

Z_1

De Paolis, *l.c.*, p. 739, 1*a*; $\mu = 3$.

Z_2

$$Zw = Wz$$
$$Xz = Ww + Z(Ax + By)$$
$$Yz = Xw + Z(Cx + Dy) + W(Ex + Fy)$$

Y

Berry, *Camb. Phil. Soc. Trans.*, Vol. XIX, p. 275; Vol. XX, pp. 109. 111.

ON THE REDUCTION OF ARITHMETICAL BINARY CUBICS WHICH HAVE A NEGATIVE DETERMINANT

By G. B. MATHEWS *and* W. E. H. BERWICK.

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1. The reduction of a binary cubic with three real roots has been discussed by Arndt, and presents no particular difficulty. In this case the Hessian is a definite form, and the unitary substitution which reduces it is, in general, unique, and converts the cubic into an equivalent form which is reduced by definition. But the case is different when the cubic has two imaginary roots, for the Hessian is then indefinite, and there is no obvious way of deducing from it a unique reduced form of the cubic. The object of this paper is to explain a method of reduction based upon other considerations, and leading, by a finite process of calculation, to a unique result.

All the forms dealt with are supposed written without the attachment of binomial coefficients: thus (a, b, c) means $ax^2+bxy+cy^2$, and so on.

2. Let the cubic considered be given by

$$f = ax^3+bx^2y+cxy^2+dy^3,$$

and let its linear factors be

$$x-\alpha y, \qquad x-\beta y, \qquad x-\gamma y,$$

where α is real, and β, γ complex. It will be supposed, in the first place, that α is irrational.

We have identically

$$f = (x-\alpha y)(ax^2+mxy+ny^2) = (x-\alpha y)\,\phi,$$

say, where ϕ is a definite form, with irrational coefficients. Now let us, as usual in the geometrical theory of quadratic forms, take a plane cut up into equivalent curvilinear triangles. The fundamental triangle, in the positive half-plane, lies outside the circle $x^2+y^2 = 1$, and between the lines $4x^2 = 1$; the others are derived from it by unitary substitutions. Now, if we plot off the points corresponding to α, β, γ, the first will be on the axis of real quantities, and the others will be within (or on the boundaries of) two conjugate triangles T, T'. To fix the ideas, suppose that β has a positive imaginary part, and that it is within the triangle T. Then there is a unique substitution S which converts T into the fundamental triangle T_0, and brings β to a position β_0 within T_0. Applying this substitution to f, we obtain an equivalent form

$$f_0 = Sf = (a_0, b_0, c_0, d_0),$$

which is unique, and by definition reduced. In the case when β is on a boundary, there are two reducing substitutions that bring β to the boundary of T_0, but there is only one of them which brings it to the left of the axis of y. Choosing this as the reducing substitution, we again have a unique form f_0. Two reduced cubics cannot be properly equivalent; so that for a given determinant the number of reduced cubics is equal to the number of classes.

3. It remains to discover S by a finite process of calculation. Since

$$f = (x-ay)\{ax^2+(b+aa)xy-d/a.y^2\}$$

identically, it follows that, if t is a sufficiently close rational approximation to a, then the roots of the rational quadratic ψ, given by

$$\psi = ax^2+(b+at)xy-\frac{d}{t}y^2,$$

will be near the roots of ϕ. Now, if β is within T, it will be possible to surround β by a circle C, of radius r, which is also within T, and if we carry the approximation so far as to bring a root of ψ within C, the substitution S which reduces ψ will also reduce ϕ, and therefore f. Practically, there is not much difficulty in a given case, for if, in any way, we get a sequence of rational approximations $t_1, t_2, t_3, \ldots$, to a, and form the corresponding quadratics $\psi_1, \psi_2, \psi_3, \ldots$, the reducing substitutions $S_1, S_2, S_3, \ldots$ must ultimately become the same; and in many cases this repetition shows itself at a very early stage. It is desirable, however, to establish a criterion enabling us to be sure that we have reached a sufficient degree of approximation. One such test will be explained in the next two articles.

4. For convenience we put $x/y = \theta$, and consider the cubic equation

$$a\theta^3+b\theta^2+c\theta+d=0$$

with roots a, β, γ. Let u_1, u_2 be two rational approximations to $\beta+\gamma$, such that $u_1 < \beta+\gamma < u_2$, and let v_1, v_2 be two rational approximations to $\beta\gamma$, such that $v_1 < \beta\gamma < v_2$. Further, let z_1, z_2, z_3, z_4 be the roots, with positive imaginary parts, of the equations

$$\theta^2-u_1\theta+v_1=0, \qquad \theta^2-u_2\theta+v_1=0,$$
$$\theta^2-u_1\theta+v_2=0, \qquad \theta^2-u_2\theta+v_2=0.$$

Then the points z_i are the vertices of a curvilinear parallelogram bounded by parts of the loci

$$2x=u_1,\, u_2, \qquad x^2+y^2=v_1,\, v_2$$

respectively. Suppose, now, that the four points z_i are reduced by the same substitution S, defined by

$$S=\begin{pmatrix} p, & q \\ r, & s \end{pmatrix};$$

then the circle $x^2+y^2=v_1$ is transformed by S into

$$(r^2v_1-p^2)(x^2+y^2)+2(rsv_1-pq)x+s^2v_1-q^2=0,$$

which has its centre on the real axis. It follows from this that S transforms the circular arc z_1z_3 into an equivalent arc which lies wholly within the fundamental triangle T_0; for, by supposition, $S(z_1)$ and $S(z_3)$ are reduced points, and if any two reduced points A, B are taken, they can be joined in one, and only one, way by a minor arc of a circle which cuts the real axis orthogonally, and this arc lies wholly within T_0. Similarly with respect to each of the other sides of the curvilinear parallelogram. Hence the transformed quadrilateral lies wholly within T_0, and the point corresponding to β is therefore reduced.

It should be observed that it is assumed here that the approximation has been carried so far that each of the quadratics in θ has complex roots: this must ultimately be the case. Moreover, if we take for granted the known properties of linear substitutions of a complex variable, it is unnecessary to find the equation of the circle into which $x^2+y^2=v_1$ is transformed by S.

5. In order to obtain a set of four auxiliary quadratics, it is convenient to approximate to a by means of an ordinary continued fraction. Thus, suppose that

$$a=\eta_1+\frac{1}{\eta_2}+\frac{1}{\eta_3}+\ldots,$$

where η_2, η_3, ... are all positive integers, and let a_1, a_2, a_3, ... be the successive convergents. Then the numbers a_1-a, a_2-a, a_3-a, ... are alternately negative and positive, and $|a_i-a|$ is a monotone sequence with limit zero.

Further, if we write

$$-\frac{b}{a}-a_r = u_r, \qquad -\frac{d}{aa_r} = v_r,$$

the numbers $\beta+\gamma-u_1$, $\beta+\gamma-u_2$, ... have signs alternately positive and negative, and so have the numbers $\beta\gamma-v_1$, $\beta\gamma-v_2$, ..., and the limits of the sequences (u_r), (v_r) are respectively $\beta+\gamma$ and $\beta\gamma$. Hence we obtain a sequence of sets of quadratics

$$\theta^2-u_r\theta+v_r = 0, \qquad \theta^2-u_{r+1}\theta+v_r = 0,$$

$$\theta^2-u_r\theta+v_{r+1} = 0, \qquad \theta^2-u_{r+1}\theta+v_{r+1} = 0,$$

and there will be a finite index r which will give us a set such as we require. It will generally happen that the first set of quadratics, or possibly more, obviously do not satisfy the proper conditions; but, as soon as this ceases to be the case, we reduce the four quadratics and examine the reducing substitutions to see whether they are identical. If so, we have reached the proper substitution for reducing f; if not, we proceed to the next set of auxiliary quadratics, and so on, until we have attained the object in view.

6. The following example illustrates the points that occur in the practical application of the theory. It is required to reduce (20, 113, 218, 146).

Here we have

$$\alpha = -3+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\dots,$$

with convergents

$$\frac{-3}{1}, \quad \frac{-2}{1}, \quad \frac{-9}{4}, \quad \frac{-20}{9}, \quad \frac{-29}{13}, \quad \dots.$$

The first set of auxiliaries is

(60, 159, 146), (20, 53, 73), (60, 219, 146), (20, 73, 73),

of which the third is indefinite; so we must go further. The second set is

(20, 73, 73), (180, 657, 584), (20, 68, 73), (45, 153, 146),

of which the first is ambiguous, and has a root on a boundary line. Without, therefore, proceeding to reduce, we go on to the next stages and obtain the auxiliaries

(45, 153, 146), (200, 680, 657). (180, 617, 584), (1800, 6170, 5913).

The chain of reduction for the first is

$$(45, 153, 146) \sim (146, 139, 38) \sim (38, 13, 20) \sim (20, -13, 38),$$

and for the second

$$(200,\ 680,\ 657) \sim (657,\ 634,\ 177) \sim (177,\ 74,\ 297),$$

with reducing substitutions

$$\begin{pmatrix} 0, & 1 \\ -1, & -1 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ -1, & -2 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0, & 1 \\ -1, & -1 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ -1, & -2 \end{pmatrix},$$

which are different; so we must still go on. The next set of auxiliaries is

$$(1800,\ 6170,\ 5913), \quad (5220,\ 17893,\ 17082),$$
$$(2600,\ 8890,\ 8541), \quad (7540,\ 25781,\ 24674),$$

and it is found that these are all reduced by the same substitution

$$\begin{pmatrix} 0, & 1 \\ -1, & -1 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ -1, & -2 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} = \begin{pmatrix} 2, & -1 \\ -1, & 1 \end{pmatrix}.$$

Finally, applying this to the cubic, we obtain the reduced form

$$f_0 = S(20,\ 113,\ 218,\ 146) = (-2,\ 12,\ -11,\ 21).$$

7. For the sake of completeness, we will consider the case when f has a rational linear factor, say,

$$f = (lx+my)(px^2+qxy+ry^2),$$

where l, m are integers prime to each other. Then p, q, r must also be integers, and the reducing substitution is simply that which reduces $(p,\ q,\ r)$. This covers all those cases where β is on a boundary of a triangle T: for, in such a case, the reduced form must be either

$$f_0 = a(x-\alpha y)(x^2 \pm xy + ny^2)$$

or

$$f_0 = a(x-\alpha y)(x^2+mxy+y^2),$$

and in each case α must be rational in order that the coefficients of f_0 may be rational. It follows incidentally that every reduced cubic which has no rational root must have one complex root *within* T_0.

8. Without attempting at present to discuss the necessary or sufficient conditions to be satisfied by the coefficients of a given cubic in order that it may be reduced, we will prove one interesting theorem about the leading coefficient a of a reduced cubic.

Let A, B, C be the points corresponding to the roots α, β, γ, and let D be the discriminant of f. Then, since

$$D = a^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2,$$

it follows that

$$|D| = a^4\,\overline{BC}^2 . \overline{CA}^2 . \overline{AB}^2 = a^4\Delta,$$

where $\overline{BC}$ is the length of the straight line BC. Now, since B, C are not outside of T_0 and its conjugate, it is obvious from a figure that we obtain a minimum value of Δ by supposing B, C at the points corresponding to $(-1 \pm i\sqrt{3})/2$, and A half-way between them. In this limiting case we have

$$\overline{BC} = \sqrt{3}, \qquad \overline{CA} = \overline{AB} = \frac{\sqrt{3}}{2}, \qquad \Delta = \frac{27}{16}.$$

Hence, in general, for a reduced form

$$|D| > 27a^4/16$$

or

$$|a| < \sqrt[4]{\frac{16\Delta}{27}}.$$

In particular, so long as $|D| < 27$, $|a| = 1$; and the only case of $|D| = 27$ with $|a| > 1$ is the form

$$(2x-y)(x^2-xy+y^2).$$

9. It will be seen that the principles of this paper apply not only to cubics, but also to higher binary forms. For instance, if we have a quintic with three real and two complex roots, there will be in general just one reducing substitution which will bring the complex roots to T_0, and this substitution can actually be discovered. If the quintic has two pairs of complex roots, there will be, in general, two distinct reducing substitutions, and in order to find them it will be necessary to approximate to each of the real irrational quadratic factors of the quintic. If the quintic has all its roots real, it will have a definite quadratic covariant, and this may be used for purposes of reduction. However, the discussion of the further applications of the theory may be reserved for another occasion.

PROPERTIES OF LOGARITHMICO-EXPONENTIAL FUNCTIONS

By G. H. HARDY.

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I.

Introduction and Summary of Results.

1. The contents of this paper are designed as a supplement to those of my recently published pamphlet, "Orders of Infinity, &c." (*Cambridge Tracts in Mathematics*, No. 12, 1910). When writing this pamphlet I was compelled by considerations of space to condense the proof of one theorem at any rate almost to the point of obscurity, and to omit the proofs of others altogether. Apart from the new results contained in this paper, the interest of the theorems that I refer to seems to me sufficient to justify a return to the subject.

2. I shall use throughout the paper the system of notation explained in the first section of my tract,* which attaches a special sense to the symbols

$$\succ,\quad \prec,\quad \asymp,\quad \overset{\smile}{\underset{\frown}{-}},\quad \sim,\quad \succcurlyeq,\quad \preccurlyeq,$$

$$O,\quad K,\quad \delta,\quad \Delta,\quad x_0,\quad \epsilon.$$

To the conventions there explained I propose to add two others.

(i) I shall frequently employ formulæ containing constants restricted to be *rational*. Such constants I shall denote by small italic letters s, t, ... When the constant is not restricted to be rational, I shall use large italic letters A, B, ... Thus Ax^s is an algebraic function, but x^A is in general transcendental.

(ii) I shall frequently (when no confusion is likely to arise from the practice) repeat a letter or letters without implying that the numbers denoted by them are the same as when the letters were previously used. For example, I shall write

$$\int^x At^s\,dt = Ax^s,$$

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* "Orders of Infinity," pp. 2–7. I shall refer to this tract henceforth simply as "O.I."

meaning thereby, "if f is a constant multiple of a rational power of x, then its integral is a function of the same kind." I might add, "unless $s = -1$"; it would, of course, be evident that this clause referred to the s on the left-hand side of the equation.

3. Let $A(x)$ denote generally a real one-valued algebraical function of x—it will, of course, be a branch of an algebraical function of a complex variable, which we suppose to be real for $x > x_0$ (*i.e.*, for all real values of x from some definite value onwards). Thus $A(x)$ might be

$$x, \quad \sqrt{x}, \quad -\sqrt{x}, \quad Ax^s, \quad \sqrt{(x+1)}-\sqrt[3]{(x+1)},$$

or be defined as an implicit function by an equation which, like

$$y^5-y-x=0,$$

has one root which has the property stated.

Then in my tract I defined a *logarithmico-exponential function* (shortly, an *L-function*) substantially as follows:—

An L-function is a real one-valued function defined by a finite combination of the functional symbols

$$A(\ldots), \quad \log(\ldots), \quad e^{(\ldots)}$$

operating on the variable x and on real constants.[*]

It is, of course, to be understood that, if necessary, we confine our attention to values of x greater than some definite value x_0.

4. I also classified L-functions according to *orders*, by a method due in principle to Liouville.[†] Thus

$$e^x, \quad \log x, \quad \frac{e^x+\sqrt{(\log x)}}{\sqrt{(x+1)}-(\log x)^2}$$

are of order 1;

$$e^{e^x}, \quad \log\log x, \quad x^x = e^{x\log x}, \quad x^{\sqrt{2}} = e^{\sqrt{2}\log x},$$

of order 2; and so on.[‡] It should be observed that it is not obvious that

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[*] "O.I.," p. 17. I there confined myself to the case in which the algebraical functions are explicit; but I pointed out that no additional difficulties arose from considering the more general case. I also pointed out (p. 18) that "the result of working out the value of the function, by substituting the value of x in the formula defining it, is to be real at all stages of the work."

[†] *Journal de Mathématiques* (1), t. 2, pp. 71 *et seq.*

[‡] "O.I.," p. 18.

(*e.g.*) $\log\log x$ is really of order 2. It is obviously of order *not greater than* 2. That it cannot be expressed as an L-function of lower order demands a proof. Such a proof was, in fact, given by Liouville[*]; another proof follows from results which will be established in this paper.

The following additional definitions will be found useful.

We shall say that f_n, an L-function of order n, is *integral* if it is of the form[†]

$$\Sigma \rho_{n-1}\, e\sigma_{n-1}\, (l\tau^{(1)}_{n-1})^{\kappa_1} (l\tau^{(2)}_{n-1})^{\kappa_2} \dots (l\tau^{(h)}_{n-1})^{\kappa_h},$$

where the functions with suffix $n-1$ are L-functions of order $n-1$ and $\kappa_1, \kappa_2, \dots, \kappa_h$ are positive integers. We call

$$\kappa_1 + \kappa_2 + \dots + \kappa_h$$

the *logarithmic degree*, or, simply, the *degree*, of the typical term of f_n; if λ is the greatest value of $\kappa_1+\kappa_2+\dots+\kappa_h$, we say that f_n is *of logarithmic degree* λ. If the number of terms of degree λ in f_n is μ, we say that f_n is *of logarithmic type* (λ, μ).[‡] We shall in general denote integral L-functions by the letter M, with or without suffixes, indices, &c.

If an integral L-function is of degree 0, *i.e.*, of the form

$$\Sigma \rho_{n-1}\, e\sigma_{n-1},$$

we shall say that it is *exponential.*

If an integral exponential L-function contains ϖ terms, we shall say that it is *of type* ϖ; if $\varpi = 1$, we shall say that it is *simply exponential.* Thus $(lx)^2 e^{e^x lx}$ is a simply exponential L-function of order 2, while

$$(lx)^2 (l_2 x)^2 e^{e^x lx}$$

is an integral function of order 2, of type (2, 1). We shall in general denote integral exponential L-functions by the letter N.

If f_n is the quotient of two integral functions, *i.e.*, of the form M_1/M_2, we shall say that it is *rational.*

If M_1 and M_2 are exponential, *i.e.*, if f_n is of the form N_1/N_2, we shall describe f_n as a *rational exponential* L-function.

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[*] *L.c.*, pp. 99 *et seq.*

[†] I write lx, l_2x, ..., ex, e_2x, ... for $\log x$, $\log\log x$, ..., e^x, e^{e^x}, ..., as in my tract.

[‡] A function of type (λ, μ) may be immediately reducible to a lower type. Thus

$$l_2 x + l_2(1+x) = l\left\{lx\, l(1+x)\right\}.$$

The left-hand side is of type (1, 2), the right-hand side of type (1, 1). In some kinds of argument it might be essential so to frame our definitions as to be free from such ambiguities; but they in no way affect the arguments of this paper. See further, § 6.

5. We shall constantly be making use of the following facts:—

(i) The derivative of a simply exponential function is a simply exponential function, with the same exponential factor.

(ii) The derivative of an integral exponential function of type ϖ is an integral exponential function of type ϖ. If one of the terms of the original function is a constant, the derivative is of type $\varpi-1$.

(iii) The derivative of an integral function of logarithmic type (λ, μ) is in general a function of the same type. If the exponential factor $\rho_{n-1}e\sigma_{n-1}$ of one of the terms of degree λ is a constant, then the derivative is of type $(\lambda, \mu-1)$; if $\mu = 1$, the derivative is of degree $\lambda-1$.

(iv) In general, the derivative of an L-function of order n is an L-function of order n. In exceptional cases the derivative may be of order $n-1$.

These facts are all immediate consequences of the formal rules of the differential calculus. The only one which possibly requires a word of additional explanation is (iii). The derivative of

$$(l\tau_{n-1}^{(1)})^{\kappa_1}(l\tau_{n-1}^{(2)})^{\kappa_2}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

where $\kappa_1+\kappa_2+\dots+\kappa_h = \lambda$, plainly consists of h terms each of degree $\lambda-1$ only. When we differentiate

$$\rho_{n-1}e\sigma_{n-1}(l\tau_{n-1}^{(1)})^{\kappa_1}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

we obtain one term of degree λ (by differentiating the exponential factor) and h of degree $\lambda-1$.

6. Before proceeding further, I wish to make a few remarks about the relation between my results and those of Liouville. The subject-matter appears very much the same, but the methods used and the results obtained are entirely dissimilar.

I regard an L-function essentially as the embodiment of an "order of infinity," as expressing a certain rate of increase or decrease or of approach to a limit; and for this reason I consider only functions of a real variable which are real and one-valued and (as I shall show) ultimately monotonic, excluding altogether oscillating functions such as $\sin x$. These ideas do not appear in Liouville's work at all. He was interested solely in problems of functional form: $\sin x$ was for him exactly on the same footing as $\log x$ or e^x. From his point of view, each of my orders of

L-functions would present itself simply as a sub-class of his corresponding order of "elementary transcendents."[*] But his classes do not, as classes, possess any of the properties with which I am concerned in this paper; and so, even when we seem to be engaged by very similar questions, our results are in reality widely divergent.

Thus Liouville proves that $\log\log x$ and $x^{\sqrt{2}}$ cannot be expressed as elementary transcendents of order one. I prove that the same functions cannot be expressed as L-functions of order 1; and in so doing I do indeed prove a part of what Liouville proved. But the substance of what I prove is something quite different from any question raised by Liouville at all, viz., that *the rates of increase specified by* $\log\log x$ and $x^{\sqrt{2}}$ cannot belong to any L-function of the first order.

It is then only to be expected that the processes of argument used by Liouville should be entirely unlike any of mine—how different a single consideration is enough to show. Liouville considers, as I do, functions of order n built up by means of the simple functions

$$e\sigma_{n-1}, \qquad l\tau_{n-1}$$

(see § 4 above). In all his arguments it is absolutely vital to suppose these simple functions genuinely independent and reduced to the smallest possible number—that is to say, to suppose that no algebraical relation connects them with one another and transcendents of lower order. This assumption plays no part in any of my arguments: it will, in point of fact, usually be satisfied, but it is in no way essential.

7. It is easy to see that any L-function (or any elementary transcendent in Liouville's sense) is a solution of an algebraic differential equation, which may without loss of generality be supposed to be of the form

$$P(x, y, y', \ldots, y^{(k)}) = 0,$$

where P is a polynomial. And in the case of an L-function (more generally, a Liouville's transcendent) of order n, this equation cannot be of order less than n. The lines of a proof have been indicated by Königsberger,[†] who has also established some simple results as to the forms of transcendents that can satisfy equations of specified order.

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[*] For a short account of his classification, see my tract, "The Integration of Functions of a Single Variable" (*Cambridge Tracts*, No. 2, 1905).

[†] "Bemerkungen zur Liouville's Classificirung der Transcendenten" (*Math. Annalen*, Bd. 28, S. 483).

The classification of transcendents adopted by Liouville, however, does not run parallel with any obvious classification of algebraic differential equations. Thus e^{e^x} satisfies an equation of the second order, viz.,

$$yy'' = (y+y')y'.$$

So do the functions of the first order

$$y = e^x + \log x, \qquad y = e^x + e^{\frac{1}{2}x^2},$$

the equations being

$$x^2(y''-y')+x+1 = 0, \qquad (x-1)y''-x^2y'+(x^2-x+1)y = 0.$$

Generally, an integral exponential L-function, of order 1 and type ϖ, satisfies a linear and homogeneous equation of order ϖ, whose coefficients are algebraic functions of x; and other particular remarks of the same nature may be made. And it is possible to obtain a certain number of more or less general results concerning the possible modes of increase of the solutions of algebraic differential equations of specified types: I shall make a few further observations on this point at the end of this paper. But it is clearly impracticable to base the fundamental properties of L-functions upon the nature of the differential equations satisfied by them. As soon as we adopt the standpoint of the differential equation, the properties of L-functions are lost in those of larger and vaguer aggregates of functions.

It would be unjustifiable to conclude from this (as Königsberger appears to have done) that Liouville's classification is in some sense illegitimate or trivial or uninteresting. It must not be forgotten that the particular is often more interesting than the general: and, in my opinion, the main interest of Liouville's classification lies in its application to two special problems—indefinite integration in finite terms on the one hand, orders of infinity on the other.

8. It will probably be convenient if I give a rapid summary of the results which I propose to prove. Some of the proofs, I am afraid, are long and rather tedious: that this should be so is, I think, inevitable from the nature of the subject. The results, too, are of the class that seem more obvious than they are.

In Section II (§§ 9–11) I prove that every L-function is continuous and monotonic from a certain value of x onwards. This theorem and its corollaries form the basis of all the subsequent work.*

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* A proof of this theorem (but of no other proved in this paper) is contained in my tract ("O.I.," pp. 18-20). The proof given here is not different in principle, but has been remodelled in such a way that I hope it will be found simpler and clearer.

In Section III (§§ 12–16) I discuss the limits of rapidity of the increase of an L-function of given order. I state in my tract,[*] but without proof, that an L-function of order n cannot increase more rapidly than $e_n(x^{\Delta})$ or (if it tends to infinity at all) more slowly than $(l_n x)^{\delta}$. I now prove more precise results of which these are corollaries.

In Section IV (§§ 17–19) I apply the results of the preceding section to determine generally the order of the integral of a given L-function.

In Section V (§§ 20–22) I consider systems of standard forms for the increase of L-functions of given order. I give a full investigation in the case of $n = 1$, which shows incidentally that such modes of increase as are given by $l_2 x$, $e_2 x$, x^x, $x^{\sqrt{2}}$, ... are impossible for functions of order 1. I then state the corresponding results for $n = 2$ and for higher orders; but I have not written out a detailed proof of these results. It would be long and tedious, and the nature of the arguments employed will be clear from the simpler discussions which precede.

In Section VI (§§ 23–26) I discuss shortly the construction of functions whose rate of increase corresponds to a gap in the logarithmico-exponential scales of infinity, and in Section VII (§§ 27–28) various topics of a miscellaneous character.

II.

Proof of the Fundamental Theorem.

9. **THEOREM 1.**—*Any L-function is ultimately continuous, of constant sign, and monotonic; and tends, as $x \to \infty$, to ∞, or to zero or some other definite limit. Further, if f and ϕ are L-functions, one or other of the relations*

$$f \succ \phi, \quad f \asymp \phi, \quad f \prec \phi,$$

holds between them.

Two preliminary remarks will be useful.

(1) If f and ϕ are L-functions, f/ϕ is an L-function, whose order is not greater than the greater of the orders of f and ϕ. Thus the second part of the theorem is a mere corollary of the first part; for it follows from the first part that f/ϕ must tend to infinity, or to zero or some other limit.

(2) The derivative of an L-function is an L-function of not higher order [§ 5 (iv)]. From this it follows that if all L-functions are ulti-

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[*] P. 20.

mately continuous and of constant sign they are all also ultimately monotonic; for their derivatives are also ultimately of constant sign.

10. The results of the theorem are certainly true of functions of zero order, that is to say, of purely algebraical L-functions. Any such function can, in fact, be expressed in the form

$$\Sigma x^{a_r} P_r(x^{-b_r}),$$

where a_r and b_r are rational, and P_r is a power series convergent for sufficiently large values of x. It is therefore sufficient to prove that, if the results of Theorem 1 are true of functions of order $n-1$, then they are true also of functions of order n.

(1) *The results of the theorem are true of any simply exponential function of order* n.

It is, in fact, obvious that

$$f_n = \rho_{n-1} e\sigma_{n-1}$$

is ultimately continuous and of constant sign; and the same is true of its derivative

$$f_n' = (\rho_{n-1}' + \rho_{n-1}\sigma_{n-1}')\, e\sigma_{n-1}.$$

(2) *The results are true of any integral exponential function of order* n.

This has just been proved when the function is of type 1 (§ 4). Let us then assume it true for functions of type $\varpi-1$: and let

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1}$$

be of type ϖ.

If $\rho_{n-1} e\sigma_{n-1}$ is any one of the terms of f_n, the function

$$\bar{f}_n = f_n/(\rho_{n-1} e\sigma_{n-1})$$

is of type ϖ, with one term a constant (unity). And so $\bar{f}_n'$ is of type $\varpi-1$ [§ 5 (ii)]. Hence $\bar{f}_n'$ is ultimately continuous and of constant sign: and so the same is true of $\bar{f}_n$, and therefore of f_n. Finally, f_n' is of type ϖ, and so (after what has just been proved) ultimately of constant sign; and so f_n is ultimately monotonic.

(3) *The results are true of any integral function of order* n.

Suppose that

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1} (l\tau_{n-1}^{(1)})^{\kappa_1} (l\tau_{n-1}^{(2)})^{\kappa_2} \dots (l\tau_{n-1}^{(h)})^{\kappa_h}$$

is of logarithmic type (λ, μ). The results have been proved true when $\lambda = 0$. Hence it is enough to prove

(i) that, if true for functions of logarithmic degree $\lambda - 1$, they are true for functions of degree λ and type $(\lambda, 1)$;

(ii) that, if true for functions of type $(\lambda, \mu - 1)$, they are true for functions of type (λ, μ).

Suppose that the typical term written above in the expression of f_n is one of the terms of degree λ, and let $\bar{f}_n = f_n/(\rho_{n-1} e\sigma_{n-1})$ as before. Then $\bar{f}_n'$ is of type $(\lambda, \mu - 1)$, unless $\mu = 1$, when it is of degree $\lambda - 1$ [§ 5 (iii)]. Hence, whichever of the inductions (i), (ii) we are engaged in proving, $\bar{f}_n'$ is ultimately continuous and of constant sign; and we deduce as before that f_n is ultimately continuous, of constant sign, and monotonic.

(4) We are now in a position to complete the proof of the theorem. Any L-function f_n is of the form

$$f_n = A\left\{e\phi_{n-1}^{(1)}, e\phi_{n-1}^{(2)}, \ldots, e\phi_{n-1}^{(r)}, l\psi_{n-1}^{(1)}, \ldots, l\psi_{n-1}^{(s)}, \chi_{n-1}^{(1)}, \ldots, \chi_{n-1}^{(t)}\right\}$$
$$= A(z_1, z_2, \ldots, z_q),$$

say, where $$q = r + s + t.$$

There is therefore an identical relation

$$F(x, y) \equiv M_0 y^p + M_1 y^{p-1} + \ldots + M_p = 0,$$

where $y = f_n$, and the coefficients $M_1, M_2, \ldots, M_p$ are integral L-functions of order n.

The derivatives of these coefficients are also integral. It therefore follows from what has already been proved that

$$\frac{\partial F}{\partial x} = \sum_{(i)} \frac{dM_i}{dx} y^{p-i}, \qquad \frac{\partial F}{\partial y} = \sum_{(i)} (p-i) M_i y^{p-i-1},$$

considered as functions of the two variables x, y, are continuous for all sufficiently large values of x and for all values of y.

Let ξ, η be a pair of values of x and y satisfying the equation $F = 0$. Then, if only ξ is large enough, $\partial F/\partial y$ cannot vanish for $x = \xi$, $y = \eta$. For, if F and $\partial F/\partial y$ both vanish for $x = \xi$, $y = \eta$, then the eliminant of y between $F = 0$ and $\partial F/\partial y = 0$ vanishes for $x = \xi$. But this eliminant is plainly an integral L-function of order n, and so cannot vanish for values of x surpassing all limit.

Now there is a well known theorem which asserts that* if F is a function of x and y which vanishes for $x=\xi$, $y=\eta$, and has derivatives $\partial F/\partial x$, $\partial F/\partial y$ continuous in a region including (ξ, η), and if $\partial F/\partial y$ does not vanish for $x=\xi$, $y=\eta$, then there is a unique continuous and differentiable function y of x, which is equal to η when x is equal to ξ, and satisfies the equation $F(x, y)=0$ identically. Hence f_n is an ultimately continuous function of x.

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* Goursat, *Cours d'Analyse*, t. i, p. 40; Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 397 *et seq.*

Moreover f_n is ultimately of constant sign. For $f_n=0$ involves $M_p=0$, and we have already seen that it is impossible that this equation should be satisfied for values of x surpassing all limit.

Finally, f_n' is a function of order n. Hence, applying the same conclusions to f_n', we see that f_n is ultimately monotonic. The proof of the theorem is thus completed.*

11. COROLLARY.—*If f, ϕ, and F are L-functions, then*

$$F(f)/F(\phi), \qquad F(f)-F(\phi)$$

tend to infinity or to some definite limit.

This requires no further proof. It is of some interest in connection with Pincherle's extensions of some of Du Bois-Reymond's work.†

III.

The Limits of the Increase of an L-Function.

12. I also stated, but without proof, the following theorems‡:—

THEOREM 2.—*An L-function of order n cannot satisfy*

$$f_n \succ e_n(x^{\Delta}).$$

THEOREM 3.—*An L-function of order n cannot satisfy*

$$1 \prec f_n \prec (l_n x)^{\delta},$$

or

$$(l_n x)^{-\delta} \prec f_n \prec 1.$$

Theorem 2 is not only obvious to the eye of common sense, but very easy to prove. Assume the result true for functions of order $n-1$: it is, of course, true for functions of zero order.

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* I should mention that the idea of attempting to construct a formal proof of this theorem was suggested to me some years ago by Mr. V. Ramaswami Aiyar.

† "O.I.," pp. 13 *et seq.*

‡ "O.I.," p. 20.

Any function of order n is an algebraical function of certain arguments $e\phi_{n-1}, \ldots, l\psi_{n-1}, \ldots, \chi_{n-1}, \ldots$, the increase of any one of which is *ex hypothesi* less than that of

$$e(e_{n-1}x^{\Delta}) = e_n(x^{\Delta})$$

for some value of Δ. Hence the increase of the function is less than that of

$$(e_n x^{\Delta})^{\Delta_1}$$

for some values of Δ and Δ_1; and so less than that of $e_n(x^{\Delta_2})$ for some value of Δ_2. Thus the theorem is established.

Theorem 3 appears to be much harder to prove. It is indeed natural to suppose that it ought to be deducible as a corollary of Theorem 2. But for such a deduction we appear to need some such theorem as the following:*—

If f_n is an L-function of order n, tending to infinity with x, and $\bar{f}$ is the function inverse to f_n, then there is an L-function ϕ_n, of degree n at most, such that

$$\bar{f} \sim \phi_n.$$

If this is true, Theorem 3 may be at once deduced from Theorem 2. For the inverse of $(l_n x)^{\delta}$ is $e_n(x^{1/\delta})$; and so, if

$$1 \prec f_n \prec (l_n x)^{\delta},$$

it follows that $\bar{f} \geqslant e_n(x^{\Delta})$ for sufficiently large values of x,† and so $\phi_n \succ e_n(x^{\Delta})$, which is impossible. I have, however, not been able to prove the result assumed in this deduction, and I am not inclined to commit myself to a definite opinion as regards the probability of its being true.‡

13. It therefore appears to be necessary to proceed to the proof of Theorem 3 by a different road. It is convenient to recall one or two theorems from my tract which will be needed in the proof.

THEOREM 4.—*The relations $f \succ \phi$, $f \asymp \phi$, $f \prec \phi$ involve the corresponding relations*

$$\int_a^x f(t)\,dt \overset{\succ}{\underset{\prec}{\asymp}} \int_a^x \phi(t)\,dt,$$

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* A less precise result might be sufficient. The possibility of the truth of the theorem was suggested to me by Mr. Littlewood.

† "O.I.," p. 16.

‡ I shall return to this point in Section VII.

if either of the integrals $\int^\infty f dx$, $\int^\infty \phi dx$ *is divergent. If both integrals are convergent, then*

$$\int_x^\infty f(t)\,dt \overset{\succ}{\underset{\prec}{\asymp}} \int_x^\infty \phi(t)\,dt.$$

Here f and ϕ are supposed to be positive, continuous and monotonic—there is, however, no difficulty in interpreting the results so as to apply them to negative functions.[*] The theorem may be stated briefly thus: *relations of the type* $f \succ \phi$, ..., *may be integrated.*

THEOREM 5. — *The relation* $f \succ \phi$ *involves* $f' \succ \phi'$, *unless* $f \asymp 1$; $f \asymp \phi$ *involves* $f' \asymp \phi'$, *unless* $f \asymp \phi \asymp 1$; $f \prec \phi$ *involves* $f' \prec \phi'$, *unless* $\phi \asymp 1$; *provided always that it is known that one of the relations* $f' \succ \phi'$, $f' \asymp \phi'$, $f' \prec \phi'$ MUST *hold between* f' *and* ϕ'.

The exceptional cases mentioned in the enunciation are in reality of a trivial character, and are due to the fact that, if $f \asymp 1$, then f, when regarded as the integral of f', is dominated by a constant of integration. Thus, if $f = 1 + e^{-x}$, $\phi = 1/x$, then $f \succ \phi$, but $f' \prec \phi'$.

Theorem 5 may be stated thus: *relations of the type* $f \succ \phi$, ... *may* (except in certain special cases) *be differentiated, provided we are assured a priori that some relation of this type must hold between the derivatives.*

When f and ϕ are L-functions, so are their derivatives: thus we obtain:

THEOREM 6.—*Relations of the type* $f \succ \phi$, ..., *holding between* L-*functions, may be integrated and differentiated, subject to certain restrictions relating to the cases in which* $f \asymp 1$ *or* $\phi \asymp 1$.

14. I shall now establish Theorem 3 as a corollary of a more precise theorem.

THEOREM 7.—*If* f_n *is an* L-*function of order* n, *and*

$$(l_{n-1}x)^{-\delta} \prec f \prec (l_{n-1}x)^{\delta},$$

then

$$f_n \asymp (l_n x)^s,$$

where s *is a rational number. In particular, if* f_n *is rational,* s *must be an integer; and if* f_n *is integral, a positive integer; and if* f_n *is an integral or rational exponential function, then* s *must be zero.*

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[*] "O.I.," pp. 36 *et seq.*; this reference applies also to Theorems 5 and 6.

I shall base my proof upon two lemmas:

Lemma A.—*If* $$(l_m x)^{-\delta} \prec f \prec (l_m x)^{\delta},$$

and it is not true that $f \asymp 1$, *then*

$$(l_m x)^{-\delta} \prec x\, lx \ldots l_m x\,.\,f' \prec (l_m x)^{\delta}.$$

The truth of this follows immediately upon differentiation.

Lemma B.—*If* $$(l_m x)^{-\delta} \prec f/\phi \prec (l_m x)^{\delta},$$
while, for some positive γ,

$$\phi \prec (l_m x)^{-\gamma} \quad or \quad \phi \succ (l_m x)^{\gamma},$$

then $$(l_m x)^{-\delta} \prec f'/\phi' \prec (l_m x)^{\delta}.$$

For

(1) $$\phi\,(l_m x)^{-\delta} \prec f \prec \phi\,(l_m x)^{\delta}.$$

It is evident from the conditions that no one of the relations

$$\phi(l_m x)^{-\delta} \asymp 1, \quad f \asymp 1, \quad \phi(l_m x)^{\delta} \asymp 1$$

is possible, if δ is small enough.

Hence (Theorem 5) we may differentiate, and so we obtain

(2) $$\frac{d}{dx}\left\{\phi\,(l_m x)^{-\delta}\right\} \prec f' \prec \frac{d}{dx}\left\{\phi\,(l_m x)^{\delta}\right\}.$$

Now $$\frac{d}{dx}\left\{\phi(l_m x)^{-\delta}\right\} = \phi\,(l_m x)^{-\delta}\left(\frac{\phi'}{\phi} - \frac{\delta}{x\,lx \ldots l_m x}\right)$$

But $$l\phi \succcurlyeq \gamma\, l_{m+1} x,$$

and so $$\frac{\phi'}{\phi} \succcurlyeq \frac{\gamma}{x\,lx \ldots l_m x}.$$

When γ is fixed we can find K and x_0, so that

$$\left|\frac{\phi'}{\phi}\right| > \frac{K}{x\,lx \ldots l_m x} \quad (x \geqslant x_0)$$

Thus, if $\delta < \tfrac{1}{2}K$, the ratio of the functions

$$\frac{\phi'}{\phi} - \frac{\delta}{x\,lx \ldots l_m x}, \quad \frac{\phi'}{\phi}$$

lies, for $x > x_0$, between $\frac{1}{2}$ and $\frac{3}{2}$. As they are L-functions, we must have

$$\frac{\phi'}{\phi} - \frac{\delta}{x\,lx \dots l_m x} \asymp \frac{\phi'}{\phi}.$$

Thus
$$\frac{d}{dx}\{\phi(l_m x)^{-\delta}\} \asymp \phi'(l_m x)^{-\delta};$$

and a similar result holds when $-\delta$ is replaced by δ. The truth of the lemma now follows immediately from (2).

15. We shall now assume Theorem 7 to be true for functions of order $n-1$, and prove it true for functions of order n. As it is plainly true for $n = 0$ (when $l_{n-1}x$ is e^x and $l_n x$ is x), it will follow that it is true generally.

(1) *If* f_n *is a simply exponential function, and*

$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

then $f_n \asymp 1$.

Suppose, *e.g.*, that $f_n \succ 1$. Then

$$1 \prec lf_n \prec l_n x,$$

and so
$$\frac{1}{x\,lx \dots l_{n-2}x\,(l_{n-1}x)^{1+\delta}} \prec \frac{f_n'}{f_n} \prec \frac{1}{x\,lx \dots l_{n-1}x};$$

the second relation following from the differentiation of $lf_n \prec l_n x$, the first being a consequence of the fact that

$$\int^{\infty} \frac{dx}{x\,lx \dots l_{n-2}x\,(l_{n-1}x)^{1+\delta}}$$

is convergent.

Hence, if $f_n = \rho_{n-1}e^{\sigma_{n-1}}$, we obtain

$$(l_{n-1}x)^{-\delta} \prec (x\,lx \dots l_{n-1}x)\left(\frac{\rho_{n-1}'}{\rho_{n-1}} + \sigma_{n-1}'\right) \prec 1,$$

which is impossible, as the function in the middle is of order $n-1$.

(2) *The same result holds for any integral exponential function.*

Suppose that
$$f_n = \Sigma\rho_{n-1}e^{\sigma_{n-1}}$$

is of type ϖ, and that the result has been proved for functions of type $\varpi - 1$.

Two cases are conceivable. Either *all* the terms of f_n satisfy

$$\rho_{n-1}e^{\sigma_{n-1}} \asymp 1,$$

or there is at least one which does not. We consider the latter case first.

(i) Let
$$\rho_{n-1}e\sigma_{n-1}=1/\phi$$
be one of the terms which does not. Then, by what has already been proved,
$$\phi \prec (l_{n-1}x)^{-\gamma} \quad \textit{or} \quad \phi \succ (l_{n-1}x)^{\gamma}$$
for some positive γ.

Let $f_n=\bar{f}_n/\phi$, so that
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n/\phi \prec (l_{n-1}x)^{\delta}.$$
The conditions of Lemma B are satisfied, and so
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n'/\phi' \prec (l_{n-1}x)^{\delta}.$$
But $\bar{f}_n'/\phi'$ is clearly of type $\varpi-1$. Hence these last relations are only possible if
$$\bar{f}_n' \asymp \phi',$$
i.e., if $\bar{f}_n \asymp \phi$ or $f_n \asymp 1$.

(ii) If every $\rho_{n-1}e\sigma_{n-1}$ satisfies $\rho_{n-1}e\sigma_{n-1} \asymp 1$, we define ϕ as above by means of any one of them.

In this case it is plain that $f_n \preccurlyeq 1$.

Now
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n \prec (l_{n-1}x)^{\delta},$$
and so, by Lemma A,
$$(l_{n-1}x)^{-\delta} \prec x\,lx \ldots l_{n-1}x\,.\bar{f}_n' \prec (l_{n-1}x)^{\delta},$$
unless $\bar{f}_n \asymp 1$ or $f_n \asymp 1$. But $x\,lx \ldots l_{n-1}x\,.\bar{f}_n'$ is of type $\varpi-1$. Hence we must have
$$x\,lx \ldots l_{n-1}x\,.\bar{f}_n' \asymp 1,$$
which involves
$$f_n \asymp \bar{f}_n \asymp l_n x,$$
and this is impossible, since $f_n \preccurlyeq 1$. Thus we are driven back on the conclusion that $f_n \asymp 1$.

(3) *If f_n is any integral function, and*
$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$
then
$$f_n \asymp (l_n x)^q,$$
where q is a positive integer.

We shall prove this by establishing, as in § 10 (3), inductions from functions of degree $\lambda-1$ to functions of degree λ and type $(\lambda, 1)$, and from functions of type $(\lambda, \mu-1)$ to functions of type (λ, μ).

Suppose, then, that f_n is of type (λ, μ), and consider the factors $\rho_{n-1}e\sigma_{n-1}$ of the μ terms in f_n of degree λ. We must distinguish two cases exactly as under (2) above.

(i) Suppose that among these μ factors there is one of which it is not true that

$$\rho_{n-1}e\sigma_{n-1} = 1/\phi \asymp 1,$$

and let $f_n = \bar{f}_n/\phi$. Arguing precisely as above, we find

$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n'/\phi' \prec (l_{n-1}x)^{\delta}.$$

The function in the middle is of type $(\lambda, \mu-1)$, or if $\mu = 1$, of degree $\lambda-1$. Hence we must have

$$\bar{f}_n' \asymp (l_n x)^q \phi',$$

where q is a positive integer. Now

$$\phi \prec (l_{n-1}x)^{-\gamma} \quad \text{or} \quad \phi \succ (l_{n-1}x)^{\gamma},$$

so that

$$l\phi \succcurlyeq \gamma l_n x,$$

$$\frac{\phi'}{\phi} \succ \frac{1}{x\,lx \ldots l_{n-1}x} \succ \frac{1}{x\,lx \ldots l_n x}.$$

Thus

$$\frac{\phi'}{\phi} \succ \frac{d}{dx}\log\{(l_n x)^q\},$$

or

$$(l_n x)^q \phi' \sim \frac{d}{dx}\{(l_n x)^q \phi\}.$$

Hence

$$\bar{f}_n' \asymp \frac{d}{dx}\{(l_n x)^q \phi\},$$

$$\bar{f}_n \asymp (l_n x)^q \phi,$$

$$f_n = \bar{f}_n/\phi \asymp (l_n x)^q.$$

Thus in this case the required induction is established.

(ii) Suppose that all the μ functions $\rho_{n-1}e\sigma_{n-1}$ satisfy $\rho_{n-1}e\sigma_{n-1} \asymp 1$. Then

$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n \prec (l_{n-1}x)^{\delta},$$

and so, by Lemma A,

$$(l_{n-1}x)^{-\delta} \prec x\,lx \ldots l_{n-1}x \,.\, \bar{f}_n' \prec (l_{n-1}x)^{\delta},$$

unless $\bar{f}_n \asymp 1$ or $f_n \asymp 1$. Apart from this case we must have

$$x\,lx \ldots l_{n-1}x \,.\, \bar{f}_n' \asymp (l_n x)^q ;$$

and, dividing by the logarithmic factors on the left and integrating, we obtain

$$f_n \asymp \bar{f}_n \asymp (l_n x)^{q+1}.$$

Thus again our induction is established, and the proof of (3) is completed.

(4) *If f_n is any rational function, and*

$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

then

$$f_n \asymp (l_n x)^q,$$

where q is a positive (or negative) integer.

Suppose that

$$f_n = M_1/M_2,$$

where M_1 and M_2 are integral. When M_2 is simply exponential, the proposition reduces to one already proved. We shall prove first that it is true when M_2 is any integral exponential function. In these cases q may be restricted to be positive.

Let $\rho_{n-1} e\sigma_{n-1}$ be one of the ϖ terms of M_2; let

$$\rho_{n-1} e\sigma_{n-1} = 1/\phi, \qquad M_1 = \bar{M}_1/\phi, \qquad M_2 = \bar{M}_2/\phi ;$$

and let us assume that the proposition has been proved when M_2 is of type $\varpi - 1$. Then

(1) $$(l_{n-1}x)^{-\delta} \prec \bar{M}_1/\bar{M}_2 \prec (l_{n-1}x)^{\delta}.$$

If

$$(l_{n-1}x)^{-\delta} \prec \bar{M}_2 \prec (l_{n-1}x)^{\delta},$$

we must, by what has already been proved, have

$$\bar{M}_2 \asymp (l_n x)^{q_2}$$

(q_2 a positive integer). In this case M_1 satisfies similar relations, and so

$$\bar{M}_1 \asymp (l_n x)^{q_1}, \qquad f_n = \bar{M}_1/\bar{M}_2 \asymp (l_n x)^{q_1 - q_2},$$

which is what we want to prove. Thus this case may be dismissed, and we may assume that

$$\bar{M}_2 \prec (l_{n-1}x)^{-\gamma} \quad \textit{or} \quad \bar{M}_2 \succ (l_{n-1}x)^{\gamma}.$$

Then it follows from (1), by Lemma B, that

$$(2) \qquad (l_{n-1}x)^{-\delta} \prec \bar{M}_1'/\bar{M}_2 \prec (l_{n-1}x)^{\delta}.$$

As $\bar{M}_2'$ is of type $\varpi-1$, we must have

$$\bar{M}_1'/\bar{M}_2' \asymp (l_n x)^q,$$

$$\bar{M}_1' \asymp (l_n x)^q \bar{M}_2',$$

where q is a positive integer. It then follows, just as at the end of (3) (i), that

$$\bar{M}_1 \asymp (l_n x)^q \bar{M}_2 \qquad or \qquad f_n \asymp (l_n x)^q.$$

Thus the induction from $\varpi-1$ to ϖ is established; and (4) is proved when M_2 is any integral exponential function, q being so far necessarily positive or zero.

We now suppose M_2 to be of logarithmic type (λ, μ), and we have to establish our customary inductions from degree $\lambda-1$ to degree λ and type $(\lambda, 1)$, and from type $(\lambda, \mu-1)$ to type (λ, μ).

Suppose, then, that the relations (1) are established. The assumption that $M_2 \asymp (l_n x)^{q_2}$ leads, precisely as before, to the conclusion that

$$f_n \asymp (l_n x)^{q_1-q_2}.$$

Rejecting this hypothesis, and pursuing the same train of argument, we arrive at the relations (2). As $\bar{M}_2'$ is of type $(\lambda, \mu-1)$, or, if $\mu = 1$, of degree $\lambda-1$, we must have

$$\bar{M}_1'/\bar{M}_2' \asymp (l_n x)^q.$$

If $q = 0$, we have

$$\bar{M}_1' \asymp \bar{M}_2', \quad \bar{M}_1 \asymp \bar{M}_2, \quad f_n \asymp 1.$$

Suppose then $q > 0$, so that

$$\bar{M}_1' \asymp (l_n x)^q \bar{M}_2'.$$

If

$$\bar{M}_2' \asymp (l_n x)^{q_2},$$

we have

$$\bar{M}_1' \asymp (l_n x)^{q+q_2},$$

$$\bar{M}_1 \asymp x(l_n x)^{q+q_2}, \qquad \bar{M}_2 \asymp x(l_n x)^{q_2},^*$$

$$f \asymp (l_n x)^q.$$

---

[*] It is easy to see that $\int (l_n x)^q\, dx \sim x\,(l_n x)^q$: see Section IV.

If it is not true that $\bar{M}_2' \asymp (l_n x)^{q_2}$,

we must have $\bar{M}_2' \prec (l_{n-1}x)^{-\gamma}$ *or* $\bar{M}_2' \succ (l_{n-1}x)^{\gamma}$,

and then we prove that $f_n \asymp (l_n x)^q$ precisely as at the end of (3) (i). The case in which q is negative may be treated similarly, and so the proof of (4) is completed.

Before proceeding further let us point out that it is easily proved (by reasoning of the same character) that, *if f_n is a rational exponential function, then q must be zero.*

16. We are now in a position to complete the proof of Theorem 7.

If f_n is any L-function of order n, there is [§ 10, (4)] an equation

$$M_0 f_n^p + M_1 f_n^{p-1} + \ldots + M_p = 0,$$

wherein the coefficients are integral L-functions of order n. If we denote by $X_0, X_1, \ldots, X_p$ the various terms of this equation, any pair X_i, X_j must satisfy one of the relations

$$X_i \succ X_j, \quad X_i \asymp X_j, \quad X_i \prec X_j$$

(Theorem 1). Moreover, as their sum is zero, there must be *at least one* pair such that

$$X_i \asymp X_j.$$

Thus $$f_n^{j-i} \asymp M_j/M_i.$$

Hence, if $$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

we have also $$(l_{n-1}x)^{-\delta} \prec M_j/M_i \prec (l_{n-1}x)^{\delta},$$

and so $$M_j/M_i \asymp (l_n x)^q,$$

where q is an integer. Therefore

$$f_n \asymp (l_n x)^s,$$

where s is rational, being in fact an integral multiple of $1/(j-i)$.

It is possible to obtain, for the different cases which we have considered, more precise information as to the indices of the powers $(l_n x)^q$, $(l_n x)^s$ which figure in the theorem. To do so would however, lead us into a more elaborate analysis of the possible forms of L-functions of order n; and the theorem embodies all that we shall require in the sequel.

IV.

The Integration of L-Functions.

17. We are now in a position to establish a general system of rules which define the orders of greatness or smallness of the integrals

$$\int_a^x f_n(t)\,dt, \qquad \int_x^\infty f_n(t)\,dt.$$

We choose the first form of the integral or the second, of course, according as $\int^\infty f_n(x)\,dx$ is divergent or convergent. In either case we denote the integral by $F(x)$.

We observe first, that if f is an L-function, then either (i) $f \succ x^\Delta$ or (ii) $f \prec x^{-\Delta}$ or (iii) $f = x^a f_1$, where

$$x^{-\delta} \prec f_1 \prec x^\delta.\text{*}$$

THEOREM 8*a*.—*If* $f \succ x^\Delta$ *or* $f \prec x^{-\Delta}$, *then*

$$F \sim f^2/f'.$$

If $f = x^a f_1$, *then*
$$F \sim \frac{x^{a+1}}{a+1} f_1,$$

unless $a = -1$, *when further investigation is necessary.*

(1) If $f \succ x^\Delta$ the integral is obviously divergent, and

$$\int_a^x f\,dt = \int_a^x f' \frac{f}{f'}\,dt = \frac{\{f(x)\}^2}{f'(x)} - \frac{\{f(a)\}^2}{f'(a)} - \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt.\dagger$$

Now, since $f \succ x^\Delta$, we have

$$\log f \succ \log x, \qquad f'/f \succ 1/x,$$

$$x \succ f/f', \qquad 1 \succ \frac{d}{dx}\left(\frac{f}{f'}\right)$$

$$\int_a^x f\,dt \succ \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt,$$

and so
$$\int_a^x f\,dt \sim \frac{\{f(x)\}^2}{f'(x)}.$$

---

* "O.I.," p. 21.

† We suppose a chosen so that the functions are continuous for $x \geqslant a$.

The case in which $f \prec x^{-\Delta}$, when the integral is obviously convergent, may be settled in the same way.

(2) If $f = x^{\alpha} f_1$, where $\alpha > -1$, the integral is divergent. We have then

$$\int_a^x t^{\alpha} f_1\,dt = \frac{x^{\alpha+1}}{\alpha+1} f_1(x) - \frac{a^{\alpha+1}}{\alpha+1} f_1(a) - \frac{1}{\alpha+1}\int_a^x t^{\alpha+1} f_1'\,dt.$$

But $x^{-\delta} \prec f_1 \prec x^{\delta}$, and so

$$\log f_1 \prec \log x, \qquad f_1'/f_1 \prec 1/x,$$

$$x^{\alpha+1} f_1' \prec x^{\alpha} f_1,$$

$$\int_a^x t^{\alpha} f_1\,dt \sim \frac{x^{\alpha+1}}{\alpha+1} f_1(x).$$

The case in which $\alpha < -1$, when the integral is convergent, may be treated similarly. But when $\alpha = -1$ (when the integral may be convergent or divergent) further analysis is required.

18. Suppose now that $\quad f = f_1/x \quad (x^{-\delta} \prec f_1 \prec x^{\delta}).$

Then either (i) $f_1 \succ (lx)^{\Delta}$ or (ii) $f_1 \prec (lx)^{-\Delta}$ or (iii) $f_1 = (lx)^{\alpha_1} f_2$, where

$$(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}.$$

THEOREM 8*b*.—*If* $\quad f = f_1/x \quad (x^{-\delta} \prec f_1 \prec x^{\delta}),$

then, if $f_1 \succ (lx)^{\Delta}$ *or* $f_1 \prec (lx)^{-\Delta}$, *we have*

$$F \sim f_1^2/xf_1'.$$

If $f_1 = (lx)^{\alpha_1} f_2$, *where* $(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}$, *we have*

$$F \sim \frac{(lx)^{\alpha_1+1}}{\alpha_1+1} f_2,$$

unless $\alpha_1 = -1$, *when further investigation is needed.*

We have, in fact (taking the integral to be divergent),

$$\int_a^x \frac{f_1}{t}\,dt = \int_{\log a}^{\log x} f_1(e^{\tau})\,d\tau = \int_{\log a}^{\log x} F_1(\tau)\,d\tau,$$

say. If, *e.g.*, $f_1 \succ (lx)^{\Delta}$, we have clearly $F_1 \succ \tau^{\Delta}$, and so

$$\int_a^x \frac{f_1}{t}\,dt \sim \frac{\{F_1(\log x)\}^2}{F_1'(\log x)} = \frac{f_1^2}{xf_1'}.$$

Similarly in the other cases.

It is evident that this process may be continued: thus we obtain

THEOREM 8*c*.—*If* $f = f_2/x\,lx$, $(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}$,

then if $$f_2 \succ (l_2x)^{\Delta} \quad or \quad f_2 \prec (l_2x)^{-\Delta},$$

we have $$F \sim \frac{f_2^2}{x\,lx\,f_2'}.$$

If $f_2 = (l_2x)^{\alpha_2} f_3$, *where* $(l_2x)^{-\delta} \prec f_3 \prec (l_2x)^{\delta}$, *we have*

$$F \sim \frac{(l_2x)^{\alpha_2+1}}{\alpha_2+1} f_3;$$

unless $\alpha_2 = -1$, *when further investigation is necessary. And so on generally.*

19. THEOREM 9.—*The various forms of Theorem* 8 $(a, b, c, \ldots)$ *apply to* ALL *L-functions.*

Consider first L-functions of order 1. The only case not settled by 8*a* is that in which $$f = f_1/x,$$

where $$x^{-\delta} \prec f_1 \prec x^{\delta}.$$

By Theorem 7 we must then have

$$f_1 \sim A\,(lx)^{\alpha_1}.$$

The only case not settled by Theorem 8*b* is that in which

$$f_1 \sim \frac{A}{lx}, \quad f \sim \frac{A}{x\,lx}, \quad F \sim A\,l_2x.$$

Similarly L-functions of order 2 are dealt with by 8*a* or 8*b* or 8*c*, unless

$$f_2 \sim \frac{A}{l_2x}, \quad f_1 \sim \frac{A}{lx\,l_2x}, \quad f \sim \frac{A}{x\,lx\,l_2x}, \quad F \sim A\,l_3x;$$

and so on generally.

V.

Standard Forms for the Increase of L-functions.

20. The preceding theorems enable us to establish systems of standard forms for the increase of L-functions of a given order.

THEOREM 10.—*Any L-function* f_1 *of the first order, ultimately positive,* *may be expressed in one or other of the forms*

$$e^{Ax\,(1+\epsilon)}, \qquad Ax^s(\log x)^t(1+\epsilon).^*$$

If f_1 is rational, t must be an integer; if f_1 is integral, a positive integer; if f_1 is an exponential function (integral or rational), t must be zero.

(1) *The conclusions of Theorem* 10 *are valid when f_1 is a simply exponential function $\rho_0 e\sigma_0$.*

If $f_1 \asymp 1$, there is nothing to prove. If $f_1 \succ 1$ or $f_1 \prec 1$, then

$$\log f_1 \succ 1:$$

the only difference between the two cases lies in the fact that in the first case $\log f_1$ is ultimately positive and in the second ultimately negative. Let us suppose, to fix our ideas, that $f_1 \succ 1$. We have then

$$\log f_1 = \int^x \left(\frac{\rho_0'}{\rho_0} + \sigma_0'\right) dt = A\int^x t^s(1+\epsilon)\,dt,$$

since the subject of integration is algebraical. Since $\log f_1 \succ 1$, $s \geqslant -1$. It is evident that, unless $s = -1$, we obtain[†]

$$\log f_1 = Ax^s(1+\epsilon), \qquad f_1 = e^{Ax\,(1+\epsilon)}.$$

Here, of course, A and s are positive. If $s = -1$, we have

$$\frac{\rho_0'}{\rho_0} + \sigma_0' = \frac{A}{x}(1+\epsilon).$$

But ϵ is algebraical. Hence we must have

$$\epsilon = O(x^{-a}) \quad (a > 0),$$

$$\frac{f_1'}{f_1} = \frac{A}{x} + O(x^{-1-a})$$

$$\log f_1 = A\log x + B + \epsilon,$$

$$f_1 = A_1 x^A(1+\epsilon).$$

It remains to be seen that A is rational and may be replaced by s.

Now σ_0', being the derivative of an algebraical function, cannot, when expanded in descending powers of x, contain any term in $1/x$. And ρ_0'/ρ_0, being the logarithmic derivative of an algebraical function, can contain such a term only with a rational coefficient.[‡] Thus A must be rational.

---

[*] As explained in § 2, s, t denote *rational* numbers.

[†] Not, of course, with the *same* A and s; see the remarks in § 2. Of course, too, "unless $s = -1$" refers to the *first* s.

[‡] The logarithm of an algebraical function is necessarily of the form $a\log x + \ldots$, where a is rational, and the remaining terms involve negative powers of x.

The case in which $f \prec 1$ may be treated in the same way.

(2) *The conclusions of the theorem are valid for any integral exponential function*

$$f_1 = \Sigma \rho_0 e\sigma_0.$$

Assume this proved when f_1 is of type $\varpi - 1$. If we divide f_1 by $\rho_0 e\sigma_0$, differentiate, and restore $\rho_0 e\sigma_0$ to the other side, we obtain

$$f_1' + s_0 f_1 = \phi_1, \tag{1}$$

where ϕ_1 is an integral exponential function of type $\varpi - 1$ and

$$s_0 = -\frac{d}{dx}\log(\rho_0 e\sigma_0) = -\frac{\rho_0'}{\rho_0} - \sigma_0'$$

is algebraical.

In the equation (1) there must be at least one pair of terms which can be connected by the symbol $\asymp$. We have thus to distinguish *three* possibilities.

(i) We may have $$s_0 f_1 \asymp \phi_1.$$

As ϕ_1/s_0 is of type $\varpi - 1$, our induction from $\varpi - 1$ to ϖ is in this case immediately established.

(ii) We may have $$f_1' \asymp \phi_1.$$

As ϕ_1 is of type $\varpi - 1$, it follows that f_1' must have one of the forms

$$e^{Ax^s(1+\epsilon)}, \qquad Ax^s(1+\epsilon);$$

and it then follows immediately, from Theorem 8*a*, that f_1 itself has one of these forms, except in the special case when in the second form $s = -1$. In this case

$$f_1' \sim A/x, \qquad f_1 \sim A \log x.$$

But this is impossible, obviously if $f_1 \prec 1$, and, by Theorem 7, if $f_1 \succ 1$.

(iii) We may have $$f_1' \asymp s_0 f_1,$$

in which case $$f_1'/f_1 = Ax^s(1+\epsilon).$$

Unless $s = -1$, we at once see that f_1 is of the form $e^{Ax^s(1+\epsilon)}$. If $s = -1$, we have

$$\frac{f_1'}{f_1} = \frac{A}{x}(1+\epsilon),$$

and the same argument as was used under (1) above may be used again to show that A is rational.

Again, $$\epsilon = (xf_1' - Af_1)/Af_1$$

is a rational exponential function, and must therefore,* by Theorem 7, be of the form

$$O(x^{-a}) \quad (a > 0).$$

---

* See the last remark in § 15.

Hence
$$\frac{f_1'}{f_1} = \frac{s}{x} + O(x^{-1-a}),\text{*}$$
$$\log f_1 = s \log x + A + \epsilon,$$
$$f_1 = Ax^s(1+\epsilon).$$

Thus (2) is completely established.

(3) *The conclusions of the theorem are valid for any integral function*
$$f_1 = \Sigma \rho_0 e\sigma_0 (l\tau_0^{(1)})^{\kappa_1} (l\tau_0^{(2)})^{\kappa_2} \dots (l\tau_0^{(\lambda)})^{\kappa_\lambda}.$$

We shall prove this by our usual process of induction from degree $\lambda-1$ to type $(\lambda, 1)$, and from type $(\lambda, \mu-1)$ to type (λ, μ).

Dividing by $\rho_0 e\sigma_0$, differentiating, and restoring $\rho_0 e\sigma_0$ to the other side, we obtain

(1)
$$f_1' + s_0 f_1 = \phi_1,$$

where s_0 is the same function as before, and ϕ_1 is of type $(\lambda, \mu-1)$ or, if $\mu = 1$, of degree $\lambda - 1$. We have again three possibilities.

(i) If $s_0 f_1 \asymp \phi_1$, our conclusion follows immediately.

(ii) If $f_1' \asymp \phi_1$, we have $f_1' = e^{Ax^s(1+\epsilon)}$

or
$$f_1' = Ax^s (lx)^t (1+\epsilon).$$

It then follows, from Theorem 8*a* and Theorem 8*b*, that f_1 itself has one of these forms. The case of exception to Theorem 8*b*, viz., $s = -1$, $t = -1$, cannot occur, since t is *ex hypothesi* zero or a positive integer.

(iii) If $f_1' \asymp s_0 f_1$, we have
$$f_1'/f_1 = Ax^s(1+\epsilon),$$
and our conclusion again follows at once unless $s = -1$.

Now
$$\epsilon = (xf_1' - Af_1)/Af_1$$
is rational, and so, either
$$\epsilon \prec x^{-\gamma},$$
for some positive γ, or
$$\epsilon \asymp (lx)^{-q},$$
where q is a positive integer. If $\epsilon \prec x^{-\gamma}$ or $q > 1$, we have
$$\frac{f_1'}{f_1} = \frac{A}{x} + O\left(\frac{1}{x^{1+\gamma}}\right) \quad or \quad \frac{A}{x} + O\left\{\frac{1}{x(lx)^q}\right\},$$
$$\log f_1 = A \log x + B + \epsilon,$$

(2)
$$f_1 = A_1 x^A (1+\epsilon).$$

---

* We have replaced A, proved to be rational, by s.

If $q = 1$, we have $$\frac{f_1'}{f_1} = \frac{A}{x} + \frac{B}{x\,lx}(1+\epsilon),$$

where now $$\epsilon = O\{(lx)^{-q}\} \quad (q \geqslant 1).$$

Hence we obtain $$lf_1 = A\,lx + B\,llx + C + \epsilon,$$

$$f_1 = A_2 x^A (lx)^B (1+\epsilon). \tag{3}$$

It may be shown as before that A is rational: that B is rational follows at once from Theorem 7. Thus in any case the proof of our induction is completed.

(4) *The conclusions of the theorem are valid for any rational function of order n.*

Let $$f_1 = M_1/M_2.$$

The truth of (4) has already been established when M_2 is simply exponential. We require to establish inductions

(i) from the case in which M_2 is exponential and of type $\varpi-1$ to the case in which it is exponential and of type ϖ,

(ii) from the case in which M_2 is of logarithmic degree $\lambda-1$ to the case in which it is of degree λ and type $(\lambda, 1)$,

(iii) from the case in which is of type $(\lambda, \mu-1)$ to the case in which it is of type (λ, μ).

In any case let $$\rho_0 e\sigma_0 = 1/\phi,$$
a factor of one of the terms of M_2, chosen as usual, and let

$$M_1 = \bar{M}_1/\phi, \qquad M_2 = \bar{M}_2/\phi, \qquad f_1 = \bar{M}_1/\bar{M}_2.$$

Each of $\bar{M}_1$, $\bar{M}_2$ is of one of the forms

$$e^{Ax^s(1+\epsilon)}, \qquad Ax^s(lx)^t(1+\epsilon),$$

where s is rational and t is integral.

There is only one case in which it is not evident that f_1 itself is of one of these forms. This case occurs when $\bar{M}_1$ and $\bar{M}_2$ are both of the first form, with the same values of A and s. Then

$$\log \bar{M}_1 \sim \log \bar{M}_2 \sim Ax^s,$$

$$\bar{M}_1'/\bar{M}_1 \sim \bar{M}_2'/\bar{M}_2,$$

and so $$f_1 \sim \bar{M}_1'/\bar{M}_2'.$$

Now $\bar{M}_2'$ is of type $(\lambda, \mu-1)$, or of degree $\lambda-1$, or of type $\varpi-1$, according to the induction which we are proving. Thus in any case the induction is established.

(5) We can now complete the proof of Theorem 10. For f_1 satisfies an equation

$$M_0 f_1^p + M_1 f_1^{p-1} + \dots + M_p = 0$$

whose coefficients are integral functions of order 1. Of the terms $X_0, X_1, \dots, X_p$ of this equation, one pair at least must satisfy a relation $X_i \asymp X_j$. Thus

$$f_1^{j-i} \asymp M_j / M_i ;$$

and the theorem follows now as a corollary of (4).

21. Theorem 10 enables us to recognize at once that certain types of function cannot be L-functions of order 1. Thus

$$x^a = e^{a \log x},$$

where a is irrational, is of order 2. This was proved quite differently by Liouville.[*] The same is true of

$$\log\log x,\text{[†]} \quad e^{e^x}, \quad x^x.$$

Again :

THEOREM 11.—*No function of order* 1 *can satisfy*

$$x^{\Delta} \prec f \prec e^{x^{\delta}}$$

or

$$(\log x)^{\Delta} \prec f \prec x^{\delta}.$$

Thus

$$e^{(\log x)^a} \quad (a \neq 1)$$

is of order 2.

22. Let us pass to functions of order 2. It is easy to recognize

$$e^{e^{ax^s}}, \qquad e^{ax^s (lx)^t}, \qquad ax^s (lx)^t (l_2 x)^u$$

as fundamentally distinct types of increase for such functions. By arguments similar to those of § 20 we can establish the theorem—

THEOREM 12.—*An L-function of order* 2, *positive and tending to infinity with x, can be expressed in one or other of the following forms :—*

(i) $e^{e^{ax^s(1+\epsilon)}}$,

(ii) $e^{ax^s(lx)^t(1+\epsilon)}$,

(iii) $x^s e^{\pm a(lx)^t(1+\epsilon)}$,

(iv) $ax^s(lx)\,(l_2 x)^u(1+\epsilon)$,

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[*] *Journal de Math.*, T. 2, pp. 94–98.

[†] *Ibid.*, pp. 99–102.

where a is positive, and s, t, u rational, and, moreover,

(i) $s > 0$;

(ii) $s > 0$ *or* $s = 0,\ t > 1$;

(iii) $s > 0,\ 0 < t < 1$ *or* $s = 0,\ 0 < t < 1$ (*when the positive sign must be taken*), *or* $s = 0,\ t = 1$ (*when again the positive sign must be taken, and a is irrational**);

(iv) $s > 0$, *or* $s = 0,\ t > 0$, *or* $s = 0,\ t = 0,\ u > 0$.

By means of this theorem we recognize, for example, that

$$l_3 x, \quad e_3 x, \quad x^{x^x}, \quad (lx)^a \quad (a\ \textit{irrational}),$$

$$e^{e^{(\log x)^a}}\ (a \neq 1), \quad e^{(\log\log x)^a}\ (a \neq 1),$$

$$\dots \quad \dots \quad \dots \quad \dots$$

are in reality of order *three*.

Again, *no L-function of order* 2 *can satisfy*

$$e^{(lx)^\Delta} \prec f \prec e^{x^\delta}.$$

It is, of course, possible to proceed further in this direction. Thus for functions of order 3 the standard forms are

$$e^{e^{e^{ax^s(1+\epsilon)}}},$$

$$e^{e^{ax^s(lx)^t(1+\epsilon)}},$$

$$e^{x^s e^{\pm a(lx)^t(1+\epsilon)}},$$

$$e^{ax^s(lx)^t(l_2x)^u(1+\epsilon)},$$

$$x^s e^{\pm a(lx)^t(l_2x)^u(1+\epsilon)},$$

$$x^s(lx)^t e^{\pm a(l_2x)^u(1+\epsilon)},$$

$$x^s(lx)^t(l_2x)^u(l_3x)^v(1+\epsilon);$$

and so on generally. No function of order 3 can satisfy

$$e^{(lx)^\Delta} \prec f \prec e^{e^{(lx)^\delta}}.$$

No function of order 4 can satisfy

$$e^{e^{(llx)^\Delta}} \prec f \prec e^{e^{(lx)^\delta}};$$

no function of order 5 can satisfy

$$e^{e^{(llx)^\Delta}} \prec f \prec e^{e^{e^{(llx)^\delta}}};$$

and so on generally. We shall make use of this remark in the sequel.†

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* Were a rational, $e^{a\log x}$ could be replaced by x^a.

† On the subject of the functions $e_r(l_s x)^\mu$, see "O.I.," pp. 21 *et seq.*

VI.

On Gaps in the Logarithmico-Exponential Scales.

23. I have in my tract considered the question of the construction of functions whose rate of increase cannot be measured by functions of the logarithmico-exponential scales; *i.e.*, of functions f such that the relation

$$f \mathrel{\overset{\smile}{\underset{\frown}{-}}} \phi$$

is impossible for any L-function ϕ.

Such functions are furnished, for example, by any function f such that

$$f \succ e_\Delta(x) \quad \text{or} \quad 1 \prec f \prec l_\Delta(x).$$

A simple example of the first type is given in my tract.* An example of the second type can be constructed by means of the functional equation

$$\phi(e^x) = e\phi(x).$$

It is easy to define, by means of a geometrical construction, a solution of this equation which tends steadily and continuously to infinity with x. The fact that

$$l_k(e^x) \succ l_k x,$$

for all values of k, at once suggests that the increase of ϕ must be slower than that of any logarithm; and it is easy to prove that this is in fact so.

A still more interesting question is that of defining functions whose increase cannot be measured by any L-function, although it falls within the limits of the logarithmico-exponential scales. I showed in my tract† how we could, by a geometrical construction, define a continuous and monotonic solution of the equation

$$\phi\phi(x) = e^x.$$

I was compelled, by considerations of space, to content myself with the briefest indications of a proof, which I will now complete.

24. The solution of the equation was defined as follows. Let

$$x_0 = 0, \quad x_1 = \tfrac{1}{2}, \quad x_2 = 1, \quad x_3 = \sqrt{e}, \quad x_4 = e, \quad \ldots,$$

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* "O.I.," p. 33.

† "O.I.," p. 34.

and generally $$x_{2n} = e_n(0) = e_{n-1}(1),$$

$$x_{2n+1} = e_{n-1}(\sqrt{e}).$$

Then $\phi(x)$ is defined by the equations

$$\phi(x) = e_{-2}(l_{-2}x)^{\sqrt{e}} = x+\tfrac{1}{2} \quad (x_0 < x < x_1),^{*}$$

$$\phi(x) = e_{-1}(l_{-2}x)^{1/\sqrt{e}} = e^{x-\frac{1}{2}} \quad (x_1 < x < x_2),$$

$$\phi(x) = e_{-1}(l_{-1}x)^{\sqrt{e}} = x\sqrt{e} \quad (x_2 < x < x_3),$$

and generally $$\phi(x) = \lambda_{n-2}(x) = e_{n-2}\,(l_{n-2}x)^{\sqrt{e}} \quad (x_{2n} < x < x_{2n+1}),$$

$$\phi(x) = \mu_{n-2}(x) = e_{n-1}\,(l_{n-2}x)^{1/\sqrt{e}} \quad (x_{2n+1} < x < x_{2n+2}).$$

Then[†] $$\lambda_0 \prec \lambda_1 \prec \lambda_2 \prec \ldots \prec \mu_2 \prec \mu_1 \prec \mu_0.$$

I shall now prove that (as was asserted in my tract)[‡]

$$\lambda_p \prec \phi \prec \mu_q$$

for all values of p and q.

25. In order to prove that $\phi \succ \lambda_p$ for all values of p, it is plainly sufficient to prove that $$\phi \geqslant \lambda_{n-2} \quad (x \geqslant x_{2n}).$$

Now, if $x > x_{2n}$, ϕ is equal either to one of the functions $\lambda_{m-2}\,(m \geqslant n)$ or to one of the functions $\mu_{m-2}\,(m \geqslant n)$.

Now $$\lambda_{n-1} \geqslant \lambda_{n-2},$$

if $$e(l_{n-1}x)^{\sqrt{e}} \geqslant (l_{n-2}x)^{\sqrt{e}};$$

or, putting $y = l_{n-2}x$, if $$(ly)^{\sqrt{e}} \geqslant \sqrt{e}\,ly,$$

$$ly \geqslant (\sqrt{e})^{1/(\sqrt{e}-1)}.$$

Now, for $x = x_{2n} = e_{n-1}(1)$, we have $y = e$. And

$$e > (\sqrt{e})^{1/(\sqrt{e}-1)},$$

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[*] We may agree to interpret $l_{-k}x$ as $e_k x$ and $e_{-k}x$ as $l_k x$.

[†] "O.I.," pp. 21 *et seq.*

[‡] "O.I.," p. 35.

if

$$\sqrt{e}-1 > \tfrac{1}{2},$$

which is obviously true. Hence $\lambda_{n-1} \geqslant \lambda_{n-2}$ for $x \geqslant x_{2n}$; similarly $\lambda_n \geqslant \lambda_{n-1}$ for $x \geqslant x_{2n+2}$, and so on. It follows that

$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}),$$

certainly throughout the intervals in which ϕ *coincides with one of the functions* λ.

Again

$$\mu_{n-2} \geqslant \lambda_{n-2},$$

if

$$e(y^{1/\sqrt{e}}) \geqslant y^{\sqrt{e}},$$

or, putting $y = z^{\sqrt{e}}$, if

$$e^z \geqslant z^e.$$

Now the equation $e^z = z^e$ has a root when $z = e$: and $e^z > z^e$ if $z > e$. Thus $\mu_{n-2} \geqslant \lambda_{n-2}$, if $y \geqslant e^{\sqrt{e}}$, or

$$x \geqslant e_{n-1}(\sqrt{e}) = x_{2n+1}.$$

Similarly $\mu_{n-1} \geqslant \lambda_{n-1}$, if $x > x_{2n+3}$, and, *a fortiori*, $\mu_{n-1} \geqslant \lambda_{n-2}$, and so on. Thus

$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}),$$

also in the intervals in which ϕ *is equal to one of the functions* μ. And so

$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}).$$

Similarly we can prove that $\phi(x) \leqslant \mu_{n-2}(x) \quad (x \geqslant x_{2n+1})$, and from this it at once follows that

$$\lambda_p \prec \phi \prec \mu_q,$$

for all values of p and q.

Thus the function ϕ enables us to divide all L-functions into two classes, such that if L_1 and L_2 denote any members of the respective classes, then

$$L_1 \prec \phi \prec L_2.$$

The increase of ϕ would, in Borel's notation,[*] be denoted by

$$\sqrt{\omega}.$$

We could similarly define the mode of increase $\omega^{n+\frac{1}{2}}$, where n is any integer, positive or negative: and indeed ω^{a} for any rational a. *All these modes of increase are distinct from those of all L-functions.*

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[*] *Leçons sur la théorie de la croissance*, pp. 14 *et seq.*

26. There is another method of constructing functions whose increase, so to say, hits off a gap in the logarithmico-exponential scales. This method is simpler but less interesting than that just discussed, and I shall content myself with some very summary indications.

Consider the function $$f = e^{x+1/x}.$$

Since $$e^{x+1/x} \sim e^{x},$$

the increase of the function is the same as that of the simpler function e^{x}.

A series of closer approximations to the function f would be furnished by

$$e^{x}\left(1+\frac{1}{x}\right), \quad e^{x}\left(1+\frac{1}{x}+\frac{1}{2x^{2}}\right), \quad \dots .$$

But we cannot in this way find a function ϕ such that

$$f = \phi+\epsilon.$$

However many terms of the series for $e^{1/x}$ we retain, the difference $f-\phi$ remains of order greater than x^{Δ}. To put it roughly, we can only express f with the degree of approximation implied by the equation $f = \phi+\epsilon$, by taking ϕ equal to f itself. Similar considerations apply, evidently, to any function

$$e^{x}P(1/x),$$

where P is a convergent power series.

If now $$F = e^{f} = e^{e^{x+1/x}},$$

we see that, in order to express F by a simpler function ϕ, with the degree of approximation indicated by $$F = \phi(1+\epsilon)$$

—*i.e.*, with sufficient approximation to fix the increase of F—we must take

$$\phi = F.$$

Similar considerations apply, of course, to the more general function

$$e^{e^{x}P(1/x)}.$$

Thus, by taking $P(1/x)$ to be a power series *whose sum is not an ele-*

mentary function—such a series as

$$1+\frac{1}{1^2}\,\frac{1}{x}+\frac{1}{1^2.2^2}\,\frac{1}{x^2}+\frac{1}{1^2.2^2.3^2}\,\frac{1}{x^3}+\dots$$

—we are led to functions whose increase is not equal to that of any L-function.*

These considerations can be extended to cases in which $P(1/x)$ is a summable asymptotic series, and the factor e^x is replaced by a more general factor—we are thus led to such functions as

$$e^{\Gamma(x+1)} = e^{x^{x+\frac{1}{2}}e^{-x}\sqrt{(2\pi)}P(1/x)}.$$

VII.

Miscellaneous Remarks and Conclusion.

27. *Inverse Functions.*—The present is a convenient moment at which to make a few additional remarks as to the possible modes of increase of the inverse of an L-function (a question already raised in § 12).

Suppose that y is an L-function of order 1, ultimately positive and tending to infinity with x. Then, by Theorem 10,

$$y = e^{Ax^s(1+\epsilon)}$$

or $\qquad y = Ax^s(lx)^t(1+\epsilon).$

In the first case $\qquad x = A_1(ly)^{t_1}(1+\epsilon),$

and in the second $\qquad x = A_1 y^{s_1}(ly)^{t_1}(1+\epsilon),$

unless $s = 0$, when $\qquad x = e^{A_1 y^{t_1}(1+\epsilon)}.$

Thus *the standard forms for the increase of the inverses of L-functions of order* 1 *are the same as for the L-functions themselves.*

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* Several theorems have been proved with reference to the possible forms of power series whose sum is an elementary transcendent in Liouville's sense, or, more generally, a solution of an algebraic differential equation: see, for example, Hermite, *Cours* (4th edition), p. 195, where a theorem originally due to Eisenstein is proved. This theorem has been extended widely: see Heine, *Crelle*, Bd. 45 and Bd. 48; Hurwitz, *Annales de l'École Normale*, t. 6 (series 3); Fouet, *Leçons sur la théorie des fonctions analytiques* (2nd edition), t. 2, p. 123, where further references are given.

Hermite (*l.c.*) also refers to an extremely interesting theorem of Tschebyschef, of which no proof appears to have been published (see Wassilief, *P. L. Tschebyschef und seine wissenschaftlichen Leistungen*, Leipzig, 1900, p. 42 and p. 54).

Similarly, if y is of order 2, it has one of the forms

$$(1)\ e^{e^{Ax^s(1+\epsilon)}},\qquad (2)\ e^{Ax^s(lx)^t(1+\epsilon)},\qquad (2a)\ e^{A(lx)^t(1+\epsilon)}\ (t>1),$$

$$(3)\ x^s e^{A(lx)^t(1+\epsilon)}\ (t<1),\qquad (3a)\ e^{A(lx)^t(1+\epsilon)}\ (t<1),$$

$$(4)\ Ax^s(lx)^t(l_2x)^u(1+\epsilon),\qquad (4a)\ A(lx)^t(l_2x)^u(1+\epsilon),\qquad (4b)\ A(l_2x)^u(1+\epsilon).$$

It is easily verified that the forms

1, 2, $2a$, 3, $3a$, 4, $4a$, $4b$

invert into $4b$, $4a$, $3a$, 3, $2a$, 4, 2, 1.*

Thus the standard forms are also the same in the case of order 2, and this conclusion may be extended to any order.

It is easy to see that many of the theorems proved in this paper are true not only of L-functions but also of their inverses. This is certainly so, for example, with Theorems 2, 3, 7, 10, 11, and 12. Again, if $\bar{f}$ and $\bar{\phi}$ are inverses of L-functions, one of the relations $\bar{f} \succ \bar{\phi}$, $\bar{f} \asymp \bar{\phi}$, $\bar{f} \prec \bar{\phi}$ holds between them; and if f is an L-function, and $\bar{\phi}$ the inverse of an L-function, one of the relations $f \succ \bar{\phi}$, ... holds between them.

But whether or not it is true that, given an L-function ϕ and its inverse $\bar{\phi}$, there must be an L-function ψ, such that

$$\bar{\phi} \sim \psi,$$

I cannot say; and, as I said in § 12, I am very doubtful whether this is so.

Consider, for example, the equation

$$(1)\qquad y = x\,lx:$$

an equation whose solution was proved by Liouville† not to be expressible explicitly in finite terms. Then

$$(2)\qquad x \sim y/ly.$$

If, now, we consider the equation

$$(3)\qquad y = lx\,l_2x,$$

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* The special case of $2a$ or $3a$, in which $t = 1$, A being *irrational*, inverts into itself.

† *Journal de Mathématiques*, t. iii, pp. 526–531.

we have

$$(4) \qquad x = e\left\{\frac{y}{ly}(1+\epsilon)\right\}.$$

More precisely* $x = e\left\{\frac{y}{ly} + \frac{y\,l_2y}{(ly)^2} + \frac{y\,(l_2y)^2}{(ly)^3} - \frac{y\,l_2y}{(ly)^3}(1+\epsilon)\right\}.$

But in order to express the solution of (3) in the form

$$x = \psi(y)(1+\epsilon),$$

we should have to express the solution of (1) in the form

$$x = \chi(y)+\epsilon,$$

χ being an L-function: and it seems to me very improbable that this is possible.

In any case it should be observed that all the information that we have acquired concerning the modes of increase of the inverses of L-functions has been obtained by the use of Theorem 7 and its consequences, and so cannot be used in order to simplify the proof of that theorem.

28. *Algebraic Differential Equations.*—Borel† has indicated the lines of extremely interesting researches concerning the possible modes of increase of functions defined by differential equations. Thus he has proved that the equation

$$(1) \qquad f(x,\, y,\, y') = 0,$$

where x and y are real, and f is a polynomial, cannot have an increasing solution of order as great as

$$e^{e^x}.$$

Similarly the corresponding equation of the second order cannot have a solution whose increase is as great as that of e_3x; and there can be no doubt of the truth of the corresponding general theorem.

The discussion even of the two simplest cases is rather elaborate when so general hypotheses as M. Borel's are adopted. At present I propose only to point out how, in the case of the equation of the first order, we can obtain much more precise information about the possible modes of increase of the solutions, if we assume this increase to be, so to say, sufficiently regular.

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* "O.I.," p. 46.

† "Mémoire sur les séries divergentes," *Annales de l'École Normale*, t. 16, pp. 25 *et seq.*: see also Boutroux, *Leçons sur les fonctions définies par les équations différentielles du premier ordre.*

Let us assume that the equation (1) has a solution y, such that any function of the form

$$x^m y^n (y')^p,$$

where m, n, p are integers, is ultimately monotonic. We shall express this by saying that y is *of strictly regular increase.* It is possible to prove that, in the case of the particular equation

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \tag{2}$$

where P and Q are polynomials, any solution which is ultimately continuous is of strictly regular increase; but it would carry us too far to enter upon the details of a proof at present.

It then follows that any two of the terms of (1) must satisfy a relation of one of the types $\phi \succ \psi$, $\phi \asymp \psi$, $\phi \prec \psi$, and there must be at least one pair which satisfy a relation of the second type. We must therefore have a relation of the type

$$y^\nu y' \sim Ax^\mu. \tag{3}$$

If neither μ nor ν is equal to -1, we obtain a relation

$$y \sim Ax^\alpha. \tag{4}$$

If ν (but not μ) is equal to -1, we obtain

$$y = e^{Ax^\alpha (1+\epsilon)}. \tag{5}$$

If μ (but not ν) is equal to -1, we obtain

$$y \sim A(lx)^\alpha. \tag{6}$$

If μ and ν are each equal to -1, we obtain

$$ly \sim A\,lx,$$

$$y = x^{A+\epsilon}. \tag{7}$$

This equation is less precise than the relation (4). To make it more precise, we must put $y = x^A z$ and form the equation satisfied by z.

Let us (to cut the matter short) confine ourselves to the equation (2). We find that y *must be expressible in one of the forms*

$$e^{Ax^\alpha(1+\epsilon)}, \qquad Ax^\beta (lx)^\gamma (1+\epsilon). \tag{8}$$

It is not difficult to see that α and γ must be *rational*. This is not true of β—as is obvious from the fact that $y = x^{\beta}$ satisfies

$$y' = \beta y/x,$$

whether β be rational or irrational.*

Apart from this circumstance, the forms (8) are the standard forms of Theorem 10. This parallelism at once suggests a number of further questions which demand a more careful analysis of the equation (1) and its extensions: to some of these questions I hope to return on another occasion.

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* More precisely, the possible forms are

$$y = e^{Ax^p(1+\epsilon)},$$

$$y = A\,(x^q lx)^{1/p}(1+\epsilon), \qquad A\left(\frac{x^q}{lx}\right)^{1/p}(1+\epsilon),$$

$$y = Ax^{\beta}(1+\epsilon).$$

Here p and q are integers.

THE INTEGRATION OF THE EQUATIONS OF PROPAGATION OF ELECTRIC WAVES

By H. M. MACDONALD.

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INTEGRALS of the equations of propagation of electrical disturbances have been obtained by various writers; in particular, their expression as integrals taken over a surface enclosing all the sources has been treated by Love,* and by the present writer, who proved that the electric and magnetic forces due to any number of sources can be expressed in terms of the electric and magnetic forces tangential to any surface enclosing all the sources.† The object of the present note is to give a direct analytical proof of this result.‡

Denoting the components of the magnetic force at any point x, y, z by α, β, γ, the components of the electric force by X, Y, Z, the components of the electric displacement by f, g, h, and the components of the total convection current by u, v, w, the components of the displacement current are given by

$$\dot{f} = \frac{1}{4\pi}\iiint\left[\frac{\partial^2}{\partial x^2}\frac{u_1}{r} + \frac{\partial^2}{\partial x\,\partial y}\frac{v_1}{r} + \frac{\partial^2}{\partial x\,\partial z}\frac{w_1}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{u_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

$$\dot{g} = \frac{1}{4\pi}\iiint\left[\frac{\partial^2}{\partial x\,\partial y}\frac{u_1}{r} + \frac{\partial^2}{\partial y^2}\frac{v_1}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{w_1}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{v_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

$$\dot{h} = \frac{1}{4\pi}\iiint\left[\frac{\partial^2}{\partial x\,\partial z}\frac{u_1}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{v_1}{r} + \frac{\partial^2}{\partial z^2}\frac{w_1}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{w_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

where u_1, v_1, w_1 are the values of u, v, w at the point ξ, η, ζ at time

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* *Phil. Trans.*, 1901.

† *Electric Waves*, pp. 16, 17.

‡ The proof referred to depends on the superposition of two distributions, in one of which the tangential components of the electric force vanish, and in the other the tangential components of the magnetic force vanish; the adjustment of the two distributions to fit any given case presents some difficulties (principally that of determining the signs of the various quantities), which the proof now given evades.

$t-r/V$, r is the distance of the point x, y, z from the point ξ, η, ζ, and the integrals are taken throughout the space in which there are convection currents.* The components of the magnetic force are given by

$$\alpha = \iiint \left[\frac{\partial}{\partial y}\frac{w_1}{r} - \frac{\partial}{\partial z}\frac{v_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

$$\beta = \iiint \left[\frac{\partial}{\partial z}\frac{u_1}{r} - \frac{\partial}{\partial x}\frac{w_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

$$\gamma = \iiint \left[\frac{\partial}{\partial x}\frac{v_1}{r} - \frac{\partial}{\partial y}\frac{u_1}{r}\right] d\xi\, d\eta\, d\zeta,$$

and these integrals have to be transformed into integrals taken over a surface enclosing all the convection currents.

If X', Y', Z', α', β', γ' denote the components of the electric and magnetic forces at the point ξ, η, ζ, since

$$4\pi u + \frac{1}{V^2}\dot{X} = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z},$$

$$4\pi u_1 = \left(\frac{\partial\gamma'}{\partial\eta}\right)_1 - \left(\frac{\partial\beta'}{\partial\zeta}\right)_1 - \frac{1}{V^2}\dot{X}'_1,$$

where the suffix 1 denotes that $t-r/V$ has been substituted for t after the operations have been performed.† Now

$$\frac{\partial\gamma'_1}{\partial\eta} = \left(\frac{\partial\gamma'}{\partial\eta}\right)_1 - \frac{1}{V}\dot{\gamma}'_1\frac{\partial r}{\partial\eta},$$

and

$$\frac{\partial}{\partial\eta}\frac{\gamma'_1}{r} = \frac{1}{r}\frac{\partial\gamma'_1}{\partial\eta} - \frac{\gamma'_1}{r^2}\frac{\partial r}{\partial\eta},$$

$$\frac{\partial}{\partial y}\frac{\gamma'_1}{r} = -\left(\frac{\gamma'_1}{r^2} + \frac{\dot{\gamma}'_1}{rV}\right)\frac{\partial r}{\partial y},$$

or, since

$$\frac{\partial r}{\partial y} = -\frac{\partial r}{\partial\eta},$$

$$\frac{\partial}{\partial y}\frac{\gamma'_1}{r} = \left(\frac{\gamma'_1}{r^2} + \frac{\dot{\gamma}'_1}{rV}\right)\frac{\partial r}{\partial\eta};$$

therefore

$$\frac{\partial}{\partial\eta}\frac{\gamma'_1}{r} + \frac{\partial}{\partial y}\frac{\gamma'_1}{r} = \frac{1}{r}\frac{\partial\gamma'_1}{\partial\eta} + \frac{\dot{\gamma}'_1}{rV}\frac{\partial r}{\partial\eta},$$

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* *Electric Waves*, pp. 17, 18.

† It should be observed that X' is obtained from X by substituting $t-r/V$ for t, whence

$$\frac{\partial X'_1}{\partial t} = \left(\frac{\partial X'}{\partial t}\right)_1.$$

that is, $$\frac{\partial}{\partial\eta}\frac{\gamma_1'}{r}+\frac{\partial}{\partial y}\frac{\gamma_1'}{r}=\frac{1}{r}\left(\frac{\partial\gamma'}{\partial\eta}\right)_1.$$

Hence $$\frac{4\pi u_1}{r}=\frac{\partial}{\partial\eta}\frac{\gamma_1'}{r}-\frac{\partial}{\partial\zeta}\frac{\beta_1}{r}+\frac{\partial}{\partial y}\frac{\gamma_1'}{r}-\frac{\partial}{\partial z}\frac{\beta_1'}{r}-\frac{1}{rV^2}\dot{X}_1';$$

and similarly, $$\frac{\dot{\alpha}_1'}{r}=\frac{\partial}{\partial\zeta}\frac{Y_1'}{r}-\frac{\partial}{\partial\eta}\frac{Z_1'}{r}+\frac{\partial}{\partial z}\frac{Y_1'}{r}-\frac{\partial}{\partial y}\frac{Z_1'}{r}.$$

The expression for a then becomes

$$a=\frac{1}{4\pi}\iiint\Big[\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial\xi}\frac{\beta_1'}{r}-\frac{\partial}{\partial\eta}\frac{\alpha_1'}{r}+\frac{\partial}{\partial x}\frac{\beta_1'}{r}-\frac{\partial}{\partial y}\frac{\alpha_1'}{r}\right\}$$

$$-\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial\zeta}\frac{\alpha_1'}{r}-\frac{\partial}{\partial\xi}\frac{\gamma_1'}{r}+\frac{\partial}{\partial z}\frac{\alpha_1'}{r}-\frac{\partial}{\partial x}\frac{\gamma_1'}{r}\right\}-\frac{1}{V^2}\left\{\frac{\partial}{\partial y}\frac{\dot{Z}_1'}{r}-\frac{\partial}{\partial z}\frac{\dot{Y}_1}{r}\right\}\Big]\,d\xi\,d\eta\,d\zeta,$$

whence

$$a=\frac{1}{4\pi}\iiint\Big[\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial\xi}\frac{\beta_1'}{r}-\frac{\partial}{\partial\eta}\frac{\alpha_1'}{r}\right\}-\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial\zeta}\frac{\alpha_1'}{r}-\frac{\partial}{\partial\xi}\frac{\gamma_1'}{r}\right\}\Big]\,d\xi\,d\eta\,d\zeta$$

$$+\frac{1}{4\pi}\iiint\Big[\frac{\partial^2}{\partial x\,\partial y}\frac{\beta_1'}{r}-\frac{\partial^2}{\partial y^2}\frac{\alpha_1'}{r}-\frac{\partial^2}{\partial z^2}\frac{\alpha_1'}{r}+\frac{\partial^2}{\partial x\,\partial z}\frac{\gamma_1'}{r}$$

$$-\frac{1}{V^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\frac{Z_1'}{r}-\frac{\partial}{\partial z}\frac{Y_1'}{r}\right\}\Big]\,d\xi\,d\eta\,d\zeta.$$

Again, since $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right)\frac{\alpha_1'}{r}=0,$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{\beta_1'}{r}-\frac{\partial^2}{\partial y^2}\frac{\alpha_1'}{r}-\frac{\partial^2}{\partial z^2}\frac{\alpha_1'}{r}+\frac{\partial^2}{\partial x\,\partial z}\frac{\gamma_1'}{r}-\frac{1}{V^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\frac{Z_1'}{r}-\frac{\partial}{\partial z}\frac{Y_1'}{r}\right\}$$

$$=\frac{\partial^2}{\partial x^2}\frac{\alpha_1'}{r}+\frac{\partial^2}{\partial x\,\partial y}\frac{\beta_1'}{r}+\frac{\partial^2}{\partial x\,\partial z}\frac{\gamma_1'}{r}-\frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\alpha_1'}{r}-\frac{1}{V^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\frac{Z_1'}{r}-\frac{\partial}{\partial z}\frac{Y_1'}{r}\right\},$$

hence, using the values of $\frac{\dot{\alpha}_1'}{r}$, $\frac{\dot{\beta}_1'}{r}$, $\frac{\dot{\gamma}_1'}{r}$ found above

$$\frac{\partial}{\partial t}\Big[\frac{\partial^2}{\partial x\,\partial y}\frac{\beta_1'}{r}-\frac{\partial^2}{\partial y^2}\frac{\alpha_1'}{r}-\frac{\partial^2}{\partial z^2}\frac{\alpha_1'}{r}+\frac{\partial^2}{\partial x\,\partial z}\frac{\gamma_1'}{r}-\frac{1}{V^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\frac{Z_1'}{r}-\frac{\partial}{\partial z}\frac{Y_1'}{r}\right\}\Big]$$

$$=\frac{\partial^2}{\partial x^2}\left\{\frac{\partial}{\partial\zeta}\frac{Y_1'}{r}-\frac{\partial}{\partial\eta}\frac{Z_1'}{r}\right\}+\frac{\partial^2}{\partial x\,\partial y}\left\{\frac{\partial}{\partial\xi}\frac{Z_1'}{r}-\frac{\partial}{\partial\zeta}\frac{X_1'}{r}\right\}+\frac{\partial^2}{\partial x\,\partial z}\left\{\frac{\partial}{\partial\eta}\frac{X_1'}{r}-\frac{\partial}{\partial\xi}\frac{Y_1'}{r}\right\}$$

$$-\frac{1}{V^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial\zeta}\frac{Y_1'}{r}-\frac{\partial}{\partial\eta}\frac{Z_1'}{r}\right\};$$

and therefore

$$\dot{\alpha} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iiint\left[\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial \xi}\frac{\beta_1'}{r} - \frac{\partial}{\partial \eta}\frac{\alpha_1'}{r}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial \zeta}\frac{\alpha_1'}{r} - \frac{\partial}{\partial \xi}\frac{\gamma_1'}{r}\right\}\right] d\xi\, d\eta\, d\zeta$$

$$+\frac{1}{4\pi}\iiint\left[\frac{\partial^2}{\partial x^2}\left\{\frac{\partial}{\partial \zeta}\frac{Y_1'}{r} - \frac{\partial}{\partial \eta}\frac{Z_1'}{r}\right\} + \frac{\partial^2}{\partial x\,\partial y}\left\{\frac{\partial}{\partial \xi}\frac{Z_1'}{r} - \frac{\partial}{\partial \zeta}\frac{X_1'}{r}\right\}\right.$$

$$\left. + \frac{\partial^2}{\partial x\,\partial z}\left\{\frac{\partial}{\partial \eta}\frac{X_1'}{r} - \frac{\partial}{\partial \xi}\frac{Y_1'}{r}\right\} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial \zeta}\frac{Y_1}{r} - \frac{\partial}{\partial \eta}\frac{Z_1'}{r}\right\}\right] d\xi\, d\eta\, d\zeta.$$

This integral can be immediately replaced by an integral taken over a surface enclosing all the convection currents, and the result is

$$\dot{\alpha} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial y}\frac{l\beta_1' - m\alpha_1'}{r} - \frac{\partial}{\partial z}\frac{n\alpha_1' - l\gamma_1'}{r}\right] dS$$

$$+\frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x^2}\frac{nY_1' - mZ_1'}{r} + \frac{\partial^2}{\partial x\,\partial y}\frac{lZ_1' - nX_1'}{r} + \frac{\partial^2}{\partial x\,\partial z}\frac{mX_1' - lY_1'}{r}\right.$$

$$\left. - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{nY_1' - mZ_1'}{r}\right] dS,$$

where l, m, n are the direction cosines of the normal to the surface drawn into the space in which there are no convection currents. Writing

$$m\gamma_1' - n\beta_1' = \bar{\alpha}, \qquad n\alpha_1' - l\gamma_1' = \bar{\beta}, \qquad l\beta_1' - m\alpha_1' = \bar{\gamma},$$

$$mZ_1' - nY_1' = \bar{X}, \qquad nX_1' - lZ_1' = \bar{Y}, \qquad lY_1' - mX_1' = \bar{Z},$$

the expressions for $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ are given by

$$\dot{\alpha} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial y}\frac{\bar{\gamma}}{r} - \frac{\partial}{\partial z}\frac{\bar{\beta}}{r}\right] dS$$

$$-\frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x^2}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial x\,\partial y}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial x\,\partial z}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{X}}{r}\right] dS,$$

$$\dot{\beta} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial z}\frac{\bar{\alpha}}{r} - \frac{\partial}{\partial x}\frac{\bar{\gamma}}{r}\right] dS$$

$$-\frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\,\partial y}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial y^2}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{Y}}{r}\right] dS,$$

$$\dot{\gamma} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial x}\frac{\bar{\beta}}{r} - \frac{\partial}{\partial y}\frac{\bar{\alpha}}{r}\right] dS$$

$$-\frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\,\partial z}\frac{\bar{X}}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{\bar{Y}}{r} + \frac{\partial^2}{\partial z^2}\frac{\bar{Z}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{Z}}{r}\right] dS,$$

and the corresponding expressions for the components of the displacement current are

$$4\pi\dot{f} = \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x^2}\frac{\bar{\alpha}}{r} + \frac{\partial^2}{\partial x\,\partial y}\frac{\bar{\beta}}{r} + \frac{\partial^2}{\partial x\,\partial z}\frac{\bar{\gamma}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\alpha}}{r}\right]dS$$

$$+\frac{1}{4\pi V^2}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial y}\frac{\bar{Z}}{r} - \frac{\partial}{\partial z}\frac{\bar{Y}}{r}\right]dS,$$

$$4\pi\dot{g} = \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\,\partial y}\frac{\bar{\alpha}}{r} + \frac{\partial^2}{\partial y^2}\frac{\bar{\beta}}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{\bar{\gamma}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\beta}}{r}\right]dS$$

$$+\frac{1}{4\pi V^2}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial z}\frac{\bar{X}}{r} - \frac{\partial}{\partial x}\frac{\bar{Z}}{r}\right]dS,$$

$$4\pi\dot{h} = \frac{1}{4\pi}\iint\left[\frac{\partial^2}{\partial x\,\partial z}\frac{\bar{\alpha}}{r} + \frac{\partial^2}{\partial y\,\partial z}\frac{\bar{\beta}}{r} + \frac{\partial^2}{\partial z^2}\frac{\bar{\gamma}}{r} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\frac{\bar{\gamma}}{r}\right]dS$$

$$+\frac{1}{4\pi V^2}\frac{\partial}{\partial t}\iint\left[\frac{\partial}{\partial x}\frac{\bar{Y}}{r} - \frac{\partial}{\partial y}\frac{\bar{X}}{r}\right]dS.$$

The quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are the components of a vector whose magnitude is that of the magnetic force tangential to the surface at the point ξ, η, ζ at time $t-r/V$, and whose direction forms with the normal to the surface and the tangential magnetic force a right-handed system of axes. Similarly, the quantities $\bar{X}$, $\bar{Y}$, $\bar{Z}$ are the components of a vector whose magnitude is that of the electric force tangential to the surface at the point ξ, η, ζ at time $t-r/V$, and whose direction is at right angles to the electric force. The first vector gives the electric current distribution on the surface which would be produced by the distribution of tangential magnetic force if the surface were perfectly conducting. The second vector gives the magnetic current distribution on the surface which would be produced by the distribution of tangential electric force if the surface were the boundary of a region incapable of supporting magnetic force.

ON CERTAIN VECTORS ASSOCIATED WITH AN ELECTROMAGNETIC FIELD AND THE REFLECTION OF A DISTURBANCE AT THE SURFACE OF A PERFECT CONDUCTOR

By H. Bateman.

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2. The properties of a solution of the equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial u}{\partial t}\right)^2.$$

3. The expression of an electromagnetic field in terms of four simple vectors.

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1. The transformation

$$\left.\begin{aligned} x' &= xf(u)+X(u), & z' &= zf(u)+Z(u) \\ y' &= yf(u)+Y(u), & t' &= tf(u)+T(u) \end{aligned}\right\} \tag{1}$$

admits of an interesting interpretation in connection with an electromagnetic field with a simple singularity moving through the æther.

It was shown in my former paper that if

$$\xi(u) = -\frac{X'(u)}{f'(u)}, \qquad \eta(u) = -\frac{Y'(u)}{f'(u)},$$

$$\zeta(u) = -\frac{Z'(u)}{f'(u)}, \qquad \tau(u) = -\frac{T'(u)}{f'(u)},$$

and u is defined by the equation

$$[x-\xi(u)]^2+[y-\eta(u)]^2+[z-\zeta(u)]^2 = [t-\tau(u)]^2, \tag{2}$$

then the transformation can be applied to an electromagnetic field in which the electric and magnetic forces satisfy equations of the type

$$\frac{\partial u}{\partial t}\left[\frac{\partial u}{\partial t}E_x-\frac{\partial u}{\partial y}H_z+\frac{\partial u}{\partial z}H_y\right]=\frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial x}E_x+\frac{\partial u}{\partial y}E_y+\frac{\partial u}{\partial z}E_z\right]. \tag{3}$$

Now equation (2) suggests that the moving point represented by the equations

$$x = \xi(u), \quad y = \eta(u), \quad z = \zeta(u), \quad t = \tau(u), \tag{4}$$

should have an important relation to the electromagnetic field, and this at once calls to mind Prof. Conway's investigations on the field of force due to a moving electron.*

According to the expressions obtained by Prof. Conway, the components of the electric and magnetic vectors in an electromagnetic field with a simple singularity which moves in a manner represented by equation (4), are given by the equations†

$$\left.\begin{aligned}
E_x &= -e\frac{\partial}{\partial x}\left[\frac{\tau'(u)}{M}\right]-e\frac{\partial}{\partial t}\left[\frac{\xi'(u)}{M}\right]\\
E_y &= -e\frac{\partial}{\partial y}\left[\frac{\tau'(u)}{M}\right]-e\frac{\partial}{\partial t}\left[\frac{\eta'(u)}{M}\right]\\
E_z &= -e\frac{\partial}{\partial z}\left[\frac{\tau'(u)}{M}\right]-e\frac{\partial}{\partial t}\left[\frac{\zeta'(u)}{M}\right]\\
H_x &= \quad e\frac{\partial}{\partial y}\left[\frac{\zeta'(u)}{M}\right]-e\frac{\partial}{\partial z}\left[\frac{\eta'(u)}{M}\right]\\
H_y &= \quad e\frac{\partial}{\partial z}\left[\frac{\xi'(u)}{M}\right]-e\frac{\partial}{\partial x}\left[\frac{\zeta'(u)}{M}\right]\\
H_z &= \quad e\frac{\partial}{\partial x}\left[\frac{\eta'(u)}{M}\right]-e\frac{\partial}{\partial y}\left[\frac{\xi'(u)}{M}\right]
\end{aligned}\right\}, \qquad (5)$$

where $$M = \tau'(u)\left[t-\tau(u)\right]-\xi'(u)\left[x-\xi(u)\right]$$
$$-\eta'(u)\left[y-\eta(u)\right]-\zeta'(u)\left[z-\zeta(u)\right]. \qquad (6)$$

It is easy to verify that any function of the form $V = M^{-1}\psi(u)$ is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial t^2},$$

except at the singularity. Consequently the four potentials

$$\frac{\xi'(u)}{M}, \quad \frac{\eta'(u)}{M}, \quad \frac{\zeta'(u)}{M}, \quad \frac{\tau'(u)}{M}$$

satisfy this equation. Also u is a solution of the differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial u}{\partial t}\right)^2. \qquad (7)$$

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* These *Proceedings*, Ser. 2, Vol. 1, p. 154 (1903).

† In Prof. Conway's work u is put equal to τ.

It is easy to see that $\frac{\partial M}{\partial x}$ is of the form

$$\frac{\partial M}{\partial x} = -\xi'(u) + K\frac{\partial u}{\partial x},$$

and similarly
$$\frac{\partial M}{\partial y} = -\eta'(u) + K\frac{\partial u}{\partial y},$$

$$\frac{\partial M}{\partial z} = -\zeta'(u) + K\frac{\partial u}{\partial z},$$

$$\frac{\partial M}{\partial t} = +\tau'(u) + K\frac{\partial u}{\partial t}.$$

Hence the expressions for E_x, H_x may be written

$$E_x = -e\left[\frac{\tau''(u)}{M} - \frac{K\tau'(u)}{M^2}\right]\frac{\partial u}{\partial x} - e\left[\frac{\xi''(u)}{M} - \frac{K\xi'(u)}{M^2}\right]\frac{\partial u}{\partial t},$$

$$H_x = \quad \left[\frac{\zeta''(u)}{M} - \frac{K\zeta'(u)}{M^2}\right]\frac{\partial u}{\partial g} - e\left[\frac{\eta''(u)}{M} - \frac{K\eta'(u)}{M^2}\right]\frac{\partial u}{\partial z},$$

and it is clear that equations (3) are satisfied. Moreover

$$\frac{\partial u}{\partial x}H_x + \frac{\partial u}{\partial y}H_y + \frac{\partial u}{\partial z}H_z = 0,$$

so that the vector whose components are $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$ is perpendicular to the magnetic force. It also follows that

$$E_xH_x + E_yH_y + E_zH_z = 0,$$

and this implies that the electric and magnetic forces are perpendicular to one another. These relations will be required later, but are not essential as far as the transformation is concerned.

Since equations (3) are satisfied the transformation can be used to transform the given electromagnetic field into another. We shall now show that the new electromagnetic field also possesses a simple singularity moving through the æther.

If we put $\bar{x} = \xi(u), \quad \bar{y} = \eta(u), \quad \bar{z} = \zeta(u), \quad \bar{t} = \tau(u),$

we find that the corresponding point $(\bar{x}', \bar{y}', \bar{z}', \bar{t}')$ is given by

$$\bar{x}' = X(u) + f(u)\,\xi(u), \; \ldots,$$

$$d\bar{x}' = [X'(u) + f'(u)\,\xi(u) + f(u)\,\xi'(u)]\,du = f(u)\,\xi'(u)\,du.$$

Hence

$$\left.\begin{aligned} \bar{x}' &= \int f(u)\,\xi'(u)\,du \\ \bar{y}' &= \int f(u)\,\eta'(u)\,du \\ \bar{z}' &= \int f(u)\,\zeta'(u)\,du \\ \bar{t}' &= \int f(u)\,\tau'(u)\,du \end{aligned}\right\}. \tag{8}$$

It follows from these equations that the points $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$, $(\bar{x}', \bar{y}', \bar{z}', \bar{t}')$ move along two curves in such a way that the velocities at corresponding points (determined by the same value of u) are the same both in magnitude and direction. The two curves thus have the same spherical indicatrix. We also have the relations

$$(x'-\bar{x}') = f(u)\,[x-\xi(u)], \;\ldots;$$

hence the relation

$$(x'-\bar{x}')^2+(y'-\bar{y}')^2+(z'-\bar{z}')^2 = (t'-\bar{t}')^2$$

is a consequence of the relation

$$[x-\xi(u)]^2+[y-\eta(u)]^2+[z-\zeta(u)]^2 = [t-\tau(u)]^2.$$

Writing

$$\begin{aligned} \alpha(u) &= \bar{x}', & \alpha'(u) &= f(u)\,\xi'(u), \\ \beta(u) &= \bar{y}', & \beta'(u) &= f(u)\,\eta'(u), \\ \gamma(u) &= \bar{z}', & \gamma'(u) &= f(u)\,\zeta'(u), \\ \theta(u) &= \bar{t}', & \theta'(u) &= f(u)\,\tau'(u), \end{aligned}$$

we have

$$\begin{aligned} M' = {} & \theta'(u)\,[t'-\theta(u)]-\alpha'(u)\,[x-\alpha(u)] \\ & -\beta'(u)[y-\beta(u)]-\gamma'(u)\,[z-\gamma(u)] \\ = {} & [f(u)]^2\,[\tau'(u)\,\{t-\tau(u)\}-\xi'(u)\,\{x-\xi(u)\} \\ & -\eta'(u)\,\{y-\eta(u)\}-\zeta'(u)\,\{z-\zeta(u)\}] \\ = {} & M[f(u)]^2. \end{aligned}$$

Hence if we introduce the new potentials

$$\begin{aligned} A'_x &= \frac{\alpha'(u)}{M'} = \frac{\xi'(u)}{Mf(u)} = \frac{1}{f(u)}\,A_x, \\ A'_y &= \frac{\beta'(u)}{M'} = \frac{\eta'(u)}{Mf(u)} = \frac{1}{f(u)}\,A_y, \\ A'_z &= \frac{\gamma'(u)}{M'} = \frac{\zeta'(u)}{Mf(u)} = \frac{1}{f(u)}\,A_z, \\ \Phi' &= \frac{\theta'(u)}{M'} = \frac{\tau'(u)}{Mf(u)} = \frac{1}{f(u)}\,\Phi, \end{aligned}$$

we have the relation

$$A'_x dx' + A'_y dy' + A'_z dz' - \Phi' dt' = A_x dx + A_y dy + A_z dz - \Phi dt - \frac{f'(u)\,du}{f(u)}. \quad (9)$$

For, since $$dx' = f(u)\,dx + f'(u)\,[x - \xi(u)]\,du,$$

we have $A'_x dx' + A'_y dy' + A'_z dz' - \Phi' dt'$

$$= A_x dx + A_y dy + A_z dz - \Phi dt + \frac{f'(u)}{f(u)}\,\frac{du}{M}\,[\xi'(u)\,\{x - \xi(u)\} + \eta'(u)\,\{y - \eta(u)\}$$

$$+ \zeta'(u)\,\{z - \zeta(u)\} - \tau'(u)\,\{t - \tau(u)\}],$$

and the quantity within the square brackets is equal to $-M$.

Multiplying equation (9) by the equation

$$\frac{\partial}{\partial x'}\,dx' + \frac{\partial}{\partial y'}\,dy' + \frac{\partial}{\partial z'}\,dz' + \frac{\partial}{\partial t'}\,dt' = \frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz + \frac{\partial}{\partial t}\,dt,$$

we obtain the equation

$$H'_x dy'dz' + H'_y dz'dx' + H'_z dx'dy' + E'_x dx'dt' + E'_y dy'dt' + E'_z dz'dt'$$

$$= H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt, \quad (10)$$

where $$H'_x = \frac{\partial A'_z}{\partial y'} - \frac{\partial A'_y}{\partial z'}, \; \ldots .$$

This equation shows that the electromagnetic field with the simple singularity $[\xi(u), \eta(u), \zeta(u), \tau(u)]$ *is transformed by our transformation into the electromagnetic field with the simple singularity* $[\alpha(u), \beta(u), \gamma(u), \theta(u)]$.

2. We are now led to investigate the properties of a certain system of lines in space.

Let a direction (l, m, n) be associated with the point (x, y, z) at time t according to the law

$$\frac{l}{\dfrac{\partial\theta}{\partial x}} = \frac{m}{\dfrac{\partial\theta}{\partial y}} = \frac{n}{\dfrac{\partial\theta}{\partial z}} = -\frac{1}{\dfrac{\partial\theta}{\partial t}},$$

where θ satisfies the differential equation

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = \left(\frac{\partial\theta}{\partial t}\right)^2. \quad (11)$$

If we imagine a particle to be moving in this direction with unit velocity (the velocity of light), its position at time $(t+\delta t)$ will be given by

$$x+\delta x = x+l\delta t,$$
$$y+\delta y = y+m\delta t,$$
$$z+\delta z = z+n\delta t,$$

and the new value of $\partial\theta/\partial x$ will be

$$\frac{\partial\theta}{\partial x}+\left[\frac{\partial^2\theta}{\partial x^2}l\delta t+\frac{\partial^2\theta}{\partial x\partial y}m\delta t+\frac{\partial^2\theta}{\partial x\partial z}n\delta t+\frac{\partial^2\theta}{\partial x\partial t}\delta t\right].$$

Now the quantity within the square brackets may be written in the form

$$\frac{\delta t}{\dfrac{\partial\theta}{\partial t}}\left[\frac{\partial^2\theta}{\partial x\partial t}\frac{\partial\theta}{\partial t}-\frac{\partial^2\theta}{\partial x^2}\frac{\partial\theta}{\partial x}-\frac{\partial^2\theta}{\partial x\partial y}\frac{\partial\theta}{\partial y}-\frac{\partial^2\theta}{\partial x\partial z}\frac{\partial\theta}{\partial z}\right],$$

and this vanishes on account of the equation (11). Hence the value of $\partial\theta/\partial x$ is the same as before, and similarly for the values of $\partial\theta/\partial y$, $\partial\theta/\partial z$, $\partial\theta/\partial t$. The direction associated with the point $(x+l\delta t,\ y+m\delta t,\ z+n\delta t)$ at time $t+\delta t$ is again (l, m, n), and so the particle will continue to move in a straight line with the velocity of light.*

It appears then that the lines associated with the ∞^4 space time points (x, y, z, t), form a complex of ∞^3 lines and represent the rectilinear paths of an aggregate of particles moving with the velocity of light.

The case in which the complex of lines consists of all the lines meeting a curve is of special interest, for this corresponds to the case discussed in § 1.

If we take an arbitrary point $P(x, y, z)$ at time t, and find the one position $Q(\xi, \eta, \zeta, \tau)$ of the singularity,† for which

$$[x-\xi(u)]^2+[y-\eta(u)]^2+[z-\zeta(u)]^2=[t-\tau(u)]^2 \quad [t>\tau(u)],$$

then the line determined by the equations

$$\frac{l}{\dfrac{\partial u}{\partial x}}=\frac{m}{\dfrac{\partial u}{\partial y}}=\frac{n}{\dfrac{\partial u}{\partial z}}=-\frac{1}{\dfrac{\partial u}{\partial t}}$$

is simply the line PQ.

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* This is simply a generalisation of Hamilton's theorem for the equation

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=1.$$

† Prof. Conway proves that there is only one position of the singularity for which these conditions are satisfied.

A general solution of the equation

$$\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2+\left(\frac{\partial\theta}{\partial z}\right)^2=\left(\frac{\partial\theta}{\partial t}\right)^2$$

may be obtained by putting

$$[x-\xi(\alpha,\beta,\gamma,\theta)]^2+[y-\eta(\alpha,\beta,\gamma,\theta)]^2+[z-\zeta(\alpha,\beta,\gamma,\theta)]^2 = [t-\tau(\alpha,\beta,\gamma,\theta)]^2,$$

where

$$\frac{\partial\xi}{\partial\alpha}[x-\xi]+\frac{\partial\eta}{\partial\alpha}[y-\eta]+\frac{\partial\zeta}{\partial\alpha}[z-\zeta]=\frac{\partial\tau}{\partial\alpha}[t-\tau],$$

$$\frac{\partial\xi}{\partial\beta}[x-\xi]+\frac{\partial\eta}{\partial\beta}[y-\eta]+\frac{\partial\zeta}{\partial\beta}[z-\zeta]=\frac{\partial\tau}{\partial\alpha}[t-\tau],$$

$$\frac{\partial\xi}{\partial\gamma}[x-\zeta]+\frac{\partial\eta}{\partial\gamma}[y-\eta]+\frac{\partial\zeta}{\partial\gamma}[z-\zeta]=\frac{\partial\tau}{\partial\gamma}[t-\tau],$$

or else, by putting

$$x\xi(\alpha,\beta,\gamma,\theta)+y\eta(\alpha,\beta,\gamma,\theta)+z\zeta(\alpha,\beta,\gamma,\theta) -t\tau(\alpha,\beta,\gamma,\theta)+f(\alpha,\beta,\gamma,\theta)=0,$$

where

$$x\frac{\partial\xi}{\partial\alpha}+y\frac{\partial\eta}{\partial\alpha}+z\frac{\partial\zeta}{\partial\alpha}-t\frac{\partial\tau}{\partial\alpha}+\frac{\partial f}{\partial\alpha}=0,$$

$$x\frac{\partial\xi}{\partial\beta}+y\frac{\partial\eta}{\partial\beta}+z\frac{\partial\zeta}{\partial\beta}-t\frac{\partial\tau}{\partial\beta}+\frac{\partial f}{\partial\beta}=0,$$

$$x\frac{\partial\xi}{\partial\gamma}+y\frac{\partial\eta}{\partial\gamma}+z\frac{\partial\zeta}{\partial\gamma}-t\frac{\partial\tau}{\partial\gamma}+\frac{\partial f}{\partial\gamma}=0,$$

$$\xi^2+\eta^2+\zeta^2-\tau^2=0.$$

The first solution appears to be better adapted for problems in the theory of electrons.

3. We shall now investigate the properties of a vector (v_x, v_y, v_z) which satisfies the relations

$$E_x+v_yH_z-v_zH_y=v_x(v_xE_x+v_yE_y+v_zE_z),$$

$$E_y+v_zH_x-v_xH_z=v_y(v_xE_x+v_yE_y+v_zE_z),$$

$$E_z+v_xH_y-v_yH_x=v_z(v_xE_x+v_yE_y+v_zE_z),$$

$$v_x^2+v_y^2+v_z^2=1.$$

These equations give rise to a similar set of equations

$$H_x - v_y E_z + v_z E_y = v_x(v_x H_x + v_y H_y + v_z H_z),$$

$$H_y - v_z E_x + v_x E_z = v_y(v_x H_x + v_y H_y + v_z H_z),$$

$$H_z - v_x E_y + v_y E_x = v_z(v_x H_x + v_y H_y + v_z H_z).$$

Putting

$$v_x H_x + v_y H_y + v_z H_z = \lambda,$$

$$v_x E_x + v_y E_y + v_z E_z = \mu,$$

$$E_x^2 + E_y^2 + E_z^2 = e^2, \qquad H_x^2 + H_y^2 + H_z^2 = h^2,$$

$$E_x H_x + E_y H_y + E_z H_z = (eh),$$

we have the following equations for the determination of λ and μ:

$$(eh) = \lambda\mu, \quad e^2 + [e, v, h] = \lambda^2, \quad h^2 + [e, v, h] = \mu^2;$$

therefore

$$\lambda^2 - \mu^2 = e^2 - h^2.$$

Thus

$$\lambda^4 - (e^2 - h^2)\lambda^2 - (eh)^2 = 0.$$

This equation gives four values of λ, two of which are real and two imaginary; there are consequently four vectors of the type (v_x, v_y, v_z). Two of these are real and two imaginary.

The value of v_x corresponding to a root λ_1 is

$$v_x = \frac{\lambda_1 E_x + \lambda_1^{-1}(eh) H_x + E_y H_z - E_z H_y}{\lambda_1^2 + h^2}$$

$$= \frac{\lambda_1 E_x + \sqrt{\{\lambda_1^2 - (e^2 - h^2)\}}\, H_x + E_y H_z - E_z H_y}{\lambda_1^2 + h^2},$$

and for the root $-\lambda_1$ it is

$$v_x' = \frac{-\lambda_1 E_x - \sqrt{\{\lambda_1^2 - (e^2 - h^2)\}}\, H_x + E_y H_z - E_z H_y}{\lambda_1^2 + h^2}.$$

Consequently, if $(eh) = 0$ and $\lambda_1 = 0$, the two values of v_x, v_x' are not equal, but if in addition $e^2 - h^2 = 0$, then all the roots of the equation are zero and the four vectors coincide with the direction of Poynting's vector.

In this last case the components of the electric and magnetic forces can be expressed in the form

$$H_x = v_y E_z - v_z E_y, \qquad E_x = -(v_y H_z - v_z H_y),$$

$$H_y = v_z E_x - v_x E_z, \qquad E_y = -(v_z H_x - v_x H_z),$$

$$H_z = v_x E_y - v_y E_x, \qquad E_z = -(v_x H_y - v_y H_x).$$

In the general case, when the roots are all different or when two of the roots are zero, a convenient expression for the electromagnetic field may be obtained in the following way. Let λ_1, $-\lambda_1$, λ_2, $-\lambda_2$ be the four roots of the equation for λ, and let the corresponding vectors be (l, m, n), (l', m', n'); (p, q, r), (p', q', r') respectively; then it is easy to prove that these satisfy the relations

$$\begin{aligned} lp+mq+nr &= 1, & l'p+m'q+n'r &= 1, \\ lp'+mq'+nr' &= 1, & l'p'+m'q'+n'r' &= 1, \\ l^2+m^2+n^2 &= 1, & p^2+q^2+r^2 &= 1, \\ l'^2+m'^2+n'^2 &= 1, & p'^2+q'^2+r'^2 &= 1. \end{aligned}$$

With the aid of these vectors, the components of the electric and magnetic forces can be expressed in the forms

$$\begin{aligned} H_x &= a(mn'-m'n)+b(qr'-q'r) = -a'(l'-l)-b'(p'-p), \\ H_y &= a(nl'-n'l)+b(rp'-r'p) = -a'(m'-m)-b'(q'-q), \\ H_z &= a(lm'-l'm)+b(pq'-p'q) = -a'(n'-n)-b'(r'-r), \end{aligned}$$

$$\begin{aligned} E_x &= a(l'-l)+b(p'-p) = a'(mn'-m'n)+b'(qr'-q'r), \\ E_y &= a(m'-m)+b(q'-q) = a'(nl'-n'l)+b'(rp'-r'p), \\ E_z &= a(n'-n)+b(r'-r) = a'(lm'-l'm)+b'(pq'-p'q). \end{aligned}$$

If $\qquad ll'+mm'+nn' = \cos\theta, \qquad pp'+qq'+rr' = \cos\phi,$

we have

$$\begin{aligned} \mu_1 &= lH_x+mH_y+nH_z = a'(1-\cos\theta) = -(l'H_x+m'H_y+n'H_z) \\ &= ib(1-\cos\phi), \\ \mu_2 &= pH_x+qH_y+rH_z = b'(1-\cos\phi) = -(p'H_x+q'H_y+r'H_z) \\ &= ia(1-\cos\theta), \\ \lambda_1 &= lE_x+mE_y+nE_z = -a(1-\cos\theta) = -(l'E_x+m'E_y+n'E_z) \\ &= ib'(1-\cos\phi), \\ \lambda_2 &= pE_x+qE_y+rE_z = -b(1-\cos\phi) = -(p'E_x+q'E_y+r'E_z) \\ &= ia'(1-\cos\theta), \end{aligned}$$

$$\begin{vmatrix} p & q & r \\ l & m & n \\ l' & m' & n' \end{vmatrix} = i(1-\cos\theta), \qquad \begin{vmatrix} p' & q' & r' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = -i(1-\cos\theta),$$

$$\begin{vmatrix} l & m & n \\ p & q & r \\ p' & q' & r' \end{vmatrix} = i(1-\cos\phi), \qquad \begin{vmatrix} l' & m' & n' \\ p & q & r \\ p' & q' & r' \end{vmatrix} = -i(1-\cos\phi).$$

A case of considerable interest occurs when four functions (u, v), (w, s) and a constant ρ can be found such that

$$\frac{l}{\dfrac{\partial u}{\partial x}} = \frac{m}{\dfrac{\partial u}{\partial y}} = \frac{n}{\dfrac{\partial u}{\partial z}} = -\frac{1}{\dfrac{\partial u}{\partial t}},$$

$$\frac{l'}{\dfrac{\partial v}{\partial x}} = \frac{m'}{\dfrac{\partial v}{\partial y}} = \frac{n'}{\dfrac{\partial v}{\partial z}} = -\frac{1}{\dfrac{\partial v}{\partial t}},$$

$$\frac{p}{\dfrac{\partial w}{\partial x}} = \frac{q}{\dfrac{\partial w}{\partial y}} = \frac{r}{\dfrac{\partial w}{\partial z}} = -\frac{1}{\dfrac{\partial w}{\partial t}},$$

$$\frac{p'}{\dfrac{\partial s}{\partial x}} = \frac{q'}{\dfrac{\partial s}{\partial y}} = \frac{r'}{\dfrac{\partial s}{\partial z}} = -\frac{1}{\dfrac{\partial s}{\partial t}},$$

$$a = \rho f(u, v)\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}, \qquad b = g(w, s)\frac{\partial w}{\partial t}\frac{\partial s}{\partial t},$$

$$a' = \pm f(u, v)\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}, \qquad b' = \mp \rho g(w, s)\frac{\partial w}{\partial t}\frac{\partial s}{\partial t},$$

the signs in the last two equations being chosen so that the equation

$$ab = -a'b'$$

is satisfied.

The components of the electric and magnetic forces are then given by the equations

$$H_x = \rho f(u, v)\frac{\partial(u, v)}{\partial(y, z)} + g(w, s)\frac{\partial(w, s)}{\partial(y, z)} = \mp f(u, v)\frac{\partial(u, v)}{\partial(x, t)} \pm \rho g(w, s)\frac{\partial(w, s)}{\partial(x, t)},$$

$$H_y = \rho f(u, v)\frac{\partial(u, v)}{\partial(z, x)} + g(w, s)\frac{\partial(w, s)}{\partial(z, x)} = \mp f(u, v)\frac{\partial(u, v)}{\partial(y, t)} \pm \rho g(w, s)\frac{\partial(w, s)}{\partial(y, t)},$$

$$H_z = \rho f(u, v)\frac{\partial(u, v)}{\partial(x, y)} + g(w, s)\frac{\partial(w, s)}{\partial(x, y)} = \mp f(u, v)\frac{\partial(u, v)}{\partial(z, t)} \pm \rho g(w, s)\frac{\partial(w, s)}{\partial(z, t)},$$

$$E_x = \rho f(u, v)\frac{\partial(u, v)}{\partial(x, t)} + g(w, s)\frac{\partial(w, s)}{\partial(x, t)} = \pm f(u, v)\frac{\partial(u, v)}{\partial(y, z)} \mp \rho g(w, s)\frac{\partial(w, s)}{\partial(y, z)},$$

$$E_y = \rho f(u, v)\frac{\partial(u, v)}{\partial(y, t)} + g(w, s)\frac{\partial(w, s)}{\partial(y, t)} = \pm f(u, v)\frac{\partial(u, v)}{\partial(z, x)} \mp \rho g(w, s)\frac{\partial(w, s)}{\partial(z, x)},$$

$$E_z = \rho f(u, v)\frac{\partial(u, v)}{\partial(z, t)} + g(w, s)\frac{\partial(w, s)}{\partial(z, t)} = \pm f(u, v)\frac{\partial(u, v)}{\partial(x, y)} \mp \rho g(w, s)\frac{\partial(w, s)}{\partial(x, y)}.$$

With these expressions for $(E_x, E_y, E_z, H_x, H_y, H_z)$, the fundamental equations

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial H_x}{\partial t}, \qquad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial H_z}{\partial t},$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial H_y}{\partial t}, \qquad \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0,$$

$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=\frac{\partial E_x}{\partial t}, \qquad \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\frac{\partial E_z}{\partial t},$$

$$\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\frac{\partial H_y}{\partial t}, \qquad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0,$$

are satisfied identically, and

$$E_xH_x+E_yH_y+E_zH_z=2\rho\frac{\partial(u, v, w, s)}{\partial(x, y, z, t)}f(u,v)\,g(w,s),$$

$$\begin{aligned}E_x^2+E_y^2+E_z^2&\\ =\pm f(u,v)\,g(w,s)\Bigg[&\frac{\partial(u,v)}{\partial(y,z)}\frac{\partial(w,s)}{\partial(x,t)}+\frac{\partial(u,v)}{\partial(z,x)}\frac{\partial(w,s)}{\partial(y,t)}+\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(w,s)}{\partial(z,t)}\\ &-\rho^2\left\{\frac{\partial(u,v)}{\partial(x,t)}\frac{\partial(w,s)}{\partial(y,z)}+\frac{\partial(u,v)}{\partial(y,t)}\frac{\partial(w,s)}{\partial(z,x)}+\frac{\partial(u,v)}{\partial(z,t)}\frac{\partial(w,s)}{\partial(x,y)}\right\}\Bigg],\end{aligned}$$

$$\begin{aligned}H_x^2+H_y^2+H_z^2&\\ =\mp f(u,v)\,g(w,s)\Bigg[&\frac{\partial(u,v)}{\partial(x,t)}\frac{\partial(w,s)}{\partial(y,z)}+\frac{\partial(u,v)}{\partial(y,t)}\frac{\partial(w,s)}{\partial(z,x)}+\frac{\partial(u,v)}{\partial(z,t)}\frac{\partial(w,s)}{\partial(x,y)}\\ &-\rho^2\left\{\frac{\partial(u,v)}{\partial(x,t)}\frac{\partial(w,s)}{\partial(y,z)}+\frac{\partial(u,v)}{\partial(y,t)}\frac{\partial(w,s)}{\partial(z,x)}+\frac{\partial(u,v)}{\partial(z,t)}\frac{\partial(w,s)}{\partial(x,y)}\right\}\Bigg].\end{aligned}$$

Thus $E_x^2+E_y^2+E_z^2-H_x^2-H_y^2-H_z^2=\pm f(u,v)\,g(w,s)(1-\rho^2)\dfrac{\partial(u, v, w, s)}{\partial(x, y, z, t)}$.

The expressions given above are thus only suitable for the representation of an electromagnetic field in which

$$\frac{(eh)}{e^2-h^2}=\pm\frac{2\rho}{1-\rho^2}$$

is a constant quantity. When this condition is satisfied we have a representation of the field in terms of four functions u, v, w, s satisfying the equations

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial u}{\partial t}\right)^2, \qquad \left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2=\left(\frac{\partial w}{\partial t}\right)^2,$$

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2=\left(\frac{\partial v}{\partial t}\right)^2, \qquad \left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2+\left(\frac{\partial s}{\partial z}\right)^2=\left(\frac{\partial s}{\partial t}\right)^2,$$

$$\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}=\frac{\partial u}{\partial t}\frac{\partial w}{\partial t}, \qquad \frac{\partial v}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}=\frac{\partial v}{\partial t}\frac{\partial w}{\partial t},$$

$$\frac{\partial u}{\partial x}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial s}{\partial z}=\frac{\partial u}{\partial t}\frac{\partial s}{\partial t}, \qquad \frac{\partial v}{\partial x}\frac{\partial s}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial s}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial s}{\partial z}=\frac{\partial v}{\partial t}\frac{\partial s}{\partial t}.$$

[*Added August* 21*st.*—A set of functions satisfying these equations may be obtained by constructing a relation of the form

$$dx^2+dy^2+dz^2-dt^2 = A\,du\,dv+B\,dw\,ds,$$

where A and B are functions of u, v, w, s.]

4. We shall now consider the result of supposing that when electromagnetic waves are reflected at a fixed surface, the four vectors (l, m, n), (l', m', n'), (p, q, r), (p', q', r') obey the ordinary geometrical laws of reflection.

Let (λ, μ, ν) be the direction cosines of the normal at a point (x, y, z) of the reflecting surface; then if (L, M, N), (L', M', N'), (P, Q, R), (P', Q', R') are the reflected vectors*

$$\begin{array}{ll}
-l+L = 2\lambda\cos\alpha, & -l'+L' = 2\lambda\cos\alpha', \\
-m+M = 2\mu\cos\alpha, & -m'+M' = 2\mu\cos\alpha', \\
-n+N = 2\nu\cos\alpha, & -n'+N' = 2\nu\cos\alpha', \\
-p+P = 2\lambda\cos\beta, & -p'+P' = 2\lambda\cos\beta', \\
-q+Q = 2\mu\cos\beta, & -q'+Q' = 2\mu\cos\beta', \\
-r+R = 2\nu\cos\beta, & -r'+R' = 2\nu\cos\beta',
\end{array}$$

where
$$\begin{aligned}
l\lambda+m\mu+n\nu &= -\cos\alpha = -(L\lambda+M\mu+N\nu), \\
l'\lambda+m'\mu+n'\nu &= -\cos\alpha' = -(L'\lambda+M'\mu+N'\nu), \\
p\lambda+q\mu+r\nu &= -\cos\beta = -(P\lambda+Q\mu+R\nu), \\
p'\lambda+q'\mu+r'\nu &= -\cos\beta' = -(P'\lambda+Q'\mu+R'\nu).
\end{aligned}$$

These relations give

$$\begin{aligned}
PL+QM+RN &= (p+2\lambda\cos\beta)(l+2\lambda\cos\alpha)+(q+2\mu\cos\beta)(m+2\mu\cos\alpha) \\
&\qquad +(r+2\nu\cos\beta)(n+2\nu\cos\alpha) \\
&= pl+qm+rn+2\cos\beta\,(\lambda l+\mu m+\nu n) \\
&\qquad +2\cos\alpha\,(\lambda p+\mu q+\nu r)+4\cos\alpha\cos\beta \\
&= pl+qm+rn.
\end{aligned}$$

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* Cf. Herman's *Optics*, p. 4.

Hence the four relations of the type

$$PL+QM+RN = 0$$

are a consequence of the four corresponding relations of type

$$pl+qm+rn = 0.$$

If now we assume that the electric and magnetic forces in the reflected field are given by

$$\left.\begin{aligned} H'_x &= -a(MN'-M'N)-b(QR'-Q'R) \\ H'_y &= -a(NL'-N'L)-b(RP'-R'P) \\ H'_z &= -a(LM'-L'M)-b(PQ'-P'Q) \\ E'_x &= -a(L'-L)-b(P'-P) \\ E'_y &= -a(M'-M)-b(Q'-Q) \\ E'_z &= -a(N'-N)-b(R'-R) \end{aligned}\right\} . \tag{12}$$

Then, since $L'-L = l'-l+2\lambda(\cos\alpha'-\cos\alpha)$,

$$\begin{aligned} MN'-M'N &= (m+2\mu\cos\alpha)(n'+2\nu\cos\alpha')-(m'+2\mu\cos\alpha')(n+2\nu\cos\alpha) \\ &= mn'-m'n+2\cos\alpha(\mu n'-\nu m')+2\cos\alpha'(m\nu-n\mu), \end{aligned}$$

it follows that the electric and magnetic forces (e_x, e_y, e_z), (h_x, h_y, h_z) in the superposed fields are given by

$$\begin{aligned} h_x &= -2a\cos\alpha(\mu n'-\nu m')-2a\cos\alpha'(m\nu-n\mu)-2b\cos\beta(\mu r'-\nu q') \\ &\quad +2b\cos\beta'(\mu r-\nu q), \ \ldots, \\ e_x &= -2a\lambda(\cos\alpha'-\cos\alpha)-2b\lambda(\cos\beta'-\cos\beta), \ \ldots, \end{aligned}$$

and so satisfy the conditions

$$\frac{e_x}{\lambda} = \frac{e_y}{\mu} = \frac{e_z}{\nu},$$

$$\lambda h_x+\mu h_y+\nu h_z = 0.$$

That is, the electric force is along the normal and the magnetic force perpendicular thereto.

Further, if $(e'_x, e'_y, e'_z, h'_x, h'_y, h'_z)$ denote the components of the electric and magnetic forces when the two fields are subtracted from one another, we have

$$\begin{aligned} h'_x = {} & 2a(mn'-m'n)+2a\cos\alpha(\mu n'-\nu m')+2a\cos\alpha'(m\nu-n\mu) \\ & + 2b(qr'-q'r)+2b\cos\beta(\mu r'-\nu q')+2b\cos\beta'(q\nu-r\mu), \\ e'_x = {} & 2a(l'-l)+2\lambda a(\cos\alpha'-\cos\alpha)+2\mu b(\cos\beta'-\cos\beta)+2b(p'-p). \end{aligned}$$

Now $mn'-m'n+\cos a(\mu n'-\nu m')+\cos a'(m\nu-n\mu)$

$$\begin{aligned} &= (\lambda^2+\mu^2+\nu^2)(mn'-m'n)-(l\lambda+m\mu+n\nu)(\mu n'-\nu m') \\ &\qquad -(l'\lambda+m'\mu+n'\nu)(m\nu-n\mu) \\ &= \lambda\begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n' \end{vmatrix}, \end{aligned}$$

and $l'-l+\lambda(\cos a'-\cos a)$

$$\begin{aligned} &= (l'-l)(\lambda^2+\mu^2+\nu^2)+\lambda(l\lambda+m\mu+n\nu)-\lambda(l'\lambda+m'\mu+n'\nu) \\ &= \mu(\mu l'-\lambda m')-\mu(\mu l-\lambda m)+\nu(\nu l'-\lambda n')-\nu(\nu l-\lambda n). \end{aligned}$$

Hence $h'_x = 2a\lambda\begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n' \end{vmatrix} + 2b\lambda\begin{vmatrix} \lambda & \mu & \nu \\ p & q & r \\ p' & q' & r' \end{vmatrix},$

$$\begin{aligned} e'_x = {} & 2a\left[\mu\{\mu(l'-l)-\lambda(m'-m)\}-\nu\{\lambda(n'-n)-\nu(l'-l)\}\right] \\ & +2b\left[\mu\{\mu(p'-p)-\lambda(q'-q)\}-\nu\{\lambda(r'-r)-\nu(p'-p)\}\right], \end{aligned}$$

and so
$$\frac{h'_x}{\lambda} = \frac{h'_y}{\mu} = \frac{h'_z}{\nu},$$

$$\lambda e'_x+\mu e'_y+\nu e'_z = 0.$$

Thus in the field obtained by subtracting the incident and reflected fields the magnetic force is along the normal and the electric force perpendicular thereto.

This result may also be deduced from a second set of expressions for the electric and magnetic forces in the reflected field, viz.,

$$\left.\begin{aligned} H'_x &= -a'(L'-L)-b'(P'-P) \\ H'_y &= -a'(M'-M)-b'(Q'-Q) \\ H'_z &= -a'(N'-N)-b'(R'-R) \\ E'_x &= a'(MN'-M'N)+b'(QR'-Q'R) \\ E'_y &= a'(NL'-N'L)+b'(RP'-R'P) \\ E'_z &= a'(LM'-L'M)+b'(PQ'-P'Q) \end{aligned}\right\}. \qquad (13)$$

The relations between the electric and magnetic forces in the incident and reflected fields may be derived from the equations

$$\left.\begin{array}{ll} e_x = E_x+E'_x, & e'_x = E'_x-E_x \\ \dots\ \dots\ \dots & \dots\ \dots\ \dots \\ h_x = H_x+H'_x, & h'_x = H'_x-H_x \\ \dots\ \dots\ \dots & \dots\ \dots\ \dots \\ \dfrac{e_x}{\lambda} = \dfrac{e_y}{\mu} = \dfrac{e_z}{\nu}, & \lambda h_x+\mu h_y+\nu h_z = 0 \\ \dfrac{h'_x}{\lambda} = \dfrac{h'_y}{\mu} = \dfrac{h'_z}{\nu}, & \lambda e'_x+\mu e'_y+\nu e'_z = 0 \end{array}\right\}. \qquad \text{(A)}$$

The relation $$e_x e'_x+e_y e'_y+e_z e'_z = 0$$

gives $$E_x^2+E_y^2+E_z^2 = E_x'^2+E_y'^2+E_z'^2.$$

Hence at the surface of the conductor the electric energy in the incident field is equal to the electric energy in the reflected field.[*]

In the same way the relation

$$h_x h'_x+h_y h'_y+h_z h'_z = 0$$

leads to the equation

$$H_x^2+H_y^2+H_z^2 = H_x'^2+H_y'^2+H_z'^2.$$

Hence the magnetic energies in the incident and reflected fields are also equal.

Finally we have the equation

$$E_x H_x+E_y H_y+E_z H_z = -(E'_x H'_x+E'_y H'_y+E'_z H'_z),$$

which shows that, if the electric and magnetic forces in the incident field are perpendicular, they are also perpendicular in the reflected field.

The results that have been obtained may be expressed in another form as follows.

Let (dx, dy, dz, dt), $(\delta x, \delta y, \delta z, \delta t)$ denote two arbitrary displacements of a space time point on the conducting surface. These displacements are subject to the sole conditions

$$\lambda\, dx+\mu\, dy+\nu\, dz = 0, \qquad \lambda\,\delta x+\mu\,\delta y+\nu\,\delta z = 0.$$

Writing $d(y, z)$ for $dy\,\delta z-dz\,\delta y$, $d(x, t)$ for $dx\,\delta t-dt\,\delta x$, ..., we have

$$\frac{d(y, z)}{\lambda} = \frac{d(z, x)}{\mu} = \frac{d(x, y)}{\nu},$$

$$\lambda d(x, t)+\mu d(y, t)+\nu d(z, t) = 0.$$

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[*] The conductor must probably extend to infinity and be devoid of singularities, so that there is no diffraction or scattering of the disturbance; if the conductor is a closed surface there are geometrical difficulties which make the present method inapplicable when the disturbance is outside the conductor.

The equations
$$\frac{e_x}{\lambda} = \frac{e_y}{\mu} = \frac{e_z}{\nu}, \qquad \lambda h_x + \mu h_y + \nu h_z = 0,$$
$$\frac{h'_x}{\lambda} = \frac{h'_y}{\mu} = \frac{h'_z}{\nu}, \qquad \lambda e'_x + \mu e'_y + \nu e'_z = 0,$$
may consequently be expressed in the form
$$h_x d(y, z) + h_y d(z, x) + h_z d(x, y) + e_x d(x, t) + e_y d(y, t) + e_z d(z, t) = 0,$$
$$e'_x d(y, z) + e'_y d(z, x) + e'_z d(x, y) - h'_x d(x, t) - h'_y d(y, t) - h'_z d(z, t) = 0.$$

And if the displacements on the surface are arbitrary these equations are equivalent to the preceding ones.

Now the equations may be written in the form
$$\begin{aligned} &H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + E_x d(x, t) + E_y d(y, t) + E_z d(z, t) \\ &\quad = -[H'_x d(y, z) + H'_y d(z, x) + H'_z d(x, y) + E'_x d(x, t) + E_y d(y, t) + E'_z d(z, t)]. \end{aligned} \tag{B}$$
$$\begin{aligned} &E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - H_x d(x, t) - H_y d(y, t) - H_z d(z, t) \\ &\quad = +[E_x d(y, z) + E'_y d(z, x) + E'_z d(x, y) - H'_x d(x, t) - H'_y d(y, t) - H'_z d(z, t)], \end{aligned} \tag{C}$$
from which we may deduce at once that if the equations
$$\iint [H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + E_x d(x, t) + E_y d(y, t) + E_z d(z, t)] = 0, \tag{I}$$
$$\iint [E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - H_x d(x, t) - H_y d(y, t) - H_z d(z, t)] = 0, \tag{II}$$
are satisfied for any space time region of two dimensions associated with the surface of the conductor, then the equations
$$\iint [H'_x d(y, z) + H'_y d(z, x) + H'_z d(x, y) + E'_x d(x, t) + E'_y d(y, t) + E'_z d(z, t)] = 0, \tag{I$'$}$$
$$\iint [E'_x d(y, z) + E'_y d(z, x) + E'_z d(x, y) - H'_x d(x, t) - H'_y d(y, t) - H'_z d(z, t)] = 0, \tag{II$'$}$$
are also satisfied for the same region of integration.

The equations (I) and (II) are satisfied in any electromagnetic field in free æther (*i.e.*, when $\rho = 0$), for they are direct consequences of Maxwell's equations.* Moreover the equations (I)′ and (II)′ must *necessarily* be satisfied if the quantities (E'_x, E'_y, E'_z), (H'_x, H'_y, H'_z) are to be the values on the surface of the conductor of the electric and magnetic forces in the reflected field.

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* See my paper on "The Transformation of the Electrodynamical Equations," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 223 (1910).

It has been shown then that, with our hypothesis, with regard to reflection, the calculated values of the electric and magnetic forces in the reflected field *at the surface of the conductor* satisfy the necessary conditions. We still require an existence theorem which shall enable us to assert that when the surface values (E'_x, E'_y, E'_z), (H'_x, H'_y, H'_z) satisfy these necessary conditions, there is an electromagnetic field suitable for the representation of a reflected field, in which the electric and magnetic forces have these given surface values.

This existence theorem has yet to be proved. All we can do at present is to give examples to show that the conditions considered here are actually fulfilled in special cases.

In a former paper I worked out the solution of the reflection of plane waves at a paraboloid of revolution, when the waves are propagated in a direction parallel to the axis. In this case it is easily verified, from the formulæ on p. 487, that the equations (A), (B) and (C) are all satisfied.

Again, in the particular case when the incident electromagnetic field can be expressed in the form

$$H_x = \rho f(u, v)\frac{\partial(u, v)}{\partial(y, z)} + g(w, s)\frac{\partial(w, s)}{\partial(y, z)} = +f(u, v)\frac{\partial(u, v)}{\partial(x, t)} - \rho g(w, s)\frac{\partial(w, s)}{\partial(x, t)},$$

$$E_x = \rho f(u, v)\frac{\partial(u, v)}{\partial(x, t)} + g(w, s)\frac{\partial(w, s)}{\partial(x, t)} = -f(u, v)\frac{\partial(u, v)}{\partial(y, z)} + \rho g(w, s)\frac{\partial(w, s)}{\partial(y, z)},$$

the conditions are all satisfied.

To see this we consider the expression

$$L\,dx + M\,dy + N\,dz - dt.$$

An arbitrary displacement from a point (x, y, z, t) on the surface of the conductor to a point $(x+dx, y+dy, z+dz, t+dt)$ not on the conductor, may be regarded as the sum of two displacements, one on the conductor, *i.e.*, subject to the condition

$$\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z = 0,$$

and a second one along a line

$$\frac{\delta x'}{L} = \frac{\delta y'}{M} = \frac{\delta z'}{N} = +\frac{\delta t'}{1}.$$

Putting $dx = \delta x + \delta x'$, $dy = \delta y + \delta y'$, $dz = \delta z + \delta z'$, $dt = \delta t + \delta t'$,

we have $L\,dx + M\,dy + N\,dz - dt$

$$= L\,\delta x + M\,\delta y + N\,\delta z - \delta t + L\,\delta x' + M\,\delta y' + N\,\delta z' - \delta t'$$

$$= (l + 2\lambda\cos a)\,\delta x + (m + 2\mu\cos a)\,\delta y + (n + 2\nu\cos a)\,\delta z - \delta t,$$

since $$L\,\delta x' + M\,\delta y' + N\,\delta z' - \delta t' = 0,$$

and $$L = l + 2\lambda\cos a, \ \ldots .$$

This equation reduces to

$$L\,dx + M\,dy + N\,dz - dt = l\,\delta x + m\,\delta y + n\,\delta z - \delta t.$$

Now since $$\frac{l}{\frac{\partial u}{\partial x}} = \frac{m}{\frac{\partial u}{\partial y}} = \frac{n}{\frac{\partial u}{\partial z}} = -\frac{1}{\frac{\partial u}{\partial t}},$$

the expression $l\delta x + m\delta y + n\delta z - \delta t$ is of the form $A\,\delta u$. Hence if we define a quantity U in such a way that it has a constant value along a line

$$\frac{\delta x'}{L} = \frac{\delta y'}{M} = \frac{\delta z'}{N} = \frac{\delta t'}{1},$$

and takes the value u at the surface of the conductor, then we shall have

$$L\,dx + M\,dy + N\,dz - dt = A\,dU,$$

and consequently $$\frac{L}{\frac{\partial U}{\partial x}} = \frac{M}{\frac{\partial U}{\partial y}} = \frac{N}{\frac{\partial U}{\partial z}} = -\frac{1}{\frac{\partial U}{\partial t}},$$

$$\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 = \left(\frac{\partial U}{\partial t}\right)^2.$$

The function U is not represented by the same analytical expression as the function u, although it satisfies the same differential equation and has the same values over the surface of the conductor. We know, however, that this is possible, for a similar phenomenon occurs in the case of Hamilton's equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = 1,$$

which is obtained by assuming

$$U = t + V(x, y, z).$$

Now let functions V, W, S be derived from v, w, s in the same way that

U was derived from u; then we have four new functions U, V, W, S satisfying the differential equations

$$\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2=\left(\frac{\partial U}{\partial t}\right)^2, \quad \left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+\left(\frac{\partial W}{\partial z}\right)^2=\left(\frac{\partial W}{\partial t}\right)^2,$$

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=\left(\frac{\partial V}{\partial t}\right)^2, \quad \left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=\left(\frac{\partial S}{\partial t}\right)^2,$$

$$\frac{\partial U}{\partial x}\frac{\partial W}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial W}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial W}{\partial z}=\frac{\partial U}{\partial t}\frac{\partial W}{\partial t},$$

$$\frac{\partial V}{\partial x}\frac{\partial W}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial W}{\partial z}=\frac{\partial V}{\partial t}\frac{\partial W}{\partial t},$$

$$\frac{\partial U}{\partial x}\frac{\partial S}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial S}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial S}{\partial z}=\frac{\partial U}{\partial t}\frac{\partial S}{\partial t},$$

$$\frac{\partial V}{\partial x}\frac{\partial S}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial S}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial S}{\partial z}=\frac{\partial V}{\partial t}\frac{\partial S}{\partial t},$$

and the expressions for the components of the electric and magnetic forces in the reflected field take the form

$$H'_x=-\rho f(U, V)\frac{\partial(U, V)}{\partial(y, z)}-g(W, S)\frac{\partial(W, S)}{\partial(y, z)}$$
$$=+f(U, V)\frac{\partial(U, V)}{\partial(x, t)}-\rho g(W, S)\frac{\partial(W, S)}{\partial(x, t)},$$

... ...

$$E'_x=-\rho f(U, V)\frac{\partial(U, V)}{\partial(x, t)}-g(W, S)\frac{\partial(W, S)}{\partial(x, t)}$$
$$=-f(U, V)\frac{\partial(U, V)}{\partial(y, z)}+\rho g(W, S)\frac{\partial(W, S)}{\partial(y, z)}.$$

It is easy to see that with these expressions, Maxwell's equations are satisfied identically. Also, the conditions at the surface of the mirror are satisfied, for at the surface of the conductor

$$U=u, \quad V=v, \quad W=w, \quad S=s$$

for all values of t, and so

$$\frac{\partial U}{\partial t}=\frac{\partial u}{\partial t}, \quad \frac{\partial V}{\partial t}=\frac{\partial v}{\partial t}, \quad \frac{\partial W}{\partial t}=\frac{\partial w}{\partial t}, \quad \frac{\partial S}{\partial t}=\frac{\partial s}{\partial t}$$

over the conductor. The expressions for E'_x, H'_x on the surface of the conductor then take the forms (12) and (13), and so the conditions are satisfied.

It would be unwise to assert that this is the solution of the problem of reflection in the general case, because there are certain conditions to be satisfied at infinity, and there is some doubt about the single-valuedness of the functions U, V, W, S at all space time points. It has been thought worth while, however, to give the analysis as far as it is at present complete; the points that remain to be settled will require considerable investigation, but from a physical point of view the solution appears to be right, and the mathematical analysis is sufficiently interesting to justify its being studied for its own sake.

[*Added August* 21*st*, 1911.—In the case of a moving perfect conductor the conditions to be satisfied by the components $(e_x, e_y, e_z, h_x, h_y, h_z)$ of the total electromagnetic field may be expressed in the form

$$h_x d(y, z) + h_y d(z, x) + h_z d(x, y) + e_x d(x, t) + e_y d(y, t) + e_z d(z, t) = 0,$$

where the displacements dx, δx, ... occurring in the forms $d(y, z)$, ... are such that a point on the conductor is displaced to a point on a consecutive position of the conductor, but are otherwise arbitrary.

One way of satisfying the conditions is to assume

$$h_x = f^2 \frac{\partial(\xi, \eta)}{\partial(y, z)} = g^2 \frac{\partial(\zeta, \tau)}{\partial(x, t)}, \qquad e_x = f^2 \frac{\partial(\xi, \eta)}{\partial(x, t)} = -g^2 \frac{\partial(\zeta, \tau)}{\partial(y, z)},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots \quad \cdots$$

where $dx^2+dy^2+dz^2-dt^2 = \lambda[f^2(\xi, \eta)\{d\xi^2+d\eta^2\}+g^2(\zeta, \tau)\{d\zeta^2-d\tau^2\}]$,

and λ, ξ, η, ζ, τ are functions of x, y, z, t such that the equation of the moving conductor can be expressed in the form

$$F(\xi, \eta) = 0.]$$

THE APPLICATION OF THE MATHEMATICAL THEORY OF RELATIVITY TO THE ELECTRON THEORY OF MATTER

By E. CUNNINGHAM.

[Received and Read February 9th, 1911.]

THE work of the late Hermann Minkowski on the theory of relativity has done much to stimulate the discussion of the electrodynamical relations of moving bodies from a phenomenological point of view. The cumulative effect of many fruitless efforts to obtain evidence of motion relative to an absolute æther has been to convince physicists that any results of such motion must be obscured by the limitations within which an observer is placed, in regard to the means by which phenomena are investigated, by the fact that his measures of space and time depend on material objects whose constitution appears to be largely electromagnetic.

Starting therefore with the aim of obtaining equations which shall comprehend known phenomena, including that of complete relativity for electromagnetic effects, without regard to any constitutive theory of matter, Minkowski arrives at the following scheme:

$$\operatorname{curl} H' - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{1}{c}(J + \mathrm{P}W),$$

$$\operatorname{curl} E + \frac{1}{c}\frac{\partial B}{\partial t} = 0,$$

$$\operatorname{div} D = \mathrm{P},$$

$$\operatorname{div} B = 0.$$

This set of equations will be referred to as Minkowski's.

Lorentz, on the other hand, without any reference to relativity, derives from his constitutive electron theory of matter the scheme, less simple, but rather more easy to translate into experimental significance,

$$\operatorname{curl} H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{1}{c}(J + \mathrm{P}W - \operatorname{curl}[WP]),$$

$$\operatorname{curl} E + \frac{1}{c}\frac{\partial B}{\partial t} = 0,$$

$$\operatorname{div} D = \mathrm{P},$$

$$\operatorname{div} B = 0,$$

(where $P = D - E$), a scheme identical with that of Minkowski, provided that

$$H' = H + [WP]/c.$$

In Minkowski's scheme there is no mention of *polarization* and *magnetization*; E, B, D, H' are four vectors which define the electromagnetic state at any point, and the scheme has to be completed by the addition of equations connecting them. The polarization P and magnetization M' are defined by $P = D - E$, $M' = B - H'$.

Lorentz, on the other hand, makes E, B, P, M the fundamental vectors, H and D being auxiliary vectors defined by

$$D = E + P, \qquad H = B - M,$$

the *polarization* P and *magnetization* M being defined by a process of averaging from the supposed distribution and motion of electrons within the matter. The scheme is completed by the addition of relations connecting P and M with E and B.

The difference between the actual field equations, however, is entirely one of notation.

The transformations under which Minkowski shews his scheme to be invariant are as follows.

$$X = \beta(x - vt), \quad Y = y, \quad Z = z, \quad T = \beta(t - vx/c^2), \tag{1}$$

$$\beta = (1 - v^2/c^2)^{-\frac{1}{2}},$$

$$\left.\begin{aligned} E_X &= e_x, & E_Y &= \beta(e_y - vb_z/c), & E_Z &= \beta(e_z + vb_y/c) \\ B_X &= b_x, & B_Y &= \beta(b_y + ve_z/c), & B_Z &= \beta(b_z - ve_y/c) \end{aligned}\right\}, \tag{2}$$

$$\left.\begin{aligned} D_X &= d_x, & D_Y &= \beta(d_y - vh'_z/c), & D_Z &= \beta(d_z + vh'_y/c) \\ H'_X &= h'_x, & H'_Y &= \beta(h'_y + vd_z/c), & H'_Z &= \beta(h'_z - vd_y/c) \end{aligned}\right\}, \tag{3'}$$

together with equations (4), (5) given below, connecting J, P with j, ρ.

It follows that the scheme of Lorentz is invariant under the transformations (1), (2) and (3), where

$$H' = H + [WP]/c,$$

and

$$h' = h + [wp]/c,$$

the equations for the current and density being again equations (4) and (5) below.

It is known that transformations of the type (2) form a group, so that this last transformation can be obtained by the combination of two, the change in the velocity of the matter from w to W being effected by first changing it from w to 0, and subsequently from 0 to W. It will therefore be sufficient for the present purpose to consider only transformations in which $w = 0$, so that $W = -v$. This reduces the transformations to the form

$$D_X = d_x, \quad D_Y = \beta(d_y - vh_z/c), \quad D_Z = \beta(d_z + vh_y/c),$$

$$H_X = h_x, \quad H_Y + vP_z/c = \beta(h_y + vd_z/c), \quad H_Z - vP_y/c = \beta(h_z - vd_y/c).$$

Since, however, in the Lorentz scheme the vectors P and M are more fundamental than D and H, we replace these by other equations obtained by combination with (2):

$$P_X = p_x, \quad P_Y = \beta(p_y + vm_z/c), \quad P_Z = \beta(p_z - vm_y/c),$$

$$M_X = m_x, \quad M_Y - vP_z/c = \beta(m_y - vp_z/c), \quad M_Z + vP_y/c = \beta(m_z + vp_y/c),$$

from which we obtain

$$\left.\begin{array}{lll} P_X = p_x, & P_Y = \beta(p_y + vm_z/c), & P_Z = \beta(p_z - vm_y/c) \\ M_X = m_x, & M_Y = m_y/\beta, & M_Z = m_z/\beta \end{array}\right\}. \tag{3}$$

This then must be the form of the relations connecting the *polarization* and *magnetization* $[(p, m)(P, M)]$ vectors as used by Lorentz when the point considered is at rest, and in motion with velocity $-v$, respectively. For $w = 0$, the equations for the current and density are

$$P = \beta(\rho - vj_x/c^2), \tag{4}$$

and

$$J_X = j_x/\beta, \quad J_Y = j_y, \quad J_z = j_z. \tag{5}$$

So far the divergence between Minkowski and Lorentz is one of notation only; but it is necessary now to consider a little more closely the point of view from which these writers regard the equations.

To Minkowski the field equations are at one time laws and definitions.

It is supposed that vectors exist which satisfy those equations, and that being so, these vectors are named ("elektrische oder magnetische, kraft oder erregung"). With Lorentz, however, the field equations as given above follow as deductions from prior assumptions and definitions, from the fundamental equations of the electron theory, and from a constitutive theory of matter.*

Into the bases of these assumptions it is not proposed here to enquire. The immediate purpose of the present discussion is to bring the equations (3), (4), (5), which have been seen to be necessary to preserve the invariance of the equations for material bodies, into relation with Lorentz's derivation of those equations by means of his constitutive theory.

The presentation of the constitutive theory, which is here adopted, is that which treats of electricity as distributed in discrete point charges.†

The first effort in this direction was made by Frank,‡ and a more complete investigation was outlined by the present writer shortly after,§ in which, however, an error was overlooked, the discovery of which led to the present paper. Later, the posthumous work of Minkowski, edited by Born,|| made it clear that the agreement does exist, though the results are expressed not in terms of an atomic theory of electricity, but of a continuous fluid theory. The atomic theory seems, however, to be not only more fundamental, but in the present instance more simple. The Born-Minkowski investigation defines the momentary state of a system by means of an infinite series of states which are neither the actual nor the equilibrium state, and which may be chosen in an infinite number of ways.

The purpose of the present paper is therefore to complete and correct the outline given in the author's earlier paper, and in so doing to give a treatment which is simpler in form and which keeps closer to the physical concepts upon which it is based than does that of Born and Minkowski.

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* Lorentz, "Elektronentheorie," *Enzyk. der Math. Wiss.*, Vol. 14.

† See Abraham, *Theorie der Elektrizität*, Vol. II, pp. 238 *ff.*

From this point of view, of course, the fundamental equations of the electron theory of Lorentz must themselves be looked upon as relations between mean values of the quantities concerned. Cf. Larmor, *Æther and Matter*, §§ 51 *ff.*

On the necessity of this point of view, see Pearson, *Grammar of Science*, 3rd edition, 1911, chap. x, § 8.

‡ *Ann. der Phys.*, Vol. 27 (1908), p. 1059.

§ *Proc. London Math. Soc.*, Ser. 2, Vol. 8, February, 1909. The present paper may be taken as replacing § 2 of that investigation.

|| *Math. Ann.*, Bd. 68, 1910 p. 526.

1. *The Transformation of the Density of Electricity and Current.*

If a group of electrons lie within a small element of volume δa at any instant, it contributes to the averaged density at a point of the body an amount $\Delta\rho = \Sigma e/\delta a$.

Now, if δa be a moving element of volume having velocity $(w_x w_y w_z)$, the geometrical transformation (1) transforms it into an element of volume δA given by

$$\delta A = \delta a/\beta(1-vw_x/c^2). \tag{6}$$

Since the magnitude of the charge is unaltered in the transformation, the corresponding contribution to the density in the transformed system is

$$\Delta \mathrm{P} = \Sigma e/\delta A = \beta\Sigma e(1-vw_x/c^2)/\delta a = \beta\left\{\Sigma e/\delta a-(v/c^2)\,\Sigma ew_x/\delta a\right\},$$

where it is supposed that only such electrons are considered as have the same velocity as the element of volume in question. If this were not so, δA would not at one and the same time T contain all and only those electrons which at the corresponding instant t lay within δa.

This restriction, however, may soon be removed, for, since the contribution Δj of the same group of electrons to the current intensity in the averaged system is $\Sigma ew/\delta a$, we have the equation

$$\Delta \mathrm{P} = \beta(\Delta\rho - v\Delta j_x/c^2),$$

in which form we may add together the contributions arising from groups of electrons having different velocities.

If the electrons be now divided into *free* and *bound* varieties, and we denote the contributions from the latter by the suffix 1 and the total by a dash, the contributions from the free electrons being denoted by unmarked letters, these being those which figure in the fundamental equations

$$\left.\begin{aligned} \mathrm{P} &= \beta(\rho - vj_x/c^2), && (4)\\ \mathrm{P}_1 &= \beta(\rho_1 - vj_{1x}/c^2), && (4_1)\\ \mathrm{P}' &= \beta(\rho' - vj'_x/c^2) && (4') \end{aligned}\right\}$$

[cf. equation (4)].

Similarly, since

$$W_X = \frac{w_x - v}{1-vw_x/c^2}, \quad W_Y = \frac{w_y}{\beta(1-vw_x/c^2)}, \quad W_Z = \frac{w_z}{\beta(1-vw_x/c^2)},$$

and since (W_x+v) is the velocity of the electron relative to the matter,

$$\begin{aligned} \Delta J_X &= \Sigma e(W_x+v)/\delta A \\ &= \frac{\Sigma e(1-vw_x/c^2)}{\delta a}\,\frac{w_x}{\beta(1-vw_x/c^2)} = \Sigma ew_x/\beta\delta a \\ &= \Delta j_x/\beta, \\ \Delta J_Y &= \Sigma eW_y/\delta A = \Sigma ew_y/\delta a = \Delta j_y, \\ \Delta J_Z &= \Delta j_z, \end{aligned}$$

giving, on addition, equations (5) above, viz.,

$$J_X = j_x/\beta, \qquad J_Y = j_y, \qquad J_Z = j_z, \tag{5}$$

which may apply either to free or bound electrons or both together. Applied to the free electrons, j includes both the *conduction* current and the *convection* current. Applied to the bound electrons these equations will be called (5_1), and to the total $(5')$.

Now the fundamental theorem of the theory of relativity is that the space-time transformation (1), together with (2), (4′) and (5′), leaves the equations

$$\left.\begin{aligned} \frac{1}{c}\left(\frac{\partial e}{\partial t}+j'\right) &= \operatorname{curl} b, \\ -\frac{1}{c}\,\frac{\partial b}{\partial t} &= \operatorname{curl} e, \\ \operatorname{div} e &= \rho', \\ \operatorname{div} b &= 0 \end{aligned}\right\} \tag{7}$$

invariant.

Hence (substituting p, m for e, b), if p, m are vectors such that

$$\left.\begin{aligned} \frac{1}{c}\left(\frac{\partial p}{\partial t}-j_1\right) &= -\operatorname{curl} m, \\ \operatorname{div} p &= -\rho_1 \end{aligned}\right\} \tag{8}$$

it follows from (4_1) and (5_1) that, if we define vectors P, M' such that

$$\left.\begin{aligned} P_X &= p_x, & P_Y &= \beta(p_y+vm_z/c), & P_Z &= \beta(p_z-vm_y/c); \\ M'_X &= m_x, & M'_Y &= \beta(m_y-vp_z/c), & M'_Z &= \beta(m_z+vp_y/c), \end{aligned}\right\} \tag{9}$$

then the equations

$$\left.\begin{aligned} \frac{1}{c}\left(\frac{\partial P}{\partial T}-J_1\right) &= -\operatorname{curl} M', \\ \operatorname{div} P &= -\mathrm{P}_1 \end{aligned}\right\} \tag{10}$$

are also satisfied.

Remembering that $M' = M-[WP]/c$ and that $W = -v$, equations (9) become identical with (3). By adding (7) and (8) we obtain Minkowski's modified form of the Lorentz field equations, and we see, whatever be the physical concept in terms of which we define p and m, that the equations (4) and (5), *obtained from the physical definition of current and density given by Lorentz*, together with the equations (9) which define the vectors P, M', are sufficient to ensure the invariance of those field equations.

Alternative Method of obtaining Equations (4) *and* (5).

The method used just above is in essence a process of averaging at a given instant over an element of volume, sufficiently large to contain a large number of electrons. Exactly the same results are obtained if we take a time-average for an infinitesimal element of volume over an interval of time sufficiently long to allow a large number of electrons to pass through it.

We consider an element of volume fixed in the framework of the material body, and denote its volume by δa or δA according as it is viewed in the original or transformed system of coordinates; the element being fixed in the body $w_x = 0$, so that equation (6) gives

$$\delta A = da/\beta.$$

Next we fix attention on an electron whose velocities are w and W in the two systems of coordinates respectively as it passes through this element of volume, which is supposed so small that w may be considered as constant during the passage.

If δt, δT are the times of passage in the two systems, we find that

$$\delta T = \beta\left(\delta t - \frac{v\,\delta x}{c^2}\right) = \beta\delta t(1 - vw_x/c^2).$$

The mean densities of electricity ρ, P in the element of volume during corresponding intervals θ, Θ would now be defined to be

$$\Sigma(e\,\delta t)/\theta\,\delta a \quad \text{and} \quad \Sigma(e\,\delta T)/\Theta\,\delta A,$$

the summations applying to all electrons which pass through the element during the respective intervals, and therefore to exactly the same electrons in the two cases.

Now $\Theta = \beta\theta$.* Thus

$$\mathrm{P} = \frac{\Sigma e\beta(1 - vw_x/c^2)\,\delta t}{\theta\,\delta a} = \beta\left\{\Sigma e\delta t/\theta\delta a - v\Sigma ew_x\delta t/c^2\theta\delta a\right\}.$$

The current intensity j arising from the same electrons averaged over the same interval is

$$\Sigma ew\,\delta t/\theta\,\delta a.$$

Hence

$$\mathrm{P} = \beta(\rho - vj_x/c^2).$$

---

* Θ and θ are the two estimates by the respective observers of the interval of time elapsing between two given occurrences (*e.g.*, the entries of two marked electrons into the element of volume). The equation

$$T = \beta\left(t - \frac{vx}{c^2}\right)$$

gives

$$\Theta = \beta\left(\theta - v\,\frac{x_2 - x_1}{c^2}\right),$$

x_2, x_1 being the coordinates of two points in an infinitesimal element of volume. $x_2 - x_1$ being neglected, this gives the result used.

In the same way the current intensity J in the transformed system of coordinates is given by

$$J_x = \frac{\Sigma e(W_x+v)\delta T}{\Theta\delta A} = \frac{1}{\theta\delta a}\,\Sigma\frac{ew_x}{\beta}\,\delta t = j_x/\beta,$$

$$J_y = \Sigma e W_y \delta T/\Theta\delta A = \Sigma e w_y \delta t/\theta\delta a = j_y,$$

and $$J_z = j_z,$$

Thus we have again arrived at equations (4) and (5), the results being clearly true for any particular group of electrons or for the whole.

2. *The Transformation of the Polarization and Magnetization.*

The physical meanings attached to the words *current* and *density* by Lorentz having been thus shewn to be in accord with the transformations required by the mathematical theory of relativity, it remains to consider what physical meaning may be attached to the vectors p, m, P, M' occurring above in order that the relation (9) may follow directly by geometrical transformation only.

We have to examine, as in the preceding section, whether the *polarization* and *magnetization* as defined by Lorentz will satisfy this requirement.

Consider, therefore, a single electron which is bound to move within narrow limits in the neighbourhood of a given point of the framework which we call the material body.* In the system of coordinates in which this point is at rest, let the coordinates and velocity of the electron at time t be (x, y, z), (w_x, w_y, w_z). Using the transformation

$$X = \beta(x-vt), \qquad T = \beta(t-vx/c^2),$$

we find that, at time $\tau = \beta(t-vx/c^2)$, the new coordinates of the point and electron are $-v\tau$ and $(x/\beta - v\tau)$, so that, denoting the displacement of the electron at time τ from the point of the body by (ξ, η, ζ), we have

$$\xi = x/\beta, \qquad \eta = y, \qquad \zeta = z.$$

In what follows we have, as in the concluding part of the previous section, to take time-averages over corresponding intervals of time in the two systems of coordinates. If δt is an interval of time during which the electron moves in the first system of coordinates through distances

$$\delta x\,(=w_x\delta t), \qquad \delta y\,(=w_y\delta t), \qquad \delta z\,(=w_z\delta t),$$

the corresponding interval of time is given by

$$\delta T = \beta\left(\delta t - \frac{v\,\delta x}{c^2}\right) = \beta\,\delta t\left(1-\frac{vw_x}{c^2}\right).$$

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* This framework, of course, has not necessarily any physical reality. It is for our purpose only a geometrical concept giving the mean motion of the constituent elements.

Let this result be applied to obtain the mean displacement $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ over an interval τ_1 corresponding to an interval t_1 in the stationary system

$$\begin{aligned}\int_{\tau_1} \xi\, dT &= \int_{t_1} x\left(1-\frac{vw_x}{c^2}\right)dt\\ &= \int_{t_1} x\,dt - \frac{v}{c^2}\int_{t_1} xw_x\,dt\\ &= \int_{t_1} x\,dt - \frac{v}{c^2}\int_{t_1} \frac{d}{dt}\left(\tfrac{1}{2}x^2\right)dt,\end{aligned}$$

$$\begin{aligned}\int_{\tau_1} \eta\, dT &= \int \beta y\left(1-\frac{vw_x}{c^2}\right)dt\\ &= \beta\int_{t_1} y\,dt - \frac{\beta v}{c^2}\int_{t_1} yw_x\,dt\\ &= \beta\int_{t_1} y\,dt - \frac{\beta v}{2c^2}\int_{t_1}(yw_x - xw_y)\,dt - \frac{\beta v}{2c^2}\int_{t_1}\frac{d}{dt}(xy)\,dt,\end{aligned}$$

with a similar expression for $\int_{\tau_1} \zeta\, dT$.

Supposing, now, that we introduce the supposition that the displacement (x, y, z) of the electron is confined within very narrow limits, we may take a *physically small** element of time t_1 which is yet so large that

$$\frac{1}{t_1}\int_{t_1}\frac{d}{dt}(x^2)\,dt \quad \text{and} \quad \frac{1}{t_1}\int_{t_1}\frac{d}{dt}(xy)\,dt$$

are negligible, the changes in x^2 and xy during t_1 having a very small upper limit which is independent of t_1.

The interval of time τ_1 corresponding to the interval t_1 is given by

$$\tau_1 = \beta\left(t_1 - \frac{v\,\delta x}{c^2}\right),$$

where δx is the total displacement of the electron during the interval, and, by the same hypothesis, the second term in this expression is negligible.

On division by the respective time intervals, we therefore get the following *time-mean* relations:

$$\begin{aligned}\bar{\xi} &= \frac{\bar{x}}{\beta},\\ \bar{\eta} &= \bar{y} - \frac{v}{2c^2}\,(\overline{yw_x - xw_y}),\\ \bar{\zeta} &= \bar{z} - \frac{v}{2c^2}\,(\overline{zw_x - xw_z}).\end{aligned}$$

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* Following Lorentz, it is supposed that the value of t_1 large enough to justify this step in the argument is yet a very small interval in our gross measure. See *Enzyk. der Math. Wiss.*, Vol. v*d*, p. 201.

Similar equations being obtained for all the electrons whose mean or undisturbed positions lie within a given element of the material body, corresponding terms may be added together. But, if δa is the element of volume when the matter is considered as stationary and δA when it is looked upon as moving, then, as above

$$\delta A = \delta a/\beta.$$

Hence, on division by δA, we obtain

$$\frac{\Sigma e\bar{\xi}}{\delta A} = \frac{\Sigma \overline{ex}}{\delta a},$$

$$\frac{\Sigma e\dot{\bar{\eta}}}{\delta A} = \beta\left\{\frac{\Sigma \overline{ey}}{\delta a} - \frac{v}{2c^2}\,\frac{\Sigma \overline{e(yw_x - xw_y)}}{\delta a}\right\},$$

$$\frac{\Sigma e\bar{\xi}}{\delta A} = \beta\left\{\frac{\Sigma \overline{ez}}{\delta a} - \frac{v}{2c^2}\,\frac{\Sigma \overline{e(zw_x - xw_z)}}{\delta a}\right\}.$$

If, therefore, we define the vectors $\mathfrak{p}$, $\mathfrak{m}$ by the equations

$$\mathfrak{p} = \Sigma e\bar{\mathfrak{r}}/\delta a,$$

$$\mathfrak{m} = \Sigma \overline{e[\mathfrak{r}w]}/2c\,\delta a,$$

where $\mathfrak{r}$ is the vector (x, y, z), and a vector $\mathfrak{P}$ by the equation

$$\mathfrak{P} = \Sigma e\bar{\mathfrak{R}}/\delta A.$$

we have

$$\mathfrak{P}_X = \mathfrak{p}_x,$$

$$\mathfrak{P}_Y = \beta(\mathfrak{p}_y + v\mathfrak{m}_z/c),$$

$$\mathfrak{P}_Z = \beta(\mathfrak{p}_z - v\mathfrak{m}_y/c),$$

equations which are identical with the first set of (3). But the vectors $\mathfrak{p}$, $\mathfrak{m}$ so defined are the polarization and magnetization intensities in the element δa according to the electron theory.*

In the same way we treat the magnetization $\mathfrak{M}$. If U_X, U_Y, U_Z are the transformed velocities of an electron *relative to the matter*,

$$\frac{\beta U_X}{w_x} = \frac{U_Y}{w_y} = \frac{U_Z}{w_z} = \frac{1}{\beta(1 - vw_x/c^2)},$$

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* Cf. Abraham, *Theorie der Elektriz.*, pp. 245 *ff.*

We may note that, though Abraham does not introduce the time-mean value in the original definition of the polarization, he introduces it in the derivation of the equations—p. 249, *loc. cit.*

It is also assumed by Lorentz in his treatment when he speaks of the distribution of charge remaining unaltered. *Enzyk. der Math. Wiss.*, Vol. v*d*, p. 181, § 15.

so that
$$\delta T(\eta U_Z-\zeta U_Y)=\delta t(yw_z-zw_y),$$
$$\delta T(\zeta U_X-\xi U_Z)=\delta t(zw_x-xw_z)/\beta,$$
$$\delta T(\xi U_Y-\eta U_X)=\delta t(xw_y-yw_x)/\beta,$$

Integrating and dividing by the time-intervals τ_1, t_1, adding for the electrons within a given element, and dividing by the volume-elements δA, δa, we obtain
$$\mathfrak{M}_X=\mathfrak{m}_x,$$
$$\mathfrak{M}_Y=\mathfrak{m}_y/\beta,$$
$$\mathfrak{M}_Z=\mathfrak{m}_z/\beta.$$
where
$$\mathfrak{M}=\Sigma e[\mathfrak{R}U]/2c\delta A.$$

The definitions given above have thus been shewn to lead to the equations (3).

If, therefore, we identify the vectors p, m, which were not physically defined at the outset, with the vectors $\mathfrak{p}$, $\mathfrak{m}$, which are the polarization and magnetization at a stationary element of matter according to the electronic concept, then P, M as they occur in equations (3) must also be identified with $\mathfrak{P}$, $\mathfrak{M}$, the corresponding quantities for an element of matter moving with velocity $-v$.

Thus it is possible, at the expense of sacrificing the simplicity of the Minkowski form of the equations and adhering to the original Lorentz equations, to maintain the theory of relativity and the electronic concept of matter; as, indeed, seems inevitably possible from the invariance of the primal equations. The vector M' introduced by Minkowski is not so simply defined in terms of the motions of the constituent electrons as the Lorentz vector M.

Note on the Constitutive Equations of Moving Media.

The transformations (2), (3), (4), (5) have an important bearing on the *constitutive* equations of a moving material medium, that is, on the equations connecting E, B, P, M, J which are characteristic of the particular medium, and which are required to complete the equations determining the course of electromagnetic changes in the medium.

Any relation which is assumed to hold between e, b, p, m, j at a stationary point of the medium may, by means of those transformations, be interpreted as a relation between the corresponding quantities at the same point when it is supposed to be in motion.

For example, taking the simplest common form of such relations,
$$p=(\epsilon-1)e, \qquad m=\left(1-\frac{1}{\mu}\right)b, \qquad j=\sigma e,$$

we obtain from (2) and (3) equations which in their simplest form are

$$P-[WM']/c = (\epsilon-1)(E-[WB]/c),^*$$

$$M = M'+[WP]/c = \left(1-\frac{1}{\mu}\right)(B+[WE]/c),^*$$

symmetry being attainable, as in the field equations, by the use of the vector

$$M' = M-[WP]/c\,;$$

and from (5) we have

$$J_X = \sigma E_X/\beta,$$

$$J_Y = \beta\sigma(E_Y+vB_Z/c),$$

$$J_Z = \beta\sigma(E_Z-vB_Y/c).$$

If the conditions in which the material medium is placed to an observer moving with it are such that the equations $p = (\epsilon-1)e$, ..., are not appropriate, then equally the derived equations do not hold, but must be replaced by others obtained in the same way from the constitutive equations suitable to the conditions in question. The important result is that the validity of the theorem of relativity is not dependent on any unique or particular form of the constitutive equations.

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[*] Cf. Abraham, *Theorie der Elek.*, Vol. II, § 50.

ON THE REDUCTION AND CLASSIFICATION OF BINARY CUBICS WHICH HAVE A NEGATIVE DISCRIMINANT

By G. B. Mathews.

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1. Let f' be a binary cubic, with real integral coefficients, and roots α', β', γ', of which the first is real and the others complex. Then, as proved in a former paper, f' is properly equivalent to one, and (with a certain convention) only one reduced form f, the roots of which satisfy the conditions

$$\beta\gamma-1 \geqslant 0,$$

$$\beta+\gamma-1 \leqslant 0 \leqslant \beta+\gamma+1.$$

It has also been shown that in any limiting case when one at least of the inequalities becomes an equality, the form f, and therefore f', has a rational linear factor. Consequently we consider only the strict inequalities

$$\beta\gamma-1 > 0, \quad \beta+\gamma-1 < 0, \quad \beta+\gamma+1 > 0.$$

Since β, γ are conjugate

$$(\alpha\beta-1)(\alpha\gamma-1) = |\,\alpha\beta-1\,|,$$

a positive quantity; so the first condition is equivalent to

$$\Pi(\beta\gamma-1) > 0,$$

where the product is a symmetrical function of the roots. Let

$$f = ax^3+bx^2y+cxy^2+dy^3 = (a,\, b,\, c,\, d)\,;$$

then the condition last written reduces to

$$C_1 = d(d-b)+a(c-a) > 0. \tag{1}$$

Similarly the other two conditions may be replaced by

$$\Pi(\beta+\gamma-1) < 0, \qquad \Pi(\beta+\gamma+1) > 0\,;$$

or, in terms of the coefficients,

$$C_2 = ad-(a+b)(a+b+c) < 0, \tag{2}$$

$$C_3 = ad+(a-b)(a-b+c) > 0. \tag{3}$$

These, then, are the necessary and sufficient conditions that (a, b, c, d) may be reduced.

As an example, take the form $(2, -12, 11, -21)$ obtained by a different criterion in a former paper; here

$$C_1 = -21(-9)+2.9 > 0,$$
$$C_2 = -21.2-(-10)1 < 0,$$
$$C_3 = -21.2+14.25 > 0,$$

and the form is reduced.

2. In order to reduce a given form, we proceed as follows. Let S, T denote the substitutions

$$\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}, \quad \begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix},$$

respectively. Then, if $C_1 < 0$, we apply T, and thus obtain a form with $C_1 > 0$. If now $C_2 > 0$, this means that $\beta+\gamma-1 > 0$, and the application of S will diminish C_2; on the other hand, if $C_3 < 0$, the application of S^{-1} will increase C_3, so there is always a substitution $S^{\pm h}$, which will make the new form satisfy $C_2 < 0$ and $C_3 > 0$. If in this form $C_1 > 0$, it is reduced; if not, we apply T and proceed as before. From the known theory of definite quadratics it follows that the process must end after a finite number of operations; and we shall then have obtained a reduced form properly equivalent to the given one.

For instance, suppose

$$f = (97, 471, 762, 411).$$

Here $C_1 > 0$, $C_2 < 0$, $C_3 < 0$, so we must apply S^{-1}; the result is

$$f_1 = (97, 180, 111, 23),$$

for which $C_1 < 0$; hence we apply T and find

$$f_2 = (23, -111, 180, -97).$$

Proceeding thus, we have successively

$$\begin{aligned} f_3 &= Sf_2 &&= (23, -42, 27, -5),\\ f_4 &= Tf_3 &&= (-5, -27, -42, -23),\\ f_5 &= S^{-1}f_4 &&= (-5, -12, -3, -3),\\ f_6 &= Tf_5 &&= (-3, 3, -12, 5),\end{aligned}$$

the last of which is reduced. Since $f(-x, -y) \sim f(x, y)$ we may replace it by $(3, -3, 12, -5)$, so as to make the first coefficient positive.

3. We now pass on to the classification of cubics for a given discriminant $-\Delta$. Since the discriminant of the Hessian is 3Δ, and this must be of the form $4n$ or $4n+1$, it follows that Δ must be of the form $4n$ or $4n+3$. Further, if $\Delta = 3n^2$, the Hessian breaks up into rational factors, and, as this case requires special treatment, it will be postponed for the present.

Before proceeding to the general theory, it will be well to consider a special case. Suppose, then, that $\Delta = 23$. Then the discriminant of the Hessian is 69, and this quadratic must belong to one of the classes represented by

$$(1,\ 1,\ -17),\quad (-1,\ 1,\ 17).$$

Now all the cubics which have a Hessian of the form $\mu(x^2+xy-17y^2)$, where μ is an integer, are given by

$$f = mx^3+nx^2y+(51m+n)\,xy^2+(17m+6n)\,y^3,$$

with a Hessian

$$h = (153m^2+3mn-n^2)(x^2+xy-17y^2)$$
$$= G(m,\ n)\,H(x,\ y),$$

say; so that all the associated cubics which have a discriminant -23 must be derivable from the integral solutions of

$$153m^2+3mn-n^2 = \pm 1.$$

Since -1 is not a quadratic residue of 3, the upper sign is inadmissible; taking the lower sign, we have solutions $(0,\ \pm 1)$, and all the others are given by putting

$$P = \begin{pmatrix} 11 & 1 \\ 153 & 14 \end{pmatrix},$$

$$P^h = \begin{pmatrix} \alpha_h & \beta_h \\ \gamma_h & \delta_h \end{pmatrix} \quad (h = \pm 1,\ \pm 2,\ \ldots),$$

and then taking $m = \pm\beta_h$, $n = \pm\delta_h$. In fact, P is the fundamental automorph of $G(m,\ n)$.

Now the primary automorph of $H(x,\ y)$ is given by

$$Q = \begin{pmatrix} 11 & 51 \\ 3 & 14 \end{pmatrix},$$

and if this be applied to f the result is found to be the same as if we apply the substitution P^3 to m and n. Hence there cannot be more than three classes of cubics, and it is sufficient to consider the cases

$$(m,\ n) = (0,\ 1),\ (1,\ 14),\ (1,\ -11),$$

the last of which is derived from P^{-1}. The first values give

$$f_1 = (x^2+xy+6y^2)\,y,$$

the second $$f_2 = x^3+14x^2y+65xy^2+101y^3,$$

and the third $$f_3 = x^3-11x^2y+40xy^2-49y^3.$$

The first cubic is reduced; with regard to the others we find that

$$(1,\ 14,\ 65,\ 101) \sim (1,\ 2,\ 1,\ 1) \sim (1,\ -1,\ 2,\ -1),$$

and $$\begin{aligned}(1,\ -11,\ 40,\ -49) &\sim (1,\ -2,\ 1,\ -1)\\ &\sim (-1,\ -1,\ -2,\ -1)\\ &\sim (1,\ 1,\ 2,\ 1).\end{aligned}$$

The result, then, is that, for $\Delta = 23$, there are three classes of cubics represented by the reduced forms

$$\begin{aligned}g_1 &= (x^2+xy+6y^2)\,y,\\ g_2 &= x^3-x^2y+2xy^2-y^3,\\ g_3 &= x^3+x^2y+2xy^2+y^3;\end{aligned}$$

all of these are derived from the Hessian $(1,\ 1,\ -17)$.

Similarly, if we start from $(-1,\ 1,\ 17)$, we find

$$f = mx^3+nx^2y+(51m-n)\,xy^2+(-17m+6n)\,y^3,$$

with $$h = (n^2+3mn-153m^2)(-x^2+xy+17y^2).$$

Comparing this with the previous case, we see that the values of $(m,\ n)$ to be considered are given by

$$(m,\ n) = (0,\ 1),\ (-1,\ 14),\ (-1,\ -11),$$

leading to the forms

$$\begin{aligned}f_1 &= (x^2-xy+6y^2)\,y,\\ f_2 &= -x^3+14x^2y-65xy^2+101y^3,\\ f_3 &= -x^3-11x^2y-40xy^2-49y^3.\end{aligned}$$

But these do not give any more reduced forms;* in fact,

$$f_1 \sim (x^2+xy+6y^2)\,y,$$

$$f_2 \sim (-1,\ 2,\ -1,\ -1) \sim (1,\ 1,\ 2,\ 1),$$

$$f_3 \sim (-1,\ -2,\ -1,\ -1) \sim (-1,\ 1,\ -2,\ 1) \sim (1,\ -1,\ 2,\ -1).$$

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* This might have been foreseen, because $(-1,\ -1,\ 17) \sim (-1,\ 1,\ 17)$.

Finally, then, we see that for $\Delta = 23$, there are three classes represented by

$$(0,\ 1,\ 1,\ 6),\quad (1,\ \pm 1,\ 2,\ \pm 1),$$

the last pair being improperly equivalent.

It may be added that 23 is the least value of Δ for which there exists a cubic form which has no linear factor.

4. For the purpose of explaining the general theory it will be necessary to apply a proposition first stated by Eisenstein. If (A, B, C) is the Hessian of (a, b, c, d), then

$$-A^2 = Ab^2 - 3Bba + 9Ca^2, \tag{4}$$

identically, in virtue of the relations $A = 3ac - b^2$, ..., which define the Hessian. Let L denote the form $(A, -3B, 9C)$, and suppose that (A, B, C) is an assigned quadratic H, of determinant 3Δ. Then it follows from (4) that the first two coefficients (a, b) of every cubic which has a Hessian H supply an integral solution of

$$L(b, a) = -A^2.$$

Now it can be proved without difficulty that conversely if (b, a) is any integral solution of (4) there is a corresponding cubic (a, b, c, d), and only one, which has a Hessian H. Thus there is a one-one correspondence between the cubics and the representations of $-A^2$ by L.

5. By means of the automorphs of L, these cubics may be arranged in a finite number of sequences as follows. Let (T, U) be the primary solution of

$$T^2 - 27\Delta U^2 = 4,$$

and let (a, b, c, d) be a cubic derived from any one solution of (4). The primary automorph of L is given by

$$P = \begin{bmatrix} \tfrac{1}{2}(T + 3BU), & -9CU \\ AU, & \tfrac{1}{2}(T - 3BU) \end{bmatrix}, \tag{5}$$

and if this be applied to (a, b) and the new values of c, d calculated it is found, without much difficulty, that

$$P(a, b, c, d) = (a', b', c', d'),$$

or, say,

$$Pf = f',$$

where
$$\left.\begin{aligned} a' &= \tfrac{1}{2}(T-3BU)\,a+AUb \\ b' &= -9CUa+\tfrac{1}{2}(T+3BU)\,b \\ c' &= \tfrac{1}{2}(T-3BU)\,c+9AUd \\ d' &= -CUc+\tfrac{1}{2}(T+3BU)d \end{aligned}\right\}. \tag{6}$$

Hence f is a member of a sequence of forms

$$\dots f_{-2},\ f_{-1},\ f,\ f_1,\ f_2,\ \dots,$$

where f_n means $P^n f$. It will now be proved that the forms of this sequence are either all equivalent, or else fall into three distinct classes. Let (t, u) be the primary solution of

$$t^2-3\Delta u^2 = 4\,;$$

then the primary automorph of H is given by

$$Q = \begin{bmatrix} \tfrac{1}{2}(t-Bu), & -Cu \\ Au, & \tfrac{1}{2}(t+Bu) \end{bmatrix},$$

and if this is applied to the variables in f the result is

where
$$\left.\begin{aligned} Qf &= (a'', b'', c'', d'') \\ a'' &= \tfrac{1}{2}(t_3-Bu_3)\,a+\tfrac{1}{3}Au_3b \\ b'' &= -3Cu_3a+\tfrac{1}{2}(t_3+Bu_3)\,b \\ c'' &= \tfrac{1}{2}(t_3-Bu_3)c+3Au_3d \\ d'' &= -\tfrac{1}{3}Cu_3c+\tfrac{1}{2}(t_3+Bu_3)d \end{aligned}\right\}. \tag{7}$$

Here (t_3, u_3) are found from

$$\frac{t_3+u_3\sqrt{3\Delta}}{2} = \left(\frac{t+u\sqrt{3\Delta}}{2}\right)^3,$$

so that
$$t_3 = \tfrac{1}{4}t(t^2+3\Delta u^2), \qquad u_3 = \tfrac{3}{4}(t^2+\Delta u^2)\,u, \tag{8}$$

whence we see that u_3 is divisible by 3, and t_3 is not. Putting, therefore, $t_3 = T_h$, $u_3 = 3U_h$, we have (T_h, U_h) a solution of $T^2-27\Delta U^2 = 4$, and

$$Q_{x,y}(f) = P^h_{a,b}(f),$$

where the suffixes are used to emphasise the fact that P affects the coefficients and Q the variables of f. The question now is—what is the value of h? We observe that since

$$T^2-27\Delta U^2 = T^2-3\Delta(3U)^2,$$

it follows that the solutions of $T^2-27\Delta U^2 = 4$ are simply $(t_n, \tfrac{1}{3}u_n)$, where

(t_n, u_n) are those solutions of $t_n^2 - 3\Delta u_n^2 = 4$ in which u_n is divisible by 3. There are two, and only two, possible cases. If u_1 is prime to 3 so is u_2, and $(t_3, \frac{1}{3}u_3)$ is the primary solution (T, U); in this case $h = 1$, and all the forms f_n are equivalent. But it may happen that u_1 is divisible by 3: in this case $(T, U) = (t_1, \frac{1}{3}u_1)$, and $h = 3$, so that any three consecutive forms f_{n-1}, f_n, f_{n+1} are non-equivalent. The case of $\Delta = 23$ is one of those in question, for

$$t^2 - 69u^2 = 4$$

has the primary solution (25, 3), whence $(T, U) = (25, 1)$; so that every possible Hessian leads to three classes of cubics, as, in fact, was proved in Art. 3.

If the solutions of $L(b, a) = -A^2$ are not all associated with the sequence (f_n), we take one of those that remain and form another sequence of cubics, and so on. From the theory of quadratics, it follows that there is only a finite number of sequences, depending upon the number of solutions of the congruence $x^2 \equiv 3\Delta \pmod{4A^2}$.

6. We are now in a position to find a complete set of representative cubics for any possible negative discriminant $-\Delta$. To do so we take a complete set, both primitive and derived, of representative quadratics $H_1, H_2, \ldots$ of determinant 3Δ. Let H, or (A, B, C), be any one of them; then if there are integral solutions (a, b) of

$$-A^2 = Ab^2 - 3Bba + 9Ca^2,$$

there will be one or three corresponding classes of cubics, which can be determined by the methods of Art. 5, for each associated set of solutions. If there are no solutions, there will be no cubics associated with H.

7. In practice, it is inconvenient to have to solve the equation

$$L(b, a) = -A^2,$$

but the difficulty can be avoided by the theory explained in my paper on the relations between cubics and their Hessians.* It is there shown that, if H is a prescribed Hessian, all the cubics which have a Hessian of the form μH, where μ is an integer, will be of the form

$$f = m\phi + n\psi,$$

where ϕ, ψ are determinate cubics, and m, n arbitrary integers. The Hessian is now

$$h(f) = G(m, n)H(x, y),$$

where $G(m, n)$ is a determinate quadratic in m and n. Suppose now that

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 200.

H has a determinant 3Δ, and we wish to find all the associated cubics of discriminant $-\Delta$: then $$G(m, n) = 1,$$

and this equation is much simpler to deal with than $L(b, a) = -A^2$, which it replaces. In fact, since $f = m\phi + n\psi$, the coefficients a, b are linear functions of m, n, which convert $L(b, a)$ into $-A^2 . G(m, n)$; conversely m, n are linear functions of a, b which are fractional in form but give integral values for m, n whenever (b, a) is a solution of

$$L(b, a) = -A^2.$$

Thus there is a one-one correspondence between the forms L, G, and it follows that a sequence of forms (f_n) derived from $G(m, n) = 1$ is obtainable by the same linear substitution applied to all the forms of a sequence derived from a solution of $L(b, a) = -A^2$. In particular, each sequence falls into one class, or three, according to the criterion found in Art. 5.

As a check to calculation, it may be remarked that if

$$a = pm + qn,$$
$$b = rm + sn,$$

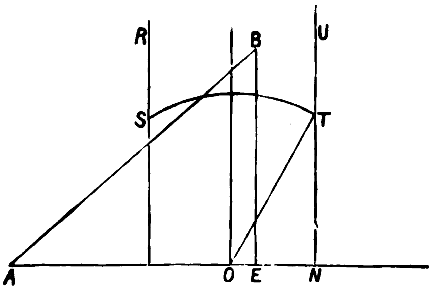
the determinant of G is $27\Delta(ps - qr)^2/A^4$.

In order that $G(m, n) = 1$ may be possible, it is necessary and sufficient that G should belong to the principal class of its determinant.

8. When, for a given discriminant, a set of representative forms has been obtained, the final process is to put for them the corresponding reduced forms, calculated by the method explained in the earlier part of this paper.

It has been shown, on a previous occasion, that, if (a, b, c, d) is a reduced form, $$27a^4 < 16\Delta;$$

Mr. Berwick has discovered other inequalities which limit the coefficients when Δ is given, and I take this opportunity of communicating them.



In the figure $RSTU$ is part of the contour of the fundamental triangle; A, B are the points representing the roots α, β of the cubic, and BE is perpendicular to AN. We have

$$AE = \tfrac{1}{2}(\beta+\gamma)-\alpha = \tfrac{3}{2}(\beta+\gamma)+\frac{b}{a}.$$

Now, if C is the point corresponding to the root γ, we have

$$AE^4 < AB^2.AC^2,$$

$$3 < BC^2;$$

and therefore $$3AE^4 < \Delta/a^4;$$

hence $$3\{2b+3a(\beta+\gamma)\}^4 < 16\Delta,$$

and, *a fortiori*, since $|\beta+\gamma| < 1$,

$$3(2|b|-3|a|)^4 < 16\Delta. \tag{9}$$

Again, if B is kept fixed, the greatest value of OB^2/AB^2 is when A is at E, and the greatest value of OB^2/BE^2 is when B is at S or T; consequently, in general, $$4AB^2 > 3OB^2.$$

Similarly $$4AC^2 > 3OC^2;$$

also $$a^2\alpha^2 < (|a|+|b|)^2;$$

hence $$9d^2 < (|a|+|b|)^2.16AB^2.AC^2,$$

and since $3 < BC^2$, we obtain by multiplication

$$27d^2 < \frac{16(|a|+|b|)^2\Delta}{a^4}. \tag{10}$$

The inequalities (9) and (10) impose limits on b, d in addition to the limits previously found for a. Now when a, b, d, Δ are assigned, there is a cubic to find possible values of c. All we have to do then is to find the sets (a, b, c, d) which give the proper value to Δ, and also satisfy the three conditions of inequality; if we reduce the resulting forms, we are certain to find representatives of every class, and we see that the number of distinct classes is limited.*

9.† Since writing the greater part of the present paper, I have looked at the two memoirs on binary cubics by Pepin, which are contained in the *Atti dell' Accad. Pontif. dei Nuovi Lincei*, Vols. 37, 39 (1884, 1886). In

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* [*Added May 4th*, 1911.—Mr. Berwick has since found the additional inequality $$27(|c|-|a|-|b|)^2 a^2 < 16\Delta.]$$

† §§ 9 and 10 and the table added May 4th, 1911.

the first of these he develops the theory on the lines laid down by Eisenstein, making use of the equation $L(b, a) = -A^2$, or rather an equivalent one, as he takes the standard form to be $(a, 3b, 3c, d)$. Of course, some of my results practically coincide with his; but he does not go into the question of reduction, nor into the geometrical theory. In the second paper he discusses classification, but confines himself to the case of a positive discriminant.

10. The following table, calculated by Mr. Berwick, gives all the non-composite reduced cubics for values of Δ which do not exceed 999. It will be understood that each entry (a, b, c, d) really stands for $(a, \pm b, c, \pm d)$, and represents two classes, improperly equivalent to each other.

Δ	f	Δ	f	Δ	f
23	(1, 1, 2, 1)	244	(2, 2, 3, 2)	432	(1, 3, 3, 5)
31	(1, 0, 1, 1)	247	(1, 3, 4, 5)	436	(1, 3, 4, 6)
44	(1, 2, 2, 2)	255	(1, 1, 0, 3)	439	(1, 2, −1, 3)
59	(1, 0, 2, 1)	268	(2, 4, 4, 3)	440	(2, 0, 1, 2)
76	(1, 1, 3, 1)	279	(1, 1, 4, 3)	451	(2, 3, 5, 3)
83	(1, 1, 1, 2)	283	(1, 0, 4, 1)	459	(2, 3, 3, 3)
87	(1, 2, 3, 3)	300	(2, 2, 4, 1)	460	(1, 1, 5, 3)
104	(2, 2, 3, 1)	304	(1, 4, 4, 4)	464	(1, 3, 5, 7)
107	(1, 1, 3, 2)	307	(1, 2, 4, 5)		(2, 1, 4, 1)
108	(1, 3, 3, 3)	324	(2, 0, 3, 1)	472	(2, 4, 5, 4)
116	(1, 1, 0, 2)	327	(3, 3, 4, 1)	484	(1, 2, 5, 6)
135	(1, 0, 3, 1)	331	(1, 1, 3, 4)	491	(1, 2, 2, 5)
139	(1, 2, 2, 3)	332	(1, 1, 2, 4)	492	(1, 2, 4, 6)
140	(1, 0, 2, 2)	335	(1, 2, 5, 5)	496	(2, 0, 2, 2)
152	(2, 3, 4, 2)	339	(1, 2, 0, 3)	499	(1, 0, 4, 3)
172	(2, 0, 2, 1)	351	(1, 0, 3, 3)	503	(2, 5, 5, 4)
175	(1, 1, 2, 3)	356	(2, 1, 2, 2)	515	(1, 4, 4, 5)
176	(1, 2, 4, 4)	364	(1, 0, 4, 2)	516	(3, 3, 4, 2)
199	(1, 1, 4, 1)	367	(1, 2, 3, 5)	519	(3, 5, 6, 3)
200	(1, 2, 3, 4)	368	(2, 2, 4, 2)	524	(1, 1, 3, 5)
204	(1, 1, 1, 3)	379	(1, 1, 1, 4)	527	(1, 0, 5, 1)
211	(2, 1, 3, 1)	411	(1, 1, 5, 2)	543	(1, 1, 2, 5)
212	(1, 1, 4, 2)	416	(1, 1, 5, 1)	547	(3, 2, 4, 1)
216	(1, 0, 3, 2)		(1, 2, 1, 4)	556	(1, 4, 3, 4)
231	(1, 2, 1, 3)	419	(2, −1, 3, 1)	560	(1, 2, 0, 4)
236	(2, −1, 2, 1)	424	(3, 4, 5, 2)	563	(1, 1, 5, 4)
239	(1, 3, 2, 3)	428	(1, 3, 2, 4)	567	(3, 0, 3, 1)
243	(1, 3, 3, 4)	431	(2, 1, 3, 2)	575	(1, 1, 4, 5)

(*Continued on p.* 138.)

Δ	*f*	Δ	*f*	Δ	*f*
588	(1, 2, 6, 6)	751	(1, 1, 6, 1)	876	(3, 2, 4, 2)
608	(1, 0, 5, 2)	755	(1, 2, 6, 7)	883	(2, 1, 1, 3)
	(1, 1, 1, 5)	756	(2, 3, 6, 3)	888	(2, 2, 5, 3)
620	(2, 0, 4, 1)	759	(1, 1, 6, 3)	891	(1, 0, 6, 1)
628	(2, 5, 6, 5)	771	(1, 1, 3, 6)	907	(2, 1, 5, 1)
643	(1, 3, 1, 4)	780	(1, 4, 4, 6)	908	(3, 4, 6, 2)
648	(2, 0, 3, 2)	783	(3, 1, 4, 1)	931	(1, 1, 5, 6)
652	(2, 2, 4, 3)		(1, 3, 6, 9)	932	(1, 0, 5, 4)
655	(1, 2, 1, 5)	800	(2, 3, 4, 4)	940	(1, 3, 1, 5)
671	(1, 3, 2, 5)		(1, 5, 5, 5)	944	(1, 2, 5, 8)
675	(1, 3, 3, 6)	804	(1, 1, 4, 6)		(2, 0, 4, 2)
676	(2, 2, 5, 2)	808	(1, 1, 2, 6)	948	(1, 2, 1, 6)
679	(1, 3, 4, 7)	812	(2, 4, 6, 5)	959	(1, 2, 7, 7)
680	(2, 2, 5, 1)	815	(3, 4, 5, 3)	964	(4, 5, 6, 2)
684	(2, 2, 2, 3)	816	(3, 3, 5, 1)	971	(3, 1, 3, 2)
687	(1, 2, 5, 7)	823	(3, −2, 3, 1)	972	(1, 0, 6, 2)
688	(1, 0, 4, 4)	835	(1, 2, 0, 5)		(1, 3, 3, 7)
695	(1, 4, 5, 7)	839	(1, 4, 3, 5)		(2, 3, 6, 4)
696	(1, 2, −1, 4)	843	(3, 3, 5, 2)		(2, 6, 6, 5)
704	(2, 4, 4, 4)	844	(1, 1, 6, 4)	976	(1, 2, 6, 8)
707	(1, 3, 5, 8)	848	(1, 4, 2, 4)		(1, 3, 4, 8)
716	(3, −1, 3, 1)		(4, 4, 5, 1)	980	(2, 4, 5, 5)
728	(1, 1, 6, 2)	856	(2, 2, 1, 3)	983	(1, 1, 6, 5)
731	(1, 2, 4, 7)	863	(1, 2, 3, 7)	984	(2, 1, 0, 3)
743	(1, 0, 5, 3)	864	(1, 3, 0, 4)	996	(1, 4, 5, 8)
744	(2, −1, 4, 1)		(3, 6, 6, 4)	999	(2, 3, 3, 4)
748	(1, 2, 2, 6)	867	(2, 1, 3, 3)		

THE ACCURACY OF INTERPOLATION BY FINITE DIFFERENCES

(SECOND PAPER)

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Note.—This is a continuation of a paper with the same title in *Proceedings of the London Mathematical Society*, Series 2, Vol. 4, pp. 320–341. For convenience of reference, the numbers of the sections and of the formulæ in the two papers are made consecutive. The table on p. 331 of the first paper is quoted as Table I.

The following is a summary of the contents of the two papers:—

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III. *Single Central-Difference Formula.*

12. For central-difference interpolation, the standard formula has [§ 5] been given in the form (16) for $0 < \theta \leqslant \frac{1}{2}$, and in the form (17) for $-\frac{1}{2} \leqslant \theta < 0$. It is usually convenient to take the interval $0 < \theta < 1$ as our basis. For the first half of the interval we have (16). For $\frac{1}{2} < \theta < 1$ we have $-\frac{1}{2} < \theta - 1 < 0$, so that the suitable formula is (17) adapted, viz.,

$$u_\theta = u_{1+(\theta-1)} = u_1 + \frac{\theta-1}{1!}\,\delta u_{\frac{1}{2}} + \frac{\theta(\theta-1)}{2!}\,\delta^2 u_1 + \frac{\theta(\theta-1)(\theta-2)}{3!}\,\delta^3 u_{\frac{1}{2}} + \dots \tag{58}$$

If we compare this formula and (16) and (17) with each other, we see that (16) and (17) are identical if we take an odd number of terms, while (16) and (58) are identical if we take an even number of terms. If, however, (17) is the suitable formula, but the number of terms is even, or, if (58) is the suitable formula, but the number of terms is odd, we can still use (16), provided that in the final term we substitute the proper central difference without altering its coefficient.

We can therefore regard (16) as the standard formula either for the whole interval $\theta = -\frac{1}{2}$ to $\theta = +\frac{1}{2}$ or for the whole interval $\theta = 0$ to $\theta = 1$, provided that, (i) if $-\frac{1}{2} \leqslant \theta < 0$, and we go up to $\delta^{2n+1}u$, we replace $\delta^{2n+1}u_{\frac{1}{2}}$ by $\delta^{2n+1}u_{-\frac{1}{2}}$, and (ii), if $\frac{1}{2} < \theta < 1$, and we go up to $\delta^{2n}u$, we replace $\delta^{2n}u_0$ by $\delta^{2n}u_1$.

It may, however, be mentioned that, in the second of these two cases, the correction need not be made if we are only considering the tabular error; *i.e.*, the failure to make the correction, although it prevents the formula used from being the true central-difference formula, does not increase the maximum-limit of the tabular error [see § 23 (ii)].

IV. *Extensions of the preceding Results.*

A. *Formulæ comprised in the Lagrange-formula.*

13. In § 9 the advancing-difference and central-difference expressions for u_θ, taken up to the difference of m-th order, were converted into expressions in terms of a series of u's; and it was found that these expressions were substantially of the same form. They are, in fact, the expressions given by Lagrange's interpolation-formula, as modified by the condition that the tabulated values of x proceed by equal intervals; the values of u which enter into the expression being in the one case u_0, u_1, u_2, ... if θ lies between 0 and 1, while in the other case, if $0 < \theta \leqslant \frac{1}{2}$, they are $u_{-\frac{1}{2}m}, u_{-\frac{1}{2}m+1}, \ldots, u_{\frac{1}{2}m}$ or $u_{-\frac{1}{2}m+\frac{1}{2}}, u_{-\frac{1}{2}m+\frac{3}{2}}, \ldots, u_{\frac{1}{2}m+\frac{1}{2}}$ according as m is even or odd. Also, we see from § 12 that the central-difference formula for $\frac{1}{2} \leqslant \theta < 1$ leads to the same expression if m is odd, while, if m is even, it gives a similar expression involving

$$u_{-\frac{1}{2}m+1},\ u_{-\frac{1}{2}m+2},\ \ldots,\ u_{\frac{1}{2}m+1}.$$

If therefore we write Lagrange's formula in the form

$$u_\theta = {}_mL_{\theta-p}, \tag{59}$$

where

$${}_mL_{\theta-p} \equiv \frac{(\theta-p)(\theta-p-1)\ldots(\theta-p-m)}{m\,!}$$

$$\times \left\{ \frac{u_{p+m}}{\theta-p-m} - {}_mC_1\,\frac{u_{p+m-1}}{\theta-p-m+1} + \ldots + (-)^m\,\frac{u_p}{\theta-p} \right\}, \tag{60}$$

then, in the notation of § 9, the advancing-difference formula is

$$u_\theta = {}_mU_\theta = {}_mL_\theta \qquad (0 < \theta < 1), \tag{61}$$

while the central-difference formula takes the different forms

$$u_\theta = {}_{2n}V_\theta = {}_{2n}L_{\theta+n} \qquad (0 < \theta \leqslant \tfrac{1}{2}), \tag{62}$$

$$u_\theta = {}_{2n}V_{\theta-1} = {}_{2n}L_{\theta+n-1} \qquad (\tfrac{1}{2} \leqslant \theta < 1), \tag{63}$$

$$u_\theta = {}_{2n+1}V_\theta = {}_{2n+1}L_{\theta+n} \qquad (0 < \theta < 1). \tag{64}$$

14. Conversely, if we start with Lagrange's formula $u_\theta = {}_mL_{\theta-p}$, we can, by giving p a suitable value, convert ${}_mL_{\theta-p}$ into any one of the above expressions, such as ${}_mU_\theta$, by working backwards and taking out differences of the m-th, $(m-1)$-th, ... orders in succession; the number of u's in the remainder being reduced at each step. It is obvious, however, that, when the value of p has been settled, this process of taking out differences can be performed in a good many different ways besides the particular one which leads to the expression we are considering, and that we shall thus arrive at other formulæ which will be equivalent to, and might be substituted for, the advancing-difference or the central-difference formula, as the case may be. Denote the difference of order n, which involves $u_q, u_{q+1}, \ldots, u_{q+n}$, by $[q, q+1, \ldots, q+n]$, so that

$$[q, q+1, \ldots, q+n] \equiv \Delta^n u_q = \delta^n u_{q+\frac{1}{2}n} = u_{q+n} - {}_nC_1 u_{q+n-1} + \ldots + (-)^n u_q. \tag{65}$$

Then from ${}_mL_{\theta-p}$ we can take out $[p, p+1, \ldots, p+m]$ in either of two ways, according as we are to get rid of u_{p+m} or of u_p; its coefficient will in the former case be $(\theta-p)(\theta-p-1)\ldots(\theta-p-m+1)/m!$, and in the latter case $(\theta-p-1)(\theta-p-2)\ldots(\theta-p-m)/m!$. Similarly, supposing we get rid of u_{p+m}, we can at the next move get rid either of u_{p+m-1} or of u_p. Proceeding in this way, we see that there are 2^m different expressions of this kind which would produce the same expression ${}_mL_{\theta-p}$; these expressions corresponding to the different ways by which we can, in the first diagram in § 4, proceed from $\Delta^m u_p$ to some value of u by steps, each of which is from a quantity in one column of the diagram to one of the two adjoining quantities in the preceding column. Of these 2^m expressions, the numbers which contain $u_p, u_{p+1}, u_{p+2}, \ldots, u_{p+m}$ are respectively $1, {}_mC_1, {}_mC_2, \ldots, 1$.

If in the diagram mentioned above we draw lines diagonally upwards and downwards to the left, so as to include $\Delta^m u_p$ but no other differences of the m-th order, and so as to meet the line which divides the column of u's from the column of x's, all the tabulated quantities which occur in any one of the equivalents of ${}_mL_{\theta-p}$ will lie within the triangle so formed. This will be called the *tabular triangle* of $\Delta^m u_p$.

15. We have next to consider the law of formation of the successive terms in any one of these expressions, proceeding from the left. The law is most easily determined by working backwards.

(i) As shown in § 14, we can remove from ${}_mL_{\theta-p}$ either

$$\{(\theta-p)(\theta-p-1)\ldots(\theta-p-m+1)/m!\}\,[p,\,p+1,\,\ldots,\,p+m]$$

or

$$\{(\theta-p-1)(\theta-p-2)\ldots(\theta-p-m)/m!\}\,[p,\,p+1,\,\ldots,\,p+m].$$

It will be found that in the one case we are left with ${}_{m-1}L_{\theta-p}$ and in the other with ${}_{m-1}L_{\theta-p-1}$. Hence we have

$${}_mL_{\theta-p} = {}_{m-1}L_{\theta-p}+\frac{(\theta-p)(\theta-p-1)\ldots(\theta-p-m+1)}{m!}\,[p,\,p+1,\,\ldots,\,p+m], \tag{66}$$

$$\begin{aligned}&{}_mL_{\theta-p}\\ &= {}_{m-1}L_{\theta-p-1}+\frac{(\theta-p-1)(\theta-p-2)\ldots(\theta-p-m)}{m!}\,[p,\,p+1,\,\ldots,\,p+m].\end{aligned} \tag{67}$$

(ii) Hence the removal of successive terms, from the right, will always leave us with an expression of the Lagrange form; and, conversely, any number of terms, from the left, will together give an expression of this form. Also ${}_{m-1}L_{\theta-p}$ contains $u_p,\,u_{p+1},\,\ldots,\,u_{p+m-1}$, and ${}_{m-1}L_{\theta-p-1}$ contains $u_{p+1},\,u_{p+2},\,\ldots,\,u_{p+m}$. We see therefore, from (66) and (67), that the first $n+1$ terms will together be equal to an expression of the form ${}_nL_{\theta-q}$, involving $u_q,\,u_{q+1},\,\ldots,\,u_{q+n}$; that the next term to be added will be either

$$\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n)}{(n+1)!}\,[q,\,q+1,\,\ldots,\,q+n+1]$$

or

$$\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n)}{(n+1)!}\,[q-1,\,q,\,\ldots,\,q+n];$$

and that the result of the addition will be in the one case ${}_{n+1}L_{\theta-q}$, and in the other case ${}_{n+1}L_{\theta-q+1}$.

(iii) Instead of regarding each coefficient as a whole, we may regard it as made up of successive factors. Continuing the terms in (ii), the coefficient of the difference of order $n+2$ will in the one case be

$$\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n-1)}{(n+2)!} = \frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n)}{(n+1)!}\,\frac{\theta-(q+n+1)}{n+2},$$

and in the other case will be

$$\frac{(\theta-q+1)(\theta-q)\ldots(\theta-q-n)}{(n+2)!} = \frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n)}{(n+1)!}\,\frac{\theta-(q-1)}{n+2}.$$

Hence (changing $n+2$ into r), if the r-th term introduces u_s, the coefficient of the difference in the $(r+1)$-th term will be found by multiplying the coefficient of the difference in the r-th term by $(\theta-s)/r$.

(iv) We may take as examples the advancing-difference formula (3) and the central-difference formulæ (16), (17), and (58), which may be written

$$u_\theta = [0]+\frac{\theta-0}{1}[0,\,1]+\frac{\theta-0}{1}\,\frac{\theta-1}{2}[0,\,1,\,2] + \frac{\theta-0}{1}\,\frac{\theta-1}{2}\,\frac{\theta-2}{3}[0,\,1,\,2,\,3]+\ldots, \tag{68}$$

$$u_\theta = [0]+\frac{\theta-0}{1}[0,\,1]+\frac{\theta-0}{1}\,\frac{\theta-1}{2}[-1,\,0,\,1] + \frac{\theta-0}{1}\,\frac{\theta-1}{2}\,\frac{\theta-(-1)}{3}[-1,\,0,\,1,\,2]+\ldots, \tag{69}$$

$$u_\theta = [0]+\frac{\theta-0}{1}[-1,\,0]+\frac{\theta-0}{1}\,\frac{\theta-(-1)}{2}[-1,\,0,\,1] + \frac{\theta-0}{1}\,\frac{\theta-(-1)}{2}\,\frac{\theta-1}{3}[-2,\,-1,\,0,\,1]+\ldots, \tag{70}$$

$$u_\theta = [1]+\frac{\theta-1}{1}[0,\,1]+\frac{\theta-1}{1}\,\frac{\theta-0}{2}[0,\,1,\,2] + \frac{\theta-1}{1}\,\frac{\theta-0}{2}\,\frac{\theta-2}{3}[-1,\,0,\,1,\,2]+\ldots. \tag{71}$$

16. It follows from the above that we need not define the central-difference formula of order m (*i.e.*, up to and including a difference of that order) for u_θ otherwise than either (i) as a formula which is algebraically equivalent to the Lagrange-formula involving u_p, u_{p+1}, ..., u_{p+m}, where p is given by the condition that $x_{p+\frac{1}{2}m-\frac{1}{2}} \leqslant x_\theta \leqslant x_{p+\frac{1}{2}m+\frac{1}{2}}$, or (ii) as a formula expressing u_θ as a series constructed in the manner explained in § 15 and ending with that difference of the m-th order which lies nearest to the horizontal line through the point corresponding to the value of x_θ.

17. We can, of course, express ${}_mL_{\theta-p}$ in a very large number of ways in terms of the quantities lying within the tabular triangle of $\Delta^m u_p$. Everett's formula, for instance, is a variant in which m is odd, and each difference of odd order is expressed in terms of the adjoining differences of the next lower order.

It will be convenient here to examine the effect of applying this process to a single term in the more general formula considered above.

Suppose that we have a term in $\Delta^n u_q$, and that we replace $\Delta^n u_q$ by $\Delta^{n-1}u_{q+1}-\Delta^{n-1}u_q$; then the result will depend on the steps by which we have reached $\Delta^n u_q$ from a difference of order $n-2$, but it will be found that the two cases, in each of which one step is by advancing and the other by receding, will give the same result. (This is obvious on general grounds; for the terms in $\Delta^{n-1}u_q$ and in $\Delta^{n-1}u_{q+1}$ are in each case together equal to ${}_nL_{\theta-q}-{}_{n-2}L_{\theta-q-1}$, and their coefficients must therefore be the same in the two cases.) There are therefore three formulæ.

(i) Where $\Delta^{n-2}u_q$ is involved,

$$
{}_nL_{\theta-q} = {}_{n-2}L_{\theta-q}+\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-n+2)}{n!}
\times\{(\theta-q-n+1)\Delta^{n-1}u_{q+1}-(\theta-q-2n+1)\Delta^{n-1}u_q\}. \quad (72)
$$

(ii) Where $\Delta^{n-2}u_{q+1}$ is involved,

$$
{}_nL_{\theta-q} = {}_{n-2}L_{\theta-q-1}+\frac{(\theta-q-1)(\theta-q-2)\ldots(\theta-q-n+1)}{n!}
\times\{(\theta-q)\Delta^{n-1}u_{q+1}-(\theta-q-n)\Delta^{n-1}u_q\}. \quad (73)
$$

(iii) Where $\Delta^{n-2}u_{q+2}$ is involved,

$$
{}_nL_{\theta-q} = {}_{n-2}L_{\theta-q-2}+\frac{(\theta-q-2)(\theta-q-3)\ldots(\theta-q-n)}{n!}
\times\{(\theta-q+n-1)\Delta^{n-1}u_{q+1}-(\theta-q-1)\Delta^{n-1}u_q\}. \quad (74)
$$

B. *Limit of Tabular Error.*

18. In §§ 3 and 6 we considered the limits of tabular error for the advancing-difference and the central-difference formulæ respectively. We have now to consider the limit for the more general formula, *i.e.*, we have to consider the limit of error involved in calculating the general expression ${}_mL_{\theta-p}$, for the purpose of (59), from the tabulated values of u, instead of from the true values. It will be convenient to regard θ as lying between 0 and 1, and p as arbitrary; so that the cases already considered are those of $p=0$ and $p+\frac{1}{2}m-\frac{1}{2}\leqslant\theta\leqslant p+\frac{1}{2}m+\frac{1}{2}$, and we have now to regard p as having any positive or negative integral value. It will be assumed throughout that the limits of error of each tabulated u are $\pm\frac{1}{2}\rho$.

19. Let a_q, as before, denote the error of the tabulated value of u_q. Then the tabular error of u_θ, as given by (59) or by any equivalent formula, is to be found by replacing each u in ${}_mL_{\theta-p}$ by the corresponding value of a. Denote the result by ${}_mA_{\theta-p}$. Then, for any given values of θ, m, and p, ${}_mA_{\theta-p}$ will have its numerically greatest value when each

value of a is $+\frac{1}{2}\rho$ or $-\frac{1}{2}\rho$, the signs being such that the terms within the bracket in (60) all have the same sign. This requires that the a's should be alternately positive and negative, except where the denominators in (60) change sign, in which case two consecutive a's must have the same sign. This, of course, will only happen when θ lies between p and $p+m$, *i.e.*, when $p \ngtr 0$ and $p+m \nless 1$. Hence, taking a_0 to be positive, the sequence of values of a which will make the tabular error a maximum (numerically) will, as in the particular cases already considered in §§ 3 and 6, be

$$\left.\begin{array}{cccccccccc} \ldots, & a_{-3}, & a_{-2}, & a_{-1}, & a_0, & a_1, & a_2, & a_3, & a_4, & \ldots, \\ \ldots & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & \ldots \end{array}\right\}. \tag{75}$$

Denote these values of a by $\ldots\ \epsilon_{-1}, \epsilon_0, \epsilon_1, \epsilon_2, \ldots,$ and the resulting value of ${}_mA_{\theta-p}$ by ${}_mE_{\theta-p}$. Then, since ${}_mE_{\theta-p}$ is an expression of the same form as ${}_mL_{\theta-p}$, it follows that the tabular error due to using any formula which is equivalent to the Lagrange-formula may be obtained by applying this formula to the calculation of a quantity ϵ_θ from a table based on the above values, as follows:—

TABLE II.

x	ϵ	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.	5th Diff.	6th Diff.	7th Diff.	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
x_{-3}	$-\frac{1}{2}\rho$		2ρ		-8ρ		32ρ		
		ρ		-4ρ		16ρ		-63ρ	
x_{-2}	$\frac{1}{2}\rho$		-2ρ		8ρ		-31ρ		
		$-\rho$		4ρ		-15ρ		56ρ	
x_{-1}	$-\frac{1}{2}\rho$		2ρ		-7ρ		25ρ		
		ρ		-3ρ		10ρ		-35ρ	
x_0	$\frac{1}{2}\rho$		$-\rho$		3ρ		-10ρ		
		0		0		0		0	...
x_1	$\frac{1}{2}\rho$		$-\rho$		3ρ		-10ρ		
		$-\rho$		3ρ		-10ρ		35ρ	
x_2	$-\frac{1}{2}\rho$		2ρ		-7ρ		25ρ		
		ρ		-4ρ		15ρ		-56ρ	
x_3	$\frac{1}{2}\rho$		-2ρ		8ρ		-31ρ		
		$-\rho$		4ρ		-16ρ		63ρ	
x_4	$-\frac{1}{2}\rho$		2ρ		-8ρ		32ρ		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Suppose, for example, that $\theta = {\cdot}2$, and that we are using the advancing-difference formula up to the 4th difference inclusive. Then the limit of tabular error is

$$\left\{\tfrac{1}{2}\rho+\frac{{\cdot}2\times{\cdot}8}{2!}\,\rho+\frac{{\cdot}2\times{\cdot}8\times1{\cdot}8}{3!}\,3\rho+\frac{{\cdot}2\times{\cdot}8\times1{\cdot}8\times2{\cdot}8}{4!}\,7\rho\right\} = {\cdot}9592\rho\,;$$

or, to three places of decimals, $\cdot960\rho$, as in Table 1 (§ 8).

20. The following are some simple properties of Table II and of the deduced values of $\epsilon_\theta \equiv {}_mE_{\theta-p}$.

$$\left.\begin{aligned} &\text{(i) (a) If } p+m \leqslant 0, \text{ then } \Delta^m\epsilon_p = (-)^{p+m}2^{m-1}\rho. \\ &\quad\text{(b) If } p>0, \text{ then } \Delta^m\epsilon_p = (-)^{p+m-1}2^{m-1}\rho. \end{aligned}\right\} \tag{76}$$

(ii) For the central-differences:—

$$\left.\begin{aligned} \delta^{2n-1}\epsilon_{\frac{1}{2}} &= 0, \\ \delta^{2n}\epsilon_0 = \delta^{2n}\epsilon_1 &= (-)^n\,{}_{2n}C_n\cdot\tfrac{1}{2}\rho, \\ \mu\delta^{2n}\epsilon_{\frac{1}{2}} &= (-)^n\,{}_{2n}C_n\cdot\tfrac{1}{2}\rho, \\ \mu\delta^{2n-1}\epsilon_0 &= (-)^{n-1}\,{}_{2n-1}C_{n-1}\cdot\tfrac{1}{2}\rho, \\ \mu\delta^{2n-1}\epsilon_1 &= (-)^n\,{}_{2n-1}C_{n-1}\cdot\tfrac{1}{2}\rho. \end{aligned}\right\} \tag{77}$$

The first two of these have already been stated in (31).

(iii) For the other differences, we may regard the table as compounded of two, in which the values of ϵ are respectively

$$\begin{array}{cccccccccc} \ldots, & \epsilon_{-3}, & \epsilon_{-2}, & \epsilon_{-1}, & \epsilon_0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & \ldots, \\ \ldots, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & -\frac{1}{2}\rho, & +\frac{1}{2}\rho, & \ldots, \\ \ldots, & 0, & 0, & 0, & 0, & +\rho, & -\rho, & +\rho, & -\rho, & \ldots. \end{array}$$

Examination of the derivation of the successive differences in each of the two tables will show that, if $p+m>0$, but $p \leqslant 0$,

$$\Delta^m\epsilon_p = (-)^{p+m-1}\left\{{}_mC'_{p+m-1}-2^{m-1}\right\}\rho, \tag{78}$$

where $${}_mC'_s \equiv {}_mC_0+{}_mC_1+\ldots+{}_mC_s \tag{79}$$

$$= \text{coeff. } x^s \text{ in expansion of } (1+x)^m/(1-x). \tag{79A}$$

It is clear that ${}_mC'_s \overset{<}{\underset{>}{=}} 2^{m-1}$, according as $s \overset{<}{\underset{>}{=}} \frac{1}{2}(m-1)$, and therefore $\Delta^m\epsilon_p$ is of sign $(-)^{p+m}$ or $(-)^{p+m-1}$ according as p is less or greater than $-\frac{1}{2}(m-1)$, being zero if $p=-\frac{1}{2}(m-1)$.

The formula (78) also holds if $p>0$, since in that case ${}_mC'_{p+m-1}=2^m$, so that $\Delta^m\epsilon_p = (-)^{p+m-1}2^{m-1}\rho$, as in (76).

(iv) (a) If $p \leqslant 0$ and $p+m>0$, so that u_0 and u_1 occur in ${}_mL_{\theta-p}$, ${}_mE_{\theta-p}$ is positive;

(b) If $p>0$, ${}_mE_{\theta-p}$ is of sign $(-)^{p+1}$;

(c) If $p+m \leqslant 0$, ${}_mE_{\theta-p}$ is of sign $(-)^{p+m}$.

(v) Since $\delta^{2n+1}\epsilon_{\frac{1}{2}}=0$, it follows from (66) and (67) that

$${}_{2n+1}E_{\theta+n} = {}_{2n}E_{\theta+n} = {}_{2n}E_{\theta+n-1}. \tag{80}$$

C. *Relative Accuracy of Formulæ.*

21. We have already found that the limit of tabular error is less for the central-difference formula than for the advancing-difference formula of the same (*i.e.*, the m-th) order. These two formulæ correspond to the use of ${}_mL_{\theta-p}$ with different values of p, *i.e.*, they correspond to two particular positions of the tabular triangle of a difference of order m. We have now to consider the limit of error in the more general case, *i.e.*, for any arbitrary position of the tabular triangle.

We can most conveniently do this by taking a particular position of the triangle, and then seeing how the corresponding value of ${}_mE_{\theta-p}$ is altered by shifting the triangle in one direction or the other.

22. Taking first the central-difference formula:—

(i) If we go up to a difference of order $2n+1$, this difference will be $\delta^{2n+1}u_{\frac{1}{2}}$; and the values of u involved will be from u_{-n} to u_{n+1}. The limit of tabular error will therefore be $\pm{}_{2n+1}E_{\theta+n}$.

(ii) Suppose we only go up to the difference of order $2n$. Then the values of u involved will be from u_{-n} to u_n if θ lies between 0 and $\frac{1}{2}$, and from u_{-n+1} to u_{n+1} if θ lies between $\frac{1}{2}$ and 1; and the limit of error will be $\pm{}_{2n}E_{\theta+n}$ in the one case, and $\pm{}_{2n}E_{\theta+n-1}$ in the other. But each of these, by (80), is equal to $\pm{}_{2n+1}E_{\theta+n}$.

(iii) Hence the limit of tabular error for the central-difference formula up to the term involving a difference of order $2n$ or $2n+1$ is $\pm{}_{2n+1}E_{\theta+n}$.

23. Next, suppose that we are using any other difference-formula, equivalent to the Lagrange-formula (59); the u's involved being from u_p to u_{p+m}, so that the u's and differences which occur in the formula lie within the tabular triangle of $\Delta^m u_p$. Then the limit of tabular error is $\pm{}_mE_{\theta-p}$. Let us see how this limit is affected by moving the tabular triangle nearer to the position which it occupies for the central-difference formula.

(i) For the central-difference formula, the vertex of the tabular triangle lies either within (m odd) or on the boundary of (m even) the central horizontal strip bounded by lines through $\theta = 0$ and $\theta = 1$; and for any other formula, except in the special case mentioned below, it lies outside this strip.

(ii) The exceptional case is that in which m is even, and $\Delta^m u_p$ is either $\delta^m u_1$, where θ lies between 0 and $\frac{1}{2}$, or $\delta^m u_0$, where θ lies between $\frac{1}{2}$ and 1. The value of ${}_mE_{\theta-p}$ is then ${}_mE_{\theta+\frac{1}{2}m-1}$ or ${}_mE_{\theta+\frac{1}{2}m}$, while for the central-difference formula it would be ${}_mE_{\theta+\frac{1}{2}m}$ or ${}_mE_{\theta+\frac{1}{2}m-1}$. But, by (80),

$${}_mE_{\theta+\frac{1}{2}m} = {}_mE_{\theta+\frac{1}{2}m-1}.$$

Hence, if we go up to a difference of even order m, it is immaterial, so far as the tabular error is concerned, whether this is $\delta^m u_0$ or $\delta^m u_1$; θ being between 0 and 1 [cf. § 12].

(iii) For any other case, the vertex of the triangle lies outside the central strip, being on the positive or the negative side of it according as

$$p > -\tfrac{1}{2}m+1 \text{ or } < -\tfrac{1}{2}m.$$

(iv) First suppose that $p > -\tfrac{1}{2}m+1$. Then the effect of shifting the triangle backwards one place is to alter the limit of error from $\pm {}_mE_{\theta-p}$ to $\pm {}_mE_{\theta-p+1}$. But, by (66) and (67),

$$ {}_mE_{\theta-p} - {}_mE_{\theta-p+1} = \frac{(\theta-p)(\theta-p-1)\dots(\theta-p-m+1)}{m!}\Delta^{m+1}\epsilon_{p-1}; \quad (81)$$

and there are three cases to be considered.

(*a*) Usually the triangle will include u_0, so that $p \leqslant 0$. Then $\Delta^{m+1}\epsilon_{p-1}$ will, by § 20 (iii), be of sign $(-)^{p+m-1}$, and its coefficient in (81) will include $-p+1$ positive, and $p+m-1$ negative, factors. The expression on the right-hand side of (81) will therefore be positive. Also, by § 20 (iv) (*a*), ${}_mE_{\theta-p}$ and ${}_mE_{\theta-p+1}$ are both positive. Hence ${}_mE_{\theta-p+1}$ is less than ${}_mE_{\theta-p}$, so that the shift will reduce the limit of error.

(*b*) Next, let p be > 1, so that u_0 will not be within the triangle, either in its old or in its new position. Then ${}_mE_{\theta-p}$ and ${}_mE_{\theta-p+1}$ will [see § 20 (iv) (*b*)] be of opposite signs. But, if they are expressed in the Lagrange form, the terms in each will all be of the same sign; and it will be found that each term in $|{}_mE_{\theta-p}|$ is greater than the corresponding term in $|{}_mE_{\theta-p+1}|$. Hence* $|{}_mE_{\theta-p}| > |{}_mE_{\theta-p+1}|$, and the shift will reduce the limit of error.

(*c*) Finally, let $p = 1$, so that the shift will bring u_0 within the triangle. Then ${}_mE_{\theta-p}$ and ${}_mE_{\theta-p+1}$ are both positive, and the reasoning of (*a*) applies.

(*d*) Hence, in every case, the effect of shifting the tabular triangle one place backwards is to reduce the limit of error.

(v) Similarly, if $p < -\tfrac{1}{2}m$, so that the vertex of the triangle lies on the negative side of the central strip, we reduce the limit of error by shifting the triangle one place forwards.

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* We might also obtain this result by regarding the tabular triangle as fixed, and θ as altering continuously to $\theta+1$. It is easy to see that, so long as $p > \theta$, $|{}_mE_{\theta-p}|$ is always of the same form, and $d|{}_mE_{\theta-p}|/d\theta$ is negative.

(vi) It follows that, if our formula is not a central-difference formula (in the general sense of § 16), we can reduce the limit of tabular error by replacing the formula by a more central one; in other words, the central-difference formula gives a smaller limit of tabular error than any other formula of this kind, involving the same number of consecutive values of u. This is subject to the exception mentioned in (ii), where the difference of highest order, instead of being the central-difference on one boundary of the central strip, is the corresponding difference on the other boundary; in this case the substitution of the correct central-difference formula does not reduce the limit of error, but it does not increase it.

24. Instead of moving the vertex of the tabular triangle directly from $\Delta^m u_p$ to $\Delta^m u_{p-1}$, we might move it by two diagonal steps, either first outwards (away from the line of values of x) and then inwards, or first inwards and then outwards. In the first case, the successive formulæ for u_θ would be ${}_mL_{\theta-p}$, ${}_{m+1}L_{\theta-p+1}$, ${}_mL_{\theta-p+1}$, and in the second case they would be ${}_mL_{\theta-p}$, ${}_{m-1}L_{\theta-p}$, ${}_mL_{\theta-p+1}$. This suggests the comparison of the limits of error for tabular triangles whose vertices are diagonally adjacent; *e.g.*, the comparison of ${}_mE_{\theta-p}$ with ${}_{m+1}E_{\theta-p+1}$ or with ${}_{m-1}E_{\theta-p}$.

(i) For movements within the central strip, we have already found [§§ 20 (v) and 22] that ${}_{2n}E_{\theta+n-1} = {}_{2n+1}E_{\theta+n} = {}_{2n}E_{\theta+n}$. Hence, by starting from one boundary of the strip, moving outwards to the centre, and then inwards to the other boundary, we make no alteration in the limit of error. If we moved first inwards and then outwards, we should first decrease it and then increase it.

(ii) Next, suppose that in the first position the vertex lies outside the central strip, on the positive side of it. Then, by (66) and (67),

$$ {}_mE_{\theta-p} = {}_{m-1}E_{\theta-p} + \frac{(\theta-p)(\theta-p-1)\dots(\theta-p-m+1)}{m!}\Delta^m\epsilon_p, \qquad (82) $$

$$ {}_mE_{\theta-p} = {}_{m+1}E_{\theta-p+1} - \frac{(\theta-p)(\theta-p-1)\dots(\theta-p-m)}{(m+1)!}\Delta^{m+1}\epsilon_{p-1}; \qquad (83) $$

and, by § 20 (iii), $\Delta^m\epsilon_p$ and $\Delta^{m+1}\epsilon_{p-1}$ are of sign $(-)^{p+m-1}$.

(*a*) If $p \leqslant 1$, so that the first triangle includes u_1, the term added in (82) is positive, and the term subtracted in (83) is negative; and, by § 20 (iv) (*a*) and (*b*), ${}_mE_{\theta-p}$, ${}_{m-1}E_{\theta-p}$, and ${}_{m+1}E_{\theta-p+1}$ are all positive. Hence ${}_mE_{\theta-p}$ is greater than each of the other two, so that the limit of error is reduced by moving diagonally towards the central strip, whether inwards or outwards.

(*b*) If $p > 1$, then ${}_mE_{\theta-p}$, ${}_{m-1}E_{\theta-p}$, and the term added in (82), are all of sign $(-)^{p+1}$. Hence in this case also the limit of error is reduced by moving toward the central strip, provided that we move inwards.

(*c*) This, however, will not necessarily be the case if $p > 1$ and we move outwards. Since ${}_mE_{\theta-p}$ and ${}_{m+1}E_{\theta-p+1}$ are respectively of signs $(-)^{p+1}$ and $(-)^p$, we have to compare the values of $(-)^{p+1}{}_mE_{\theta-p}$ and $(-)^p{}_{m+1}E_{\theta-p+1}$. We have

$$(-)^{p+1}{}_mE_{\theta-p} = \frac{(p-\theta)(p+1-\theta)\dots(p+m-\theta)}{m!}$$
$$\times\left\{\frac{{}_mC_0}{p-\theta}+\frac{{}_mC_1}{p+1-\theta}+\dots+\frac{{}_mC_m}{p+m-\theta}\right\}\tfrac{1}{2}\rho,$$

$$(-)^p{}_{m+1}E_{\theta-p+1} = \frac{(p-1-\theta)(p-\theta)\dots(p+m-\theta)}{(m+1)!}$$
$$\times\left\{\frac{{}_{m+1}C_0}{p-1-\theta}+\frac{{}_{m+1}C_1}{p-\theta}+\dots+\frac{{}_{m+1}C_{m+1}}{p+m-\theta}\right\}\tfrac{1}{2}\rho;$$

so that, denoting the expressions in the curled brackets by S_m and S'_{m+1}, we have to compare the values of $(m+1)S_m$ and $(p-1-\theta)S'_{m+1}$. Now

$$S_m = \int_0^1 x^{p-1-\theta}(1+x)^m\,dx, \qquad S'_{m+1} = \int_0^1 x^{p-2-\theta}(1+x)^{m+1}\,dx,$$
$$(m+1)S_m+(p-1-\theta)S'_{m+1} = 2^{m+1}.$$

Also

$$\frac{p-1-\theta}{m+1}S'_{m+1} = \frac{1}{m+1}+(p-1-\theta)\frac{{}_mC_0}{p-\theta}$$
$$+\frac{p-1-\theta}{2}\,\frac{{}_mC_1}{p+1-\theta}+\dots+\frac{p-1-\theta}{m+1}\,\frac{{}_mC_m}{p+m-\theta}.$$

This can obviously be greater than S_m if $p-1-\theta$ is very great, *e.g.*, if $p-1-\theta > m+1$; but, if $p = 2$, so that $p-1-\theta = 1-\theta$, the sum of the terms after the first can be made as small as we please by taking $1-\theta$ small enough, so that the sum of all the terms can be made less than $2^m/(m+1)$, and therefore less than S_m. Hence, by moving outwards towards the central strip, the limit of error will be in some cases increased and in other cases decreased.

(iii) We shall, of course, get corresponding results if the vertex of the triangle lies on the negative side of the central strip.

(iv) Hence the limit of error is in all cases reduced by moving the vertex of the tabular triangle diagonally inwards towards the central line; and it is also reduced by moving the vertex diagonally outwards towards the central line, if the triangle in its new position includes both u_0 and u_1.

25. In shifting the tabular triangle towards the centre, we are in effect taking away one of the extreme values of u and replacing it by a value at the other end, nearer the centre. It might be inferred that, in the more general case to which the general Lagrange-formula applies, *i.e.*, the case in which the differences between successive values of x are not all equal, we should reduce the tabular error by replacing one of the values of u by a value nearer the value corresponding to x. This, however, is not the case; the tabular error might be increased. Suppose, for instance, that $\theta = {\cdot}2$, and that our formula in the first instance involves u_0, u_1, and u_2. If instead of u_0, u_1, and u_2 we use u_0, $u_{\frac{1}{2}}$, and u_1, θ becomes ·4 of the new interval $\frac{1}{2}h$ in x; and it will be seen from Table I (§ 8) that the limit of tabular error is increased from $\pm{\cdot}580\rho$ to $\pm{\cdot}620\rho$. The residual error, however, is likely to be reduced by the change; and it might even become possible to dispense with 2nd differences, the limit of tabular error being thus reduced to $\pm{\cdot}500\rho$.

V. *Corrected Differences.*

26. In comparing the limits of tabular error for different formulæ, we have assumed that the table used for the interpolation is arranged so as to show the tabular differences, *i.e.*, that each quantity in any column (after that of u) is the exact difference of the two adjoining quantities in the preceding column. There is an alternative system of tabulation which is sometimes adopted; on this system the values of the differences, like those of the u's, are "corrected" in the final figure, so that each difference is given within the same limits of error $\pm\frac{1}{2}\rho$ as each u. We have to see what effect the use of these corrected differences would have on the limits of error of u_θ as given by the different formulæ.

27. If we obtain u_θ by applying a formula which involves differences to a table which shows corrected differences, the tabular error in the value so obtained will lie between $-R$ and $+R$, where R is found by replacing each difference (and each u) in the formula by $-\frac{1}{2}\rho$ or $+\frac{1}{2}\rho$ according as the coefficient of the difference (or of the u) is negative or positive. If, for instance, we use the advancing-difference formula (3), we shall have

$$R = \left\{1 + \frac{\theta}{1!} + \frac{\theta(1-\theta)}{2!} + \frac{\theta(1-\theta)(2-\theta)}{3!} + \dots\right\} \tfrac{1}{2}\rho. \qquad (84)$$

Similarly, for the standard central-difference formula (16), θ being between 0 and $\frac{1}{2}$, we shall have

$$R = \left\{1 + \frac{\theta}{1!} + \frac{\theta(1-\theta)}{2!} + \frac{(1+\theta)\,\theta(1-\theta)}{3!} + \dots\right\} \tfrac{1}{2}\rho. \qquad (85)$$

It is true that, as will be shown in more detail in § 35 below, the range of the possible error in any particular case is less than from $-R$ to $+R$; but it is simpler to take these limits first and to consider later whether, and to what extent, our results need modification. To avoid repetition of the ± sign, we may call R the limit of error.

It must be observed that formulæ which, on the system of tabular differences, are identical in result [§ 14] and have the same limit of error, may on the system of corrected differences have different limits of error. The expressions $u_0+\theta(u_1-u_0)$ and $u_1-(1-\theta)(u_1-u_0)$, for instance, are identically equal, but $u_0+\theta\Delta u_0$ and $u_1-(1-\theta)\Delta u_0$ will have limits of error $(1+\theta)\frac{1}{2}\rho$ and $(2-\theta)\frac{1}{2}\rho$ respectively. The methods used in §§ 21–24 for comparing the limits of error of different formulæ are therefore no longer applicable; and we have to take into account not only the formulæ of the ordinary kind, constructed as in § 15, but also the variants, such as Everett's formula, mentioned in § 17.

28. Considering first the advancing-difference formula, we see that if we have not to go beyond 1st or 2nd differences, we get a worse result by using corrected differences. For 1st differences the limit of error is increased from $\frac{1}{2}\rho$ to $(1+\theta)\frac{1}{2}\rho$. For 2nd differences, instead of

$$\{1+\theta(1-\theta)\}\,\tfrac{1}{2}\rho,$$

we get

$$\{1+\theta+\tfrac{1}{2}\theta(1-\theta)\}\,\tfrac{1}{2}\rho.$$

It might be supposed that the same result holds when the series is continued indefinitely. Thus one writer* says: "Il est facile de conclure de ce que précède que l'on doit également préférer les différences tabulaires pour les ordres supérieurs, lorsqu'il est nécessaire d'en faire usage." This conclusion, however, can easily be shown to be incorrect. For tabular differences the limit of error is given by (9), and the series inside the brackets is divergent; but the corresponding formula for corrected differences is (84), and the sum of the first $m+1$ terms inside the brackets is

$$2-\frac{(1-\theta)(2-\theta)\dots(m-\theta)}{m!},$$

so that the series is convergent. Hence, whatever the value of θ may be (between 0 and 1), there is always a number m such that, if we have to go up to differences of order not less than m, corrected differences give a better result than tabular differences.

This is also the case with the central-difference formula; the series in (32) is a hypergeometric series, which is divergent,† but the series in brackets in (85) is convergent [see § 33 below].

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* F. Lefort, *Proc. Roy. Soc.*, Edinburgh, Vol. VIII (1874-5), p. 610.

† G. Chrystal, *Algebra*, Pt. II (1st ed.), p. 116, Ex. 2; or T. J. I'A. Bromwich, *Introduction to the Theory of Infinite Series*, p. 35, Ex. 3.

29. Before making any further comparison between the two systems, let us compare the limits of error for the different formulæ of the kind considered in § 15, on the assumption that corrected differences are used throughout.

(i) Suppose that we have to go up to a difference of order m. Compare two formulæ, which are identical as far as a term involving $\Delta^{m-1}u_p$ inclusive. Then the difference involved in the remaining term will be $\Delta^m u_{p-1}$ in one of the formulæ and $\Delta^m u_p$ in the other. But the coefficient of this difference will be the same in both formulæ, viz.,

$$\frac{(\theta-p)(\theta-p-1)\ldots(\theta-p-m+1)}{m!};$$

and therefore the two formulæ will have the same limit of error.

(ii) Next compare two formulæ, which are identical up to a term in $\Delta^{r-1}u_q$ (inclusive) and beyond a term in $\Delta^{r+1}u_{q-1}$ (exclusive), but differ in that the intermediate term involves $\Delta^r u_{q-1}$ in the one formula and $\Delta^r u_q$ in the other. This intermediate term and the next term will in the one formula be

$$\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-r+1)}{r!}\Delta^r u_{q-1} + \frac{(\theta-q+1)(\theta-q)\ldots(\theta-q-r+1)}{(r+1)!}\Delta^{r+1}u_{q-1},$$

and in the other

$$\frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-r+1)}{r!}\Delta^r u_q + \frac{(\theta-q)(\theta-q-1)\ldots(\theta-q-r)}{(r+1)!}\Delta^{r+1}u_{q-1};$$

and the limit of error for the former will be greater or less than for the latter according as $|\theta-q+1| >< |\theta-q-r|$, *i.e.*, according as the horizontal line through x_{q-1} or through x_{q+r} is the further from that through x_θ, *i.e.*, according as $\Delta^{r-1}u_q$ lies on the negative or on the positive side of the horizontal line through x_θ. Hence we make our limit of error smaller by using that one of the two differences $\Delta^r u_{q-1}$ and $\Delta^r u_q$ which lies the nearer to this line.

(iii) It follows from (ii) that, if we have to end with a specified difference $\Delta^m u_p$, we get the best result by keeping as near as possible to the x_θ horizontal line (*i.e.*, by using u_0, $\delta u_{\frac{1}{2}}$, $\delta^2 u_0$, ..., if $0 < \theta \leqslant \frac{1}{2}$ or u_1, $\delta u_{\frac{1}{2}}$, $\delta^2 u_1$, ..., if $\frac{1}{2} \leqslant \theta < 1$) until it becomes necessary to move diagonally towards $\Delta^m u_p$. And it follows from successive applications of (i) and (ii) that the central-difference formula (in the strict sense) gives the smallest limit of error.

30. We can now go on to compare the limits of error, on the two systems, due to the principal formulæ, viz., the advancing-difference formula (3), the standard central-difference formula (16) (for $0 < \theta \leqslant \frac{1}{2}$), and the variants (18) and (19) (for $0 < \theta < 1$), and (20) and (21) (for $0 < \theta \leqslant \frac{1}{2}$). Using "$\alpha$" and "$\beta$" to denote the effects of taking tabular differences and corrected differences respectively, we obtain the following table, the factor $\frac{1}{2}\rho$ being omitted.

TABLE III.

(α = tabular differences. β = corrected differences.)

Formula.	Limit of error ($\div \frac{1}{2}\rho$) due to using					
	u	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
(3α)	1	$+0$	$+\theta(1-\theta)$	$+\theta(1-\theta)(2-\theta)$	$+\frac{7}{12}\theta(1-\theta)(2-\theta)(3-\theta)$	$+\frac{1}{4}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)$
(3β)	1	$+\frac{\theta}{1!}$	$+\frac{\theta(1-\theta)}{2!}$	$+\frac{\theta(1-\theta)(2-\theta)}{3!}$	$+\frac{\theta(1-\theta)(2-\theta)(3-\theta)}{4!}$	$+\frac{\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)}{5!}$
(16α)	1	$+0$	$+\frac{\theta(1-\theta)}{1!\,1!}$	$+0$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{2!\,2!}$	$+0$
(16β)	1	$+\frac{\theta}{1!}$	$+\frac{\theta(1-\theta)}{2!}$	$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$
(18α)		1		$+\frac{\theta(1-\theta)}{1!\,1!}$		$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{2!\,2!}$
(18β)		1		$+\frac{\theta(1-\theta)}{2!}$		$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$
(19α)	1	$+0$	$+\frac{\theta(1-\theta)}{1!\,1!}$	$+0$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{2!\,2!}$	$+0$
(19β)	1	$+\lvert\frac{1}{2}-\theta\rvert$	$+\frac{\theta(1-\theta)}{2!}$	$+\lvert\frac{1}{2}-\theta\rvert\frac{\theta(1-\theta)}{3!}$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+\lvert\frac{1}{2}-\theta\rvert\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$
(20α)	1		$+\frac{\theta(1-\theta)}{1!\,1!}$		$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{2!\,2!}$	
(20β)	1		$+\frac{\theta}{1!}$		$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$	
(21α)	1	$+\frac{\theta}{1!}$	$-\frac{\theta^2}{1!\,1!}$	$+\frac{(1+\theta)\,\theta(1-\theta)}{1!\,2!}$	$-\frac{(1+\theta)\,\theta^2(1-\theta)}{2!\,2!}$	$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{2!\,3!}$
(21β)	1	$+\frac{\theta}{1!}$	$+\frac{\theta^2}{2!}$	$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$	$+\frac{(1+\theta)\,\theta^2(1-\theta)}{4!}$	$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$

This table shows that the effect of using corrected differences is as follows :—

(i) The Everett formula (18), in which the terms are taken in pairs and recast, is the only one in which the terms of the limit of error are decreased throughout.

(ii) For the other formula in which certain terms are taken in pairs, viz. (20), the term due to 1st and 2nd differences is increased, while the other terms are decreased.

(iii) For the advancing-difference formula (3), a 1st difference term is introduced, but all the other terms are decreased.

(iv) For (16) and (19) the terms due to differences of even order are decreased (equally in the two formulæ), but new terms due to differences of odd order are introduced. In each case, however, the sum of the terms due to differences of order $2n$ and $2n+1$ is (except for $n = 0$) less in the β form than in the α form.

(v) For (21) the terms due to differences of odd order (after the first) are decreased, but those due to differences of even order are increased (algebraically). But the sum of the terms due to differences of order $2n+1$ and $2n+2$ is (except for $n = 0$) less in the β form than in the α form.

31. In the case of (3), (16), and (19), the primary disadvantage of using corrected differences is the introduction of a term due to the 1st difference term of the formula. We can avoid this by using tabular 1st differences; and, if we do this, we find that all three formulæ give better results with corrected differences than with tabular differences. This modified system, under which 1st differences are tabular and the remainder are corrected, may be denoted by "γ."

Further, we see that in (18) this result is indirectly achieved by resolving the 1st difference $\delta u_{\frac{1}{2}}$ into its components $u_1 - u_0$; and that a similar process is adopted for other differences of odd order. If in (16) or (19) we replace $\delta u_{\frac{1}{2}}$, $\delta^3 u_{\frac{1}{2}}$, $\delta^5 u_{\frac{1}{2}}$, ... by $u_1 - u_0$, $\delta^2 u_1 - \delta^2 u_0$, $\delta^4 u_1 - \delta^4 u_0$, ... we get (18). Thus we get a new system "δ," under which the differences of even order alone are corrected, while the differences of odd order may be described as "semi-corrected," being the exact differences of the corrected differences of the next lower order. On this system we find that (16) and (19) are improved by taking corrected differences.

For (3) it will be found that (3δ) is not so good as (3γ). The terms in (3) which involve $\Delta^{n-1}u_0$ and $\Delta^n u_0$ (n being >2) are respectively

$$(-)^n nf(\theta)\,\Delta^{n-1}u_0 \quad\text{and}\quad (-)^{n+1}(n-1-\theta)f(\theta)\,\Delta^n u_0,$$

where
$$f(\theta) \equiv \theta(1-\theta)(2-\theta)\dots(n-2-\theta)/n!.$$

The first of these terms, if $\Delta^{n-1}u_0$ is corrected, produces in the limit of error a term $nf(\theta)\,\tfrac{1}{2}\rho$; and the second, if $\Delta^n u_0$ is corrected, produces a term $(n-1-\theta)f(\theta)\,\tfrac{1}{2}\rho$. If, however, $\Delta^n u_0$ is the difference of the corrected values of $\Delta^{n-1}u_1$ and $\Delta^{n-1}u_0$, the two terms of (3) are [§ 17 (i)] together equal to

$$(-)^n f(\theta)\,\{(\theta-n+1)\Delta^{n-1}u_1-(\theta-2n+1)\,\Delta^{n-1}u_0\};$$

and the sum of the corresponding terms in the limit of error for (3δ) will be

$$f(\theta)\,\{(n-1-\theta)+(2n-1-\theta)\}\,\tfrac{1}{2}\rho = nf(\theta)\,\tfrac{1}{2}\rho+2(n-1-\theta)f(\theta)\,\tfrac{1}{2}\rho.$$

Hence the effect of substituting (3δ) for (3γ) would be to double those terms in the limit of error which are due to differences of odd order. On the other hand, the terms for (3δ) are still individually less than for (3α).

The formulæ (20) and (21) differ from (18) and (19) in that they are central to u_0 instead of to $u_{\frac{1}{2}}$, so that the terms which go in pairs are of odd and even order, not of even and odd. In order to keep the limit of error as low as possible, we must obviously use tabular 1st differences for both formulæ, and tabular 2nd difference for the third term in (21). For the remaining terms we have a choice of three systems: (i) all differences above the 2nd corrected; (ii) differences of the 3rd, 5th, ... orders semi-corrected and those of the 4th, 6th, ... orders corrected; (iii) differences of the 3rd, 5th, ... orders corrected and those of the 4th, 6th, ... orders semi-corrected; these systems may be denoted by "γ'," "δ'," and "ϵ." As regards (20), it is clear that (20ϵ) and ($20\gamma'$) are identical with (20γ), while ($20\delta'$) is identical with (20δ); all five systems give $\{1+\theta(1-\theta)\}\,\tfrac{1}{2}\rho$ as due to the first three terms, and for (20γ), ($20\gamma'$), and (20ϵ) the remaining terms are the same as for (20β), while for (20δ) and ($20\delta'$) they are slightly greater. As regards (21), it will be found that (21γ) and (21δ) give the same result as (21β), and ($21\gamma'$) and ($21\delta'$) give the first three terms as in (21α) and the remainder as in (21β); but (21ϵ) differs from ($21\gamma'$) and ($21\delta'$) in that it gets rid of the terms which, in (21β), are due to differences of the 4th, 6th, ... order. Up to differences of an even order, (20ϵ) and (21ϵ) are of course identical.

The more important of these results are shown in the following table. It should be specially remembered that for ($21\delta'$) we start with having all differences of even order corrected, but that we use the tabular value of $\delta^2 u_0$ in the formula, although we use the corrected values of $\delta^2 u_{-1}$ and of $\delta^2 u_1$ for determining $\mu\delta^3 u_0$.

TABLE IV.

(γ = all differences corrected except 1st; δ = 2nd, 4th, 6th, ... differences corrected, others semi-corrected; ϵ = 3rd, 5th, 7th, ... differences corrected, 1st and 2nd tabular, others semi-corrected.)

Formula.	Limit of error ($\div \frac{1}{2}\rho$) due to using						
	u	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.	6th diff.
(3γ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+\frac{\theta(1-\theta)(2-\theta)}{3!}$	$+\frac{\theta(1-\theta)(2-\theta)(3-\theta)}{4!}$	$+\frac{\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)}{5!}$	$+\frac{\theta(1-\theta)\dots(5-\theta)}{6!}$
(3δ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+2\frac{\theta(1-\theta)(2-\theta)}{3!}$	$+\frac{\theta(1-\theta)(2-\theta)(3-\theta)}{4!}$	$+2\frac{\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)}{5!}$	
(16γ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$	$+\frac{(2+\theta)\dots\theta(1-\theta)\dots(3-\theta)}{6!}$
(16δ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+0$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+0$	
(19γ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+\lvert\frac{1}{2}-\theta\rvert\,\frac{\theta(1-\theta)}{3!}$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+\lvert\frac{1}{2}-\theta\rvert\,\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$	
(19δ)	1	$+0$	$+\frac{\theta(1-\theta)}{2!}$	$+0$	$+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}$	$+0$	
(20γ) $(20\gamma')$ (20ϵ)	1	$+\frac{\theta(1-\theta)}{1!}$		$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$		$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$	
$(20\ )$ $(20\delta')$	1	$+\frac{\theta(1-\theta)}{1!}$		$+(1+\frac{1}{2}\theta)\frac{(1+\theta)\,\theta(1-\theta)}{3!}$		$+(1+\frac{1}{3}\theta)\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$	
(21ϵ)	1	$+\frac{\theta}{1!}$	$-\frac{\theta^2}{1!}$	$+\frac{(1+\theta)\,\theta(1-\theta)}{3!}$	$+0$	$+\frac{(2+\theta)(1+\theta)\,\theta(1-\theta)(2-\theta)}{5!}$	$+0$

$(18\gamma) = (18\delta) = (18\beta)$, $(21\gamma) = (21\delta) = (21\beta)$, $(21\gamma') = (21\delta')$, differing from (21β) in having $-\theta^2/1!$ in place of $+\theta^2/2!$.

32. Without going exhaustively into the various possible formulæ, we can check the more important of these results by a method which is an extension of that applied in § 31 to the comparison of (3γ) and (3δ). Suppose that we have a formula which ends at $\Delta^m u_{-p}$ with a corrected difference; let us see how the subsequent differences should be treated.

(i) Let the term in $\Delta^m u_{-p}$ be $f_1(\theta)\,\Delta^m u_{-p}$, so that it introduces into the limit of error a term $|f_1(\theta)|\,\frac{1}{2}\rho$; and let the coefficient of the next difference, which is either $\Delta^{m+1}u_{-p}$ or $\Delta^{m+1}u_{-p-1}$, be $f_2(\theta)$. Then

(α) If this difference also is corrected, it will introduce a term

$$|f_2(\theta)|\,\tfrac{1}{2}\rho.$$

(β) If it is semi-corrected, we have to combine the two terms of the formula; and they will be found to give rise to

$$\{f_1(\theta)-f_2(\theta)\}\,\Delta^m u_{-p}+f_2(\theta)\,\Delta^m u_{-p+1},$$

or to

$$\{f_1(\theta)+f_2(\theta)\}\,\Delta^m u_{-p}-f_2(\theta)\,\Delta^m u_{-p-1},$$

according as the step is forwards or backwards. In the former case the total of the corresponding terms of the limit of error is

$$\{|f_1(\theta)-f_2(\theta)|+|f_2(\theta)|\}\,\tfrac{1}{2}\rho,$$

and in the latter case it is

$$\{|f_1(\theta)+f_2(\theta)|+|f_2(\theta)|\}\,\tfrac{1}{2}\rho.$$

Also (if $0<\theta<1$, and if we start with u_0 or u_1 and Δu_0)

$$|f_1(\theta)|>|f_2(\theta)|.$$

Hence the effect of using the semi-corrected $\Delta^{m+1}u$ will be that the term of the limit of error, due to this term of the formula, will be either $2\,|f_2(\theta)|\,\frac{1}{2}\rho$ or 0; and it will be 0 if $f_1(\theta)$ and $f_2(\theta)$ are of the same sign when we are going forwards, or of different sign when we are going backwards. But

$$f_2(\theta)/f_1(\theta)=(\theta+p)/(m+1)\quad\text{or}\quad(\theta+p-m)/(m+1),$$

according as the step to $\Delta^m u_{-p}$ brought in u_{-p} or u_{-p+m}; *i.e.*, $f_1(\theta)$ and $f_2(\theta)$ are of the same sign or of opposite sign according as the step to $\Delta^m u_{-p}$ was backwards or forwards.

(ii) Hence, if the difference of order m is corrected, and if we have to take in a difference of order $m+1$, the best result is obtained by moving to it in the opposite direction to that in which we last moved, and by expressing the difference of order $m+1$ in terms of (corrected) differences of order m.

(iii) Suppose therefore that the successive differences occurring in the formula are $\Delta^{m-1}u_{-p}$, $\Delta^{m}u_{-p}$, and $\Delta^{m+1}u_{-p-1}$, the last two being respectively corrected and semi-corrected; and consider the effect of taking in another term. The difference introduced will be either $\Delta^{m+2}u_{-p-1}$ or $\Delta^{m+2}u_{-p-2}$, and its coefficient will be

$$f_1(\theta)(\theta+p-m)(\theta+p+1)/\{(m+1)(m+2)\}.$$

(α) If this difference is corrected or semi-corrected, the term introduced into the limit of error will be

$$|f_1(\theta)|(m-p-\theta)(\theta+p+1)/\{(m+1)(m+2\}.\tfrac{1}{2}\rho.$$

Call this λ.

(β) If the difference is based on the differences of order m, it will be found that the total portion of the limit of error, due to the terms of the formula which involve differences of orders m, $m+1$, and $m+2$, will be $|f_1(\theta)|\,\tfrac{1}{2}\rho+2\lambda$, whichever of the two differences of order $m+2$ we take.

We should obtain a similar result by using $\Delta^{m}u_{-p-1}$ instead of $\Delta^{m}u_{-p}$.

(γ) By using the corrected difference we can [see (i), (β)] prevent the next term from increasing the limit of error.

(iv) Hence the best result is obtained by taking the differences alternately corrected and semi-corrected; *i.e.*, by correcting alternate columns of differences, the intermediate columns being found by differencing these.

(v) The fact that after going forwards from $\Delta^{m-1}u_{-p}$ to $\Delta^{m}u_{-p}$, and then backwards to $\Delta^{m+1}u_{-p-1}$, we may again go backwards to $\Delta^{m+2}u_{-p-2}$, does not conflict with the rule that we should go alternately backwards and forwards; for the next step should be to $\Delta^{m+3}u_{-p-2}$. Each pair of steps is from one semi-corrected difference to the next in the same horizontal line, and it is immaterial whether the first step of any pair is backwards or forwards.

33. It has already been mentioned [§ 28] that the series which gives the limit of error for (3β) is convergent; and it follows from §§ 29 and 32 that the corresponding series for the other formulæ we are considering are also convergent. The accuracy of any particular formula can therefore be roughly measured, for purposes of comparison, by the limit of the sum of the series which determines the limit of error. This may be called the "maximum limit of error." It is assumed throughout that $0<\theta<1$, and that the tabular triangle of the formula always includes the interval (0, 1).

(i) For (3β) the series (omitting the factor $\frac{1}{2}\rho$) is

$$1+\theta+\frac{\theta(1-\theta)}{2!}+\ldots+\frac{\theta(1-\theta)\ldots(m-1-\theta)}{m!}+\ldots,$$

the sum of $m+1$ terms of which is

$$2-\frac{(1-\theta)(2-\theta)\ldots(m-\theta)}{m!}.$$

The maximum limit of error is therefore ρ.

(ii) The series for (3γ) only differs from the above in the omission of the second term; the maximum limit of error is therefore $(1-\frac{1}{2}\theta)\rho$.

(iii) The series for (3δ) is obtained from that for (3γ) by doubling the alternate terms. Remembering that

$$2^\theta=1+\theta+\frac{\theta(\theta-1)}{2!}+\ldots+\frac{\theta(\theta-1)\ldots(\theta-m+1)}{m!}+\ldots,$$

it will be found that the maximum limit of error is $(1-\theta+1/2^{2-\theta})\rho$.

(iv) For (18β), (18γ), and (18δ), and also for (16δ) and (19δ), we have a series

$$1+\frac{\theta(1-\theta)}{2!}+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}+\ldots.$$

Writing $\theta\equiv\frac{1}{2}+\psi$, this becomes

$$1+\frac{(\frac{1}{2})^2-\psi^2}{2!}+\frac{\{(\frac{1}{2})^2-\psi^2\}\{(\frac{3}{2})^2-\psi^2\}}{4!}+\ldots.$$

But*

$$\frac{\cos 2\psi a}{\cos a}=1+\frac{(\frac{1}{2})^2-\psi^2}{2!}(2\sin a)^2+\frac{\{(\frac{1}{2})^2-\psi^2\}\{(\frac{3}{2})^2-\psi^2\}}{4!}(2\sin a)^4+\ldots.$$

If, therefore, we take $\sin a=\frac{1}{2}$, $a=\frac{1}{6}\pi$, we find that

$$1+\frac{\theta(1-\theta)}{2!}+\frac{(1+\theta)\,\theta(1-\theta)(2-\theta)}{4!}+\ldots=\frac{2}{\sqrt{3}}\cos\tfrac{1}{3}(\theta-\tfrac{1}{2})\,\pi\,;$$

so that the maximum limit of error for any of these formulæ is

$$\frac{1}{\sqrt{3}}\cos\tfrac{1}{3}(\theta-\tfrac{1}{2})\,\pi\,.\,\rho.$$

The limit for (19β) or (19γ) need not be considered, since we should generally use (19δ) in preference.

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* E. W. Hobson, *Plane Trigonometry*, p. 265, formula (7).

(iv) The above series forms part of (16β). The remainder of (16β) is a series

$$\frac{\theta}{1!}+\frac{\theta(1^2-\theta^2)}{3!}+\frac{\theta(1^2-\theta^2)(2^2-\theta^2)}{5!}+\ldots,$$

which also occurs in (20β) and forms part of (21β) and (21ϵ). But*

$$\frac{\sin 2\theta a}{\cos a}=\theta\,.\,2\sin a+\frac{\theta(1^2-\theta^2)}{3!}(2\sin a)^3+\frac{\theta(1^2-\theta^2)(2^2-\theta^2)}{5!}(2\sin a)^5+\ldots;$$

and therefore

$$\frac{\theta}{1!}+\frac{\theta(1^2-\theta^2)}{3!}+\frac{\theta(1^2-\theta^2)(2^2-\theta^2)}{5!}+\ldots=\frac{2}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi.$$

The maximum limit of error for (16β) is therefore

$$\frac{1}{\sqrt{3}}\left\{\cos\tfrac{1}{3}(\theta-\tfrac{1}{2})\pi+\sin\tfrac{1}{3}\theta\pi\right\}\rho=\cos\tfrac{1}{3}(1-\theta)\pi\,.\,\rho.$$

For (16γ) it is

$$\left\{\cos\tfrac{1}{3}(1-\theta)\pi-\tfrac{1}{2}\theta\right\}\rho;$$

for (20β) it is

$$\left(\tfrac{1}{2}+\frac{1}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi\right)\rho;$$

and for (20γ), $(20\gamma')$, (20ϵ), and (21ϵ) it is

$$\left\{\tfrac{1}{2}(1-\theta^2)+\frac{1}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi\right\}\rho.$$

(v) In (21β), (21γ), and (21δ), we have the series considered in (iv), and also a series

$$1+\frac{\theta^2}{2!}+\frac{\theta^2(1^2-\theta^2)}{4!}+\frac{\theta^2(1^2-\theta^2)(2^2-\theta^2)}{6!}+\ldots,$$

which may similarly be shown to be equal to $2-\cos\tfrac{1}{3}\theta\pi$. Hence the maximum limit for any of these three formulæ is

$$\left(1-\tfrac{1}{2}\cos\tfrac{1}{3}\theta\pi+\frac{1}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi\right)\rho;$$

and for $(21\gamma')$ and $(21\delta')$ it is

$$\left(1-\tfrac{3}{4}\theta^2-\tfrac{1}{2}\cos\tfrac{1}{3}\theta\pi+\frac{1}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi\right)\rho.$$

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* *Ibid.*, formula (8).

For (20δ) and ($20\delta'$) the maximum limit will be found to be

$$\left(\tfrac{3}{2}-\theta^2-\cos\tfrac{1}{3}\theta\pi+\frac{1}{\sqrt{3}}\sin\tfrac{1}{3}\theta\pi\right)\rho=\left\{\tfrac{3}{2}-\theta^2-\frac{2}{\sqrt{3}}\sin\tfrac{1}{3}(1-\theta)\pi\right\}\rho.$$

The following table gives the values of the various maximum limits mentioned above, for $\theta = {\cdot}1, {\cdot}2, \ldots, {\cdot}9$. The first portion of the table relates to the systems that might be used with advancing differences, the second portion to central-difference systems that give a low limit of error, and the third to central-difference systems that are inconvenient as giving too high a limit of error, at any rate in the further portion of the interval.

TABLE V.

Formula and system.	Maximum limit of error ($\div\rho$) for θ =								
	·1	·2	·3	·4	·5	·6	·7	·8	·9
(3β)	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
(3γ)	·950	·900	·850	·800	·750	·700	·650	·600	·550
(3δ)	1·168	1·088	1·008	·930	·854	·779	·707	·636	·567
(16δ), (18β), (18γ), (18δ), (19δ)	·528	·550	·565	·575	·578	·575	·565	·550	·528
(16γ)	·538	·570	·594	·610	·617	·614	·602	·579	·545
(20γ), ($20\gamma'$) (20ϵ), (21ϵ)	·556	·601	·634	·655	·664	·660	·642	·610	·563
($21\gamma'$) ($21\delta'$)	·556	·601	·636	·659	·669	·665	·648	·615	·566
(20δ), ($20\delta'$)	·556	·602	·638	·662	·673	·671	·654	·620	·570
(20β)	·561	·621	·679	·735	·789	·840	·887	·930	·968
(21β), (21γ), (21δ)	·564	·631	·703	·779	·856	·935	1·015	1·095	1·174
(16β)	·588	·670	·744	·810	·867	·914	·952	·979	·995

34. We are now in a position to draw general conclusions as to the applicability of corrected differences to the various formulæ.

(A) For sub-tabulation (construction of tables with smaller intervals) the central-difference formula in Everett's form is still the best, and is improved by using corrected differences. The improvement, however, is not very striking in ordinary cases, since the limit of error with tabular differences is itself small, even when an appreciable number of terms is taken. This is shown by the following table.

TABLE VI.

Differences represented by last pair of terms.	System of differences.	Limit of error ($\div \rho$) due to using Everett's formula, for $\theta =$								
		·1	·2	·3	·4	·5	·6	·7	·8	·9
1st	Tabular ...	·500	·500	·500	·500	·500	·500	·500	·500	·500
	Corrected...	·500	·500	·500	·500	·500	·500	·500	·500	·500
2nd and 3rd	Tabular ...	·545	·580	·605	·620	·625	·620	·605	·580	·545
	Corrected...	·523	·540	·553	·560	·563	·560	·553	·540	·523
4th and 5th	Tabular ...	·569	·624	·664	·688	·696	·688	·664	·624	·569
	Corrected...	·527	·548	·563	·572	·575	·572	·563	·548	·527
6th and 7th	Tabular ...	·585	·653	·704	·734	·745	·734	·704	·653	·585
	Corrected...	·528	·549	·565	·574	·577	·574	·565	·549	·528
8th and 9th	Tabular ...	·597	·676	·734	·770	·782	·770	·734	·675	·597
	Corrected...	·528	·549	·565	·575	·578	·575	·565	·549	·528

Since differences of even order only are used in the formula, it is immaterial whether the differences of odd order are corrected or not.

(B) For isolated interpolations the advancing-difference (or receding-difference) formula is so much improved by the use of corrected differences as to be comparable, so far as the limit of error is concerned, with the central-difference formulæ; and it may in certain cases be found more convenient for use. The following rules may be laid down.

(i) Whatever formula is adopted, the tabular 1st difference must be used, not the corrected 1st difference.

(ii) If we have only to go to 2nd differences, it does not very much matter whether we use the tabular or the corrected 2nd difference. By a proper use of the corrected difference we can reduce the limit of error from $\{1+\theta(1-\theta)\}\,\frac{1}{2}\rho$ to $\{1+\frac{1}{2}\theta(1-\theta)\}\,\frac{1}{2}\rho$; but the difference between these is at most (viz., for $\theta = {}$·5) the difference between ·6250ρ and ·5625ρ. If we do use the corrected difference, we see from Tables III and IV that formulæ (20) and (21) are not appropriate; we can use either

$$(3)\quad u_\theta = u_0+\theta\delta u_{\frac{1}{2}}+\tfrac{1}{2}\theta\,(\theta-1)\,\delta^2 u_1,$$

or

$$(16)\quad u_\theta = u_0+\theta\delta u_{\frac{1}{2}}+\tfrac{1}{2}\theta\,(\theta-1)\,\delta^2 u_0,$$

or the mean of these, viz.,

$$(19)\quad u_\theta = \mu u_{\frac{1}{2}} - (\tfrac{1}{2}-\theta)\,\delta u_{\frac{1}{2}} + \tfrac{1}{2}\theta(\theta-1)\,\mu\delta^2 u_{\frac{1}{2}}.$$

(iii) If we have to go up to differences of an order higher than the 2nd, it will usually, as explained at the end of § 5, be found most convenient to express differential coefficients in terms of differences, and then use Taylor's theorem.

(*a*) For advancing differences the theoretically best result is obtained by having all differences after the 1st corrected, though practically (see Table VII below) we obtain nearly as good a result if those of even order only are corrected. In either case, we should use Taylor's theorem in the form given in § 5, viz.,

$$u_\theta = u_0 + \theta h u_0' + \frac{\theta^2}{2!}h^2 u_0'' + \frac{\theta^3}{3!}h^3 u_0''' + \dots \tag{25}$$

$$= u_0 + \theta\left[h u_0' + \tfrac{1}{2}\theta\left\{h^2 u_0'' + \tfrac{1}{3}\theta(h^3 u_0''' + \dots)\right\}\right]; \tag{25A}$$

and the values of hu_0', h^2u_0'', h^3u_0''', ..., would be given by

$$\left.\begin{aligned}
hu_0' &= (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \tfrac{1}{5}\Delta^5 - \tfrac{1}{6}\Delta^6 + \tfrac{1}{7}\Delta^7 - \tfrac{1}{8}\Delta^8 + \dots)\,u_0 \\
h^2u_0'' &= (\Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \tfrac{5}{6}\Delta^5 + \tfrac{137}{180}\Delta^6 - \tfrac{7}{10}\Delta^7 + \tfrac{363}{560}\Delta^8 - \dots)\,u_0 \\
h^3u_0''' &= (\Delta^3 - \tfrac{3}{2}\Delta^4 + \tfrac{7}{4}\Delta^5 - \tfrac{15}{8}\Delta^6 + \tfrac{29}{15}\Delta^7 - \tfrac{469}{240}\Delta^8 + \dots)\,u_0 \\
h^4u_0^{\text{iv}} &= (\Delta^4 - 2\Delta^5 + \tfrac{17}{6}\Delta^6 - \tfrac{7}{2}\Delta^7 + \tfrac{967}{240}\Delta^8 - \dots)\,u_0 \\
h^5u_0^{\text{v}} &= (\Delta^5 - \tfrac{5}{2}\Delta^6 + \tfrac{25}{6}\Delta^7 - \tfrac{35}{6}\Delta^8 + \dots)\,u_0 \\
h^6u_0^{\text{vi}} &= (\Delta^6 - 3\Delta^7 + \tfrac{23}{4}\Delta^8 - \dots)\,u_0 \\
h^7u_0^{\text{vii}} &= (\Delta^7 - \tfrac{7}{2}\Delta^8 + \dots)\,u_0 \\
h^8u_0^{\text{viii}} &= (\Delta^8 - \dots)\,u_0 \\
&\dots .
\end{aligned}\right\} . \tag{86}$$

We might in particular cases find it more convenient to use receding differences, working backwards from u_1, and making the necessary changes of sign of alternate columns of differences. The calculations of the differential coefficients, as shown by (86), are, however, more troublesome than the corresponding calculations for central differences.

(*b*) For central differences the standard formula (16) has to be adapted in a similar way. One method of doing this has been explained in § 5; the formula (21) is converted into (25) by using the values of the differential coefficients as given by (26). It should, however, have been mentioned that instead of (21) we might use (19). Writing $\theta \equiv \frac{1}{2}+\psi$, we obtain (23), which can be put in the form

$$u_\theta = u_{\frac{1}{2}}+\psi h u'_{\frac{1}{2}}+\frac{\psi^2}{2!}h^2u''_{\frac{1}{2}}+\frac{\psi^3}{3!}h^3u'''_{\frac{1}{2}}+\dots \tag{87}$$

$$= u_{\frac{1}{2}}+\psi\left[hu'_{\frac{1}{2}}+\tfrac{1}{2}\psi\left\{h^2u''_{\frac{1}{2}}+\tfrac{1}{3}\psi(h^3u'''_{\frac{1}{2}}+\dots)\right\}\right]; \tag{87A}$$

the values of $u_{\frac{1}{2}}$, $hu'_{\frac{1}{2}}$, $h^2u''_{\frac{1}{2}}$, $h^3u'''_{\frac{1}{2}}$, ... being given by*

$$\left.\begin{aligned}
u_{\frac{1}{2}} &= (\mu-\tfrac{1}{8}\mu\delta^2+\tfrac{3}{128}\mu\delta^4-\tfrac{5}{1024}\mu\delta^6+\tfrac{35}{32768}\mu\delta^8-\dots)\,u_{\frac{1}{2}}\\
hu'_{\frac{1}{2}} &= (\delta-\tfrac{1}{24}\delta^3+\tfrac{3}{640}\delta^5-\tfrac{5}{7168}\delta^7+\dots)\,u_{\frac{1}{2}}\\
h^2u''_{\frac{1}{2}} &= (\mu\delta^2-\tfrac{5}{24}\mu\delta^4+\tfrac{259}{5760}\mu\delta^6-\tfrac{3229}{322560}\mu\delta^8+\dots)\,u_{\frac{1}{2}}\\
h^3u'''_{\frac{1}{2}} &= (\delta^3-\tfrac{1}{8}\delta^5+\tfrac{37}{1920}\delta^7+\dots)\,u_{\frac{1}{2}}\\
h^4u^{\text{iv}}_{\frac{1}{2}} &= (\mu\delta^4-\tfrac{7}{24}\mu\delta^6+\tfrac{47}{640}\mu\delta^8-\dots)\,u_{\frac{1}{2}}\\
h^5u^{\text{v}}_{\frac{1}{2}} &= (\delta^5-\tfrac{5}{24}\delta^7+\dots)\,u_{\frac{1}{2}}\\
h^6u^{\text{vi}}_{\frac{1}{2}} &= (\mu\delta^6-\tfrac{3}{8}\mu\delta^8+\dots)\,u_{\frac{1}{2}}\\
&\dots
\end{aligned}\right\}. \tag{88}$$

As compared with (21), this method has, as will be seen from Tables III and IV, the theoretical advantage that it is immaterial whether we end with a difference of odd or of even order; but this advantage, as will be seen below, is very slight in comparison with the greater complication of the coefficients in (88). It is therefore only for values of θ very near $\frac{1}{2}$ that this method will be found useful. In ordinary cases we should use (21); taking care, with corrected differences, that the term in the limit of error which is due to the 2nd difference is $-\theta^2.\frac{1}{2}\rho$, not $+\frac{1}{2}\theta^2.\frac{1}{2}\rho$, *i.e.*, that we use ($21\gamma'$), ($21\delta'$), or ($21\epsilon$).

The limits of error for the different methods, for formulæ which go up to differences not exceeding the 6th, are shown for $\theta = \cdot 1, \cdot 2, \dots, \cdot 9$ in the following table.

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* *Proc. London Math. Soc.*, Vol. XXXI, p. 465, formulæ (75).

TABLE VII.

Highest order of differences included in formula.	Formula and system.	Limit of error ($\div \rho$) for θ =								
		·1	·2	·3	·4	·5	·6	·7	·8	·9
3rd	(3α)	·631	·724	·784	·812	·813	·788	·742	·676	·595
	(3γ)	·537	·564	·583	·592	·594	·588	·576	·556	·531
	(3δ)	·551	·588	·612	·624	·625	·616	·598	·572	·539
	(16α) or (19α)	·545	·580	·605	·620	·625	·620	·605	·580	·545
	(16δ) or (19δ)	·523	·540	·553	·560	·563	·560	·553	·540	·523
	(21α)	·570	·628	·674	·704	·719	·716	.695	·652	·588
	$(21\gamma')$ or $(21\delta')$	·554	·596	·628	·648	·657	·652	·635	·604	·560
	(21ϵ)	·554	·596	·628	·648	·657	·652	·635	·604	·560
4th	(3α)	·776	·960	1·065	1·104	1·086	1·024	·925	·800	·656
	(3γ)	·548	·581	·603	·613	·614	·605	·589	·565	·536
	(3δ)	·562	·605	·633	·645	·645	·633	·612	·581	·544
	(16α) or (19α)	·569	·624	·664	·688	·696	·688	·664	·624	·569
	(16δ) or (19δ)	·527	·548	·563	·572	·575	·572	·563	·548	·527
	(21α)	·569	·624	·664	·688	·696	·688	·664	·624	·569
	$(21\gamma')$ or $(21\delta')$	·554	·597	·630	·651	·661	·657	·640	·609	·563
	(21ϵ)	·554	·596	·628	·648	·657	·652	·635	·604	·560
5th	(3α)	1·017	1·343	1·511	1·553	1·497	1·366	1·184	1·469	1·236
	(3γ)	·556	·594	·618	·628	·627	·617	·597	·571	·538
	(3δ)	·578	·631	·662	·675	·672	·656	·629	·593	·549
	(16α) or (19α)	·569	·624	·664	·688	·696	·688	·664	·624	·569
	(16δ) or (19δ)	·527	·548	·563	·572	·575	·572	·563	·548	·527
	(21α)	·585	·655	·708	·741	·754	·746	·716	·664	·592
	$(21\gamma')$ or $(21\delta')$	·556	·600	·634	·657	·667	·663	·646	.613	·565
	(21ϵ)	·555	·600	·633	·654	·663	·658	·640	·609	·562
6th	(3α)	1·425	1·976	2·233	2·265	2·132	1·886	1·568	1·213	1·350
	(3γ)	·562	·604	·629	·640	·638	·625	·604	·575	·540
	(3δ)	·585	·641	·674	·687	·683	·665	·635	·597	·551
	(16α) or (19α)	·585	·653	·704	·734	·745	·734	·704	·653	·585
	(16δ) or (19δ)	·528	·549	·565	·574	·577	·574	·565	·549	·528
	(21α)	·585	·653	·704	·734	·745	·734	·704	·653	·585
	$(21\gamma')$ or $(21\delta')$	·556	·601	·635	·657	·667	·664	·646	·614	·566
	(21ϵ)	·555	·600	·633	·654	·663	·658	·640	·609	·562

It will be seen that the difference between (3γ) and (3δ) is very slight; and that there is not really much to choose between either of these and a central-difference formula with tabular or with corrected differences. If all the differences (after the 1st) are corrected, several courses are open to us. We may take the differences as they stand, using either (3γ) or (19γ): we may replace the 2nd difference by its tabular value, and use $(21\gamma')$; we may, in addition to this, replace the 4th, 6th, ... differences by their semi-corrected values, and use (21ϵ); or, retaining the corrected values of the 2nd, 4th, 6th, ... differences, we may replace the 3rd, 5th, 7th, ... differences by their semi-corrected values, and use (19δ). If only differences of even order are corrected, we can use either (3δ), (19δ), or $(21\delta')$; remembering that with (21δ') the 2nd difference actually used in the third term of the formula is to have its tabular value. The choice of a method, in any of these cases, is largely a matter of individual preference.

35. In the above investigation we have not taken into account the additional information with regard to the values of u that is supplied by the corrected differences. The utilisation of this information for the purpose of improving the original table as a whole belongs rather to the theory of smoothing, but it is as well to see how its existence affects our formulæ.

(i) It should be noted in the first place that any smoothing that is done will be based rather on the differences of low order, and particularly on the 1st differences, than on those of high order. We can see this by reference to an individual case, such as the following :—

x	tan x	Tabular differences.			Corrected differences.		
		Δ	Δ^2	Δ^3	Δ	Δ^2	Δ^3
		+	+	+	+	+	+
46°	1·03553						
		3684			3684		
47°	1·07237		140			141	
		3824		12	3824		11
48°	1·11061		152			151	
		3976		10	3976		12
49°	1·15037		162			163	
		4138		15	4139		13
50°	1·19175		177			176	
		4315		12	4314		14
51°	1·23490		189			190	
		4504			4504		
52°	1·27994						

Here, although the contrast between the tabular and the corrected differences is less striking for the 1st differences than for the 3rd differences, it is really more important for the former than for the latter; for we could easily obtain the more accurate differences of the higher order either by smoothing the tabular differences or (which comes to very much the same thing) by working from a table in which the intervals of x are greater.

(ii) Further, we cannot alter the determining values of one interval, in any scientific manner, without considering them in relation to the adjoining values, *i.e.*, without considering the series of values of u as a whole; and that is too complicated a matter to be considered here. It is true that, in the particular example, the two corrected 1st differences which differ from the tabular differences are consecutive, so that we might adjust the table by a guess-work alteration of tan 50°; but that is an accident. We must therefore assume that we start with the values of u as given in the table.

(iii) Now consider the case of corrected 1st differences. With the notation of § 2, the errors in the tabulation of u_0 and of u_1 are a_0 and a_1, the tabulated values being $U_0 \equiv u_0+a_0$ and $U_1 \equiv u_1+a_1$. The true value of u_1-u_0 is therefore $U_1-U_0-(a_1-a_0)$. The tabular 1st difference of u_0 and u_1 is U_1-U_0; and the corrected 1st difference will be $U_1-U_0+\rho$, U_1-U_0, or $U_1-U_0-\rho$, according as a_1-a_0 lies between $-\rho$ and $-\frac{1}{2}\rho$, between $-\frac{1}{2}\rho$ and $+\frac{1}{2}\rho$, or between $+\frac{1}{2}\rho$ and $+\rho$.

(*a*) Taking the second case first, suppose that the tabular difference and the corrected difference are the same. Then, although we have some information as to the value of a_1-a_0, we have really no more information as to the value of

$$a_0+\theta(a_1-a_0) \equiv (1-\theta)\,a_0+\theta a_1$$

than we should have had if we had been dealing with a table showing tabular differences only; for the data are quite consistent with the possibility of both U_0 and U_1 having a large error in the same direction. It might, for instance, be the case that $a_0 = a_1 = \cdot 499\rho$, and this would then be also the value of $a_0+\theta(a_1-a_0)$. The knowledge that the tabular difference and the corrected difference are the same does not therefore enable us to reduce the limit of error.

(*b*) Next, suppose that, as in the interval (49°, 50°) in the above example, a_1-a_0 lies between $-\rho$ and $-\frac{1}{2}\rho$, so that the corrected 1st difference exceeds the tabular 1st difference by ρ.

(1) We note in the first place that this means that a_0 is positive and a_1 negative, so that for interpolation towards the middle of the interval the errors will tend to counteract each other. This also applies if a_1-a_0 lies between $+\frac{1}{2}\rho$ and $+\rho$. It is therefore in those cases in which the tabular and the corrected difference are different that the ordinary formula, using the tabular difference, produces a specially good result.

(2) Now consider the actual limits of error of the result of the ordinary formula, using the corrected difference. Writing $a_1 \equiv -a_1'$, so that a_0 and a_1' are both positive, we shall have as the result of the formula (differences after the 1st being ignored)

$$u_\theta = U_0+\theta(U_1-U_0+\rho) = u_0+\theta(u_1-u_0)+a_0+\theta(-a_1'-a_0+\rho)$$

$$= u_0+\theta(u_1-u_0)+(1-\theta)\,a_0-\theta a_1'+\theta\rho\,; \quad (89)$$

and this will lie between

$$u_0+\theta(u_1-u_0)+\theta.\tfrac{1}{2}\rho \quad \text{and} \quad u_0+\theta(u_1-u_0)+(1+\theta)\tfrac{1}{2}\rho.$$

Hence the limits of error, due to using the corrected difference, are $+\theta.\frac{1}{2}\rho$ and $+(1+\theta)\frac{1}{2}\rho$.

(*c*) Similarly, if the corrected 1st difference is less than the tabular 1st difference, so that a_0 is shown to be negative and a_1 positive, the limits of error, due to using the corrected difference, are $-(1+\theta)\frac{1}{2}\rho$ and $-\theta.\frac{1}{2}\rho$.

(*d*) Comparing the three cases, we see that, while the error for 1st-difference interpolation with corrected difference may reach $-R$ on the negative side, and may reach $+R$ on the positive side, where [see § 27] $R=(1+\theta)\frac{1}{2}\rho$, it will never in any particular case have so great a range as from $-R$ to $+R$. In one of the three classes of possible cases its range will be from $-\frac{1}{2}\rho$ to $+\frac{1}{2}\rho$, in another from $R-\frac{1}{2}\rho$ to R, and in the third from $-R$ to $-R+\frac{1}{2}\rho$.

(*e*) Further, while the length of the range in the first class of cases is ρ (as it is whenever we use the tabular difference), in the other two it is only $\frac{1}{2}\rho$. Hence we can reduce the limit of error in these latter cases to $\pm\frac{1}{4}\rho$ by using a formula which is midway between the two limiting values. We should therefore, if the corrected 1st difference is greater than the tabular 1st difference, take

$$u_\theta = U_0+\theta(U_1-U_0)+(\theta-\tfrac{1}{2})\tfrac{1}{2}\rho \tag{90}$$

$$= (U_0-\tfrac{1}{4}\rho)+\theta\{(U_1+\tfrac{1}{4}\rho)-(U_0-\tfrac{1}{4}\rho)\}; \tag{90A}$$

and, if the corrected 1st difference is less than the tabular 1st difference,

$$u_\theta = U_0+\theta(U_1-U_0)-(\theta-\tfrac{1}{2})\tfrac{1}{2}\rho \tag{91}$$

$$= (U_0+\tfrac{1}{4}\rho)+\theta\{(U_1-\tfrac{1}{4}\rho)-(U_0+\tfrac{1}{4}\rho)\}; \tag{91A}$$

and in each case the limits of resulting error will be $\pm\frac{1}{4}\rho$. We might have obtained these results more directly from the information given as to the signs of a_0 and a_1. In the first of the two cases, for instance, a_0 is positive and a_1 negative, so that u_0 lies between $U_0-\frac{1}{2}\rho$ and U_0, and u_1 between U_1 and $U_1+\frac{1}{2}\rho$; and therefore by replacing U_0 and U_1 by $U_0-\frac{1}{4}\rho$ and $U_1+\frac{1}{4}\rho$, we have values which are within $\pm\frac{1}{4}\rho$ of the true values, and the limits of error for u_θ, if we interpolate by means of the tabular difference of these, will also be $\pm\frac{1}{4}\rho$.

(iv) It would be possible to extend this method to differences of the

2nd and higher orders, and to lay down rules for adjusting the values of u (within the tabular triangle of the particular formula) by means of the tabular differences in such a way as to enable us to make further reductions in the limits of error of the formula. The number of different cases would, however, increase considerably; and, unless we set to work to smooth the table as a whole, it seems simplest to confine the method to 1st differences. We can, of course, after having adjusted any values of u, continue with corrected differences as they appear in the table; the terms in the general expression for the limit of error, which are due to the terms involving these differences in the formula, will not be affected by the reduction (by $\frac{1}{4}\rho$) of the portion due to the first two terms.

VI. *The Arrangement of Mathematical Tables for Interpolation.*

36. The question then arises whether, in setting out a mathematical table, the tabular or the corrected differences should be given.

(i) The entry of the corrected differences does not involve any additional trouble, except as regards a limited number of values. To obtain a table to (say) 7 places of decimals, we must work to 9 or 10 places and must check our results by differencing; and we can correct the final figures of the differences in the same way that we correct those of the u's. The only question is with regard to those cases in which the corrected final figure of the difference is doubtful. To make the table perfect, all such cases should be specially treated, and this certainly increases the labour, especially if we go to differences of a high order. But in practice it is hardly worth while doing this, since the portions of the limit of error which are due to terms involving high differences are relatively small. A table in which all the values of u are properly corrected in the final figure, and the limits of error of the differences are stated, may be regarded as a good table.

(ii) As regards appearance, each kind of table has its disadvantages, but these are largely a matter of habit. On the corrected-difference system, the sequence of differences of a high order is more regular, but the discrepancies between values in successive columns may be disturbing.

(iii) The special advantage of the corrected-difference system is that it gives additional information. But it has been shown [§ 35] that, so far as the tabulated values are concerned, important information is only given in those cases where the tabular and the corrected 1st differences are different, and [§ 34] that, so far as interpolated values are concerned, the increase of accuracy due to corrected differences above the second, though appreciable, is not very great, if we use central-difference formulæ.

(iv) Moreover, we have found that, where the corrected 1st difference gives information as to the values of the u's, it only does so indirectly, by showing the signs of the errors. This information might be given directly, not merely for these particular u's but for all the u's in the table, by the system, sometimes adopted, of marking the final figure in those cases in which it has been increased. By paying attention to these marks we can at once reduce the limit of error of each u from $\pm\frac{1}{2}\rho$ to $\pm\frac{1}{4}\rho$.

(v) On the other hand, the system of tabular differences, besides being simpler in itself, has the special advantage of enabling us to check printers' errors, and also, in the preparation of the table for printing, to check errors of transcription. This, however, applies mainly to pairs of successive columns, and we should equally be able to check printers' errors if we had corrected and semi-corrected differences alternately.

(vi) It should be remembered that there are really two classes of mathematical tables, viz., those intended for general use, and those that will be used mainly by the specialist. In the former, the number of tabulated values is as great as possible, so as to avoid the necessity of using differences of a high order, while yet retaining a greater degree of accuracy in the tabulated values than will ordinarily be required; in the latter, considerations of economy prevent the preparation and publication of very full tables, the degree of accuracy of such tables as are published is often barely sufficient, and there is not the same objection to special methods of tabulation.

(vii) On the whole, the conclusion seems to be that in tables intended for general use, and not going beyond (say) 2nd differences, it is best to use tabular differences throughout; but that in special tables, where we go up to differences of a relatively high order, it is best to give the corrected values of the differences of even order, the tabular 1st differences, and the semi-corrected values of the remainder, and to mark those values of u in which the final figure has been altered by correction, *i.e.*, in which the tabular error of U is positive.

(viii) It is, of course, necessary that in all cases the degree of accuracy should be correctly stated, either directly or by implication. If no such statement is made, it is to be assumed that the final figure, where it professes to be corrected, is really properly corrected. Unfortunately, it cannot be certain that this is always done; there are probably a good many tables in existence which are ostensibly correct in the final figures, but in which the values have not been specially tested in doubtful cases.

A CARTESIAN THEORY OF COMPLEX GEOMETRICAL ELEMENTS OF SPACE

By G. B. MATHEWS.

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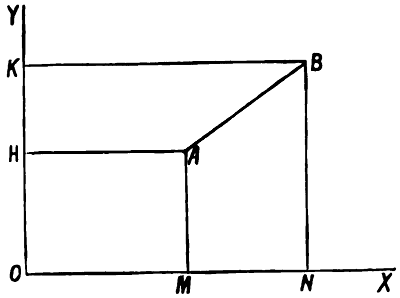
1. The object of this paper is to explain a theory of complex Cartesian coordinates in agreement with the following postulates :—

(i) A point $(a+ci, b+di)$ shall be an actual geometrical entity, reducing to the ordinary point (a, b) when c and d vanish, and similarly for other elements with complex coefficients.

(ii) The ordinary formulæ of analytical geometry shall be assumed to be valid, and lead to extended definitions of "distance," "area," and so on. These derivative quantities shall be expressible in an explicit geometrical form; that is to say, if any one of them be $\alpha+\beta i$, then α, β shall be actual geometrical quantities.



2. Taking rectangular axes OX, OY, the point $(a+ci, b+di)$ will be defined as the rectilinear segment drawn from (a, b) to $(a+c, b+d)$ and denoted by (AB), or, if convenient, by a single letter such as Z. The point (AB) must be distinguished from (BA).

If, in the above figure, $(AB) = (a+ci, b+di)$, then

$$(MN) = (a+ci, 0), \quad (HK) = (0, b+di),$$

$$(BA) = (a+c-ci, b+d-di), \quad (OA) = (ai, bi),$$

and so on.

Both (AB) and (BA) lie on the real line

$$dx-cy = ad-bc. \tag{1}$$

The conjugate to (AB) is (AB'), where the segment AB' is equal and opposite to AB.

3. *Distance.*—If (AB), (CD) are two complex points, and $\overline{AB.CD}$ denotes the distance between them, as defined by the usual formula, it is found that

$$\begin{aligned}\overline{AB.CD}^2 = {} & (AD^2+BC^2-AB^2-CD^2-BD^2) \\ & +(AD^2+BC^2-AB^2-CD^2-2AC^2)\,i, \end{aligned}\tag{2}$$

and from this various particular cases may be derived, of which the following are the most noteworthy:—

$$\left.\begin{aligned}\overline{AB.AC}^2 &= -BC^2 = i^2BC^2 \\ \overline{AC.BC}^2 &= -2iAB^2 = (1-i)^2AB^2 \\ \overline{AB.BA}^2 &= -(3+4i)AB^2 = (1-2i)^2AB^2\end{aligned}\right\}. \tag{3}$$

4. For a moment, let β, γ, δ denote the vectors AB, AC, AD; then

$$\delta^2+(\beta-\gamma)^2-\beta^2-(\gamma-\delta)^2-(\beta-\delta)^2 = S(\delta-\beta)(2\gamma-\delta+\beta),$$

$$\delta^2+(\beta-\gamma)^2-\beta^2-(\gamma-\delta)^2-2\gamma^2 = 2S\gamma(\delta-\beta-\gamma),$$

so we may write

$$\overline{AB.CD}^2 = S(\beta-\delta)(\beta+2\gamma-\delta)+2iS(\beta+\gamma-\delta)\gamma.$$

If the distance vanishes

$$S(\beta-\delta)(\beta+2\gamma-\delta) = S(\beta+\gamma-\delta)\,\gamma = 0,$$

and conversely. The latter condition expresses the fact that AC is perpendicular to the line joining the middle points of AD, BC; while the other may be replaced by $(\beta-\delta)^2 = 2\gamma^2$, which means that $BD^2 = 2AC^2$. Thus we have found the exact geometrical interpretation of two complex points being at zero distance apart.

In the case of a real point O and a complex point AB it is found that

$$\overline{O.AB}^2 = (OA^2-AB^2)+(OB^2-OA^2-AB^2)\,i, \tag{4}$$

and if this vanishes, $OA = AB$ and $OAB = \frac{1}{2}\pi$, a result which may be otherwise obtained.

5. Let Z_1, Z_2 be any two complex points, with coordinates $(a+ci, b+di)$, $(a'+c'i, b'+d'i)$, and let k be any real or complex number. Then putting

$$\xi = (1-k)(a+ci)+k(a'+c'i),$$

$$\eta = (1-k)(b+di)+k(b'+d'i),$$

we shall say that (ξ, η) divides Z_1Z_2 in the ratio $k/(1-k)$. It is convenient to put

$$k = m-ti, \quad 1-k = n+ti,$$

where m, n, t are real, and $m+n = 1$.

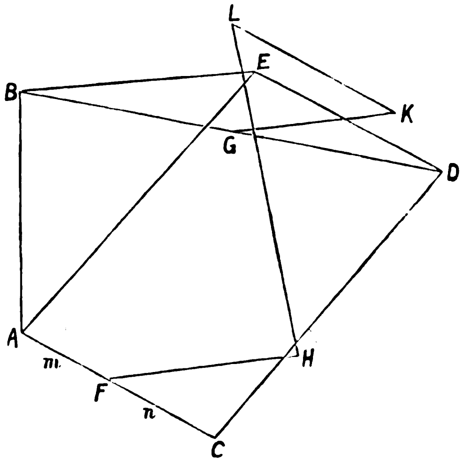
Let (AB), (CD) represent the points Z_1, Z_2. First suppose that $t = 0$, so that k is real. Then if H, L divide AC, BD respectively in the ratio $m : n$, the point (HL) divides Z_1Z_2 in the same ratio.

Now suppose t is not zero. We have

$$\begin{aligned} \xi &= (n+ti)(a+ci)+(m-ti)(a'+c'i) \\ &= (na+ma')+t(c'-c)+i\{nc+mc'+t(a-a')\}, \end{aligned}$$

$$\eta = (nb+mb')+t(d'-d)+i\{nd+md'+t(b-b')\},$$

whence we derive the following construction.



Complete the parallelogram $ACDE$, and let F, G divide AC, BD in the ratio $m : n$. Draw FH, GK each equal to $t.BE$ and parallel to BE, and KL parallel to CA, and of length $t.CA$. Then (HL) divides Z_1Z_2 in the ratio $(m-ti):(n+ti)$.

Conversely, if (HL) is on the segment Z_1Z_2, we can find the ratio of division by drawing HF parallel to BE, meeting AC in F; then the ratio of division is

$$\left(\frac{AF}{AC}-i\frac{FH}{BE}\right):\left(\frac{FC}{AC}+i\frac{FH}{BE}\right).$$

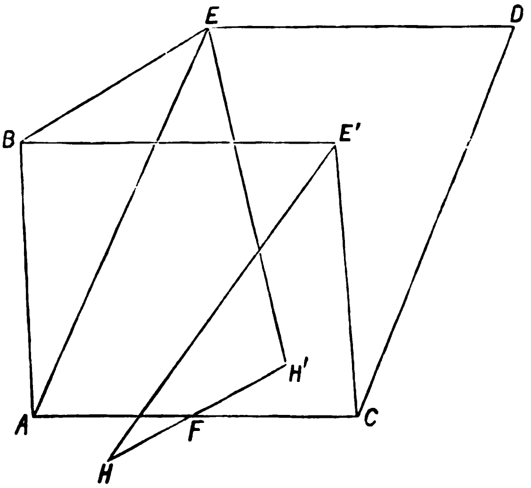
6. An interesting special case is when the ratio of division is $\pm i$. Putting

$$m-ti=\pm i(n+ti),$$

we find that

$$m=\mp t=n=\tfrac{1}{2},$$

and the figure becomes



This shows that, after completing the parallelograms $ACDE$, $ACE'B$, bisecting AC in F, and making $HF=FH'=\frac{1}{2}BE$, the points of division in question are represented by (HE') and $(H'E)$ respectively.

The four points (AB), (CD), (HE'), $(H'E)$ form a harmonic range of a special kind. Namely, since the equations

$$\xi=\frac{x+ix'}{1+i},\quad \xi'=\frac{x-ix'}{1-i}$$

lead to

$$x=\frac{\xi-i\xi'}{1-i},\quad x'=\frac{\xi+i\xi'}{1+i},$$

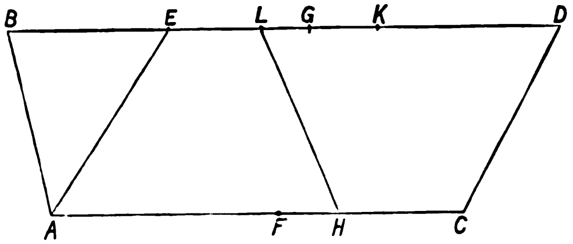
it follows that (CD), (AB) divide $[(HE'),(H'E)]$ in the ratios i and $-i$ respectively. This is easily confirmed from the figure, because $HH'DE'$, $HH'EB$ are parallelograms, and AC is equal and parallel to ED.

7. Returning to the figure of Art. 5, we see that when (AB), (CD) are

given, and H is any real point, we can find a point (HL) which is on the segment Z_1Z_2. We draw HF parallel to EB: this determines m, n, t, and we proceed as before to find L.

The line Z_1Z_2 is defined to be the aggregate of all points such as (HL) which divide Z_1Z_2 in a ratio, in the sense previously explained. It will be observed that these points form a doubly infinite collection; as a rule each real point in the plane is the base of one of the points in question.

There is, however, one exception. Suppose that AC, BD are parallel: then the figure of Art. 5 becomes



and if we put $AC = a$, $BD = b$, we have

$$AF = ma, \quad FH = t(b-a) = GK, \quad AH = ma+t(b-a),$$

$$BK = mb+t(b-a), \quad BL = mb+t(b-2a).$$

Conversely, given $$AH = c, \quad BL = d,$$

the equations $$ma+t(b-a) = c,$$

$$mb+t(b-2a) = d,$$

lead to $$m = \frac{(2c-d)a-(c-d)b}{(a-b)^2+a^2} = \frac{(2a-b)c-(a-b)d}{(a-b)^2+a^2},$$

$$t = \frac{bc-ad}{(a-b)^2+a^2},$$

so that every complex point on the line $[(AB), (CD)]$ is represented by a segment joining a real point on AC to a real point on BD, and conversely. A complex line of this sort may be called a *strip-line*. Any pair of parallel lines (in the ordinary sense) represents one or other of two complex strip-lines according as we take one or other of the parallels as the locus of the bases of the points on the strip-line. The equation of a strip-line is reducible to the form

$$ax+by = c+di,$$

where a, b, c, d are real quantities.

The intersection of two strip-lines is a diagonal of the parallelogram which they enclose. Through a given complex point we can draw a pencil of strip-lines; the angle between two strip-lines is the angle between their boundaries, and so on. In fact, an ordinary geometrical diagram is, strictly speaking, composed of more or less accurate strip-lines.

8. Let (AB), (CD) be any two complex points Z_1, Z_2, not on the same strip-line. Draw any curve S, and let P be any point upon it; then there is a definite complex point (PQ) collinear with (AB) and (CD). The locus of Q is a curve S', and it is easily seen that the coordinates of Q are linear functions of those of P, so that S' is derived from S by a linear transformation. The linear series of complex points (PQ) may be called a *chain* associated with (AB) and (CD) by means of S. If, for example, we draw all the circles which pass through A and C, the associated chains will exhaust all the points on the line joining (AB) to (CD), and we shall have, so to speak, a sequence of strip-segments joining the two given complex points.

The analytical expression for the (S, S') collineation may be found as follows. If (x, y), (x', y') are two corresponding points, there will be three real constants m, n, t, such that $m+n = 1$, and (cf. Art. 5)

$$x = (na+ma')+t(c'-c),$$

$$y = (nb+mb')+t(d'-d),$$

$$x'-x = (nc+mc')+t(a-a'),$$

which may be written

$$x-a = m(a'-a)+t(c'-c),$$

$$y-b = m(b'-b)+t(d'-d),$$

$$x'-x-c = m(c'-c)+t(a-a'),$$

whence
$$\begin{vmatrix} x'-x-c, & c'-c, & a-a' \\ x-a, & a'-a, & c'-c \\ y-b, & b'-b, & d'-d \end{vmatrix} = 0,$$

and similarly
$$\begin{vmatrix} y'-y-d, & d'-d, & b-b' \\ x-a, & a'-a, & c'-c \\ y-b, & b'-b, & d'-d \end{vmatrix} = 0.$$

Generally speaking, these give a collineation of the form

$$x' = Lx+My+N,$$
$$y' = Px+Qy+R,$$

but there is an exception when

$$\begin{vmatrix} a'-a, & c'-c \\ b'-b, & d'-d \end{vmatrix} = 0.$$

It is easily verified that the meaning of this is that AC is parallel to BD; in other words, Z_1, Z_2 lie on a strip-line, in which case, as we have seen, they do not establish a collineation.

9. We may now proceed to the consideration of the area of a triangle. Let its vertices be (A_1B_1), (A_2B_2), (A_3B_3) respectively, and let $[A_1A_2A_3]$ stand for the area of the real triangle $A_1A_2A_3$, and so on. Then for the area of the complex triangle, we find

$$[A_1B_1.A_2B_2.A_3B_3] = [A_1A_2A_3]-\Sigma Q+i\{[B_1B_2B_3]-[A_1A_2A_3]-\Sigma Q\}, \tag{5}$$

where
$$\Sigma Q = [A_2A_3B_2B_3]+[A_3A_1B_3B_1]+[A_1A_2B_1B_2], \tag{6}$$

the sum of three quadrangular areas, taken with attention to sign.

By supposing one or two of the vertices to become ordinary points, we obtain the special formulæ

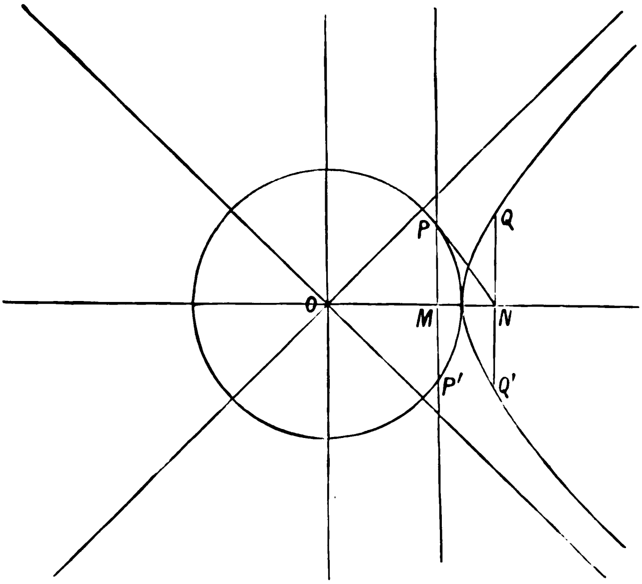
$$[O.A_1B_1.A_2B_2] = \{[OA_1B_2]+[OB_1A_2]-[OB_1B_2]\}$$
$$+i\{[OA_1B_2]+[OB_1A_2]-2[OA_1A_2]\}, \tag{7}$$

and
$$[O.A.BC] = [OAB]+i\{[OAC]-[OAB]\}. \tag{8}$$

In the last of these, the area is real if $[OAC] = [OAB]$; that is, if OA, BC are parallel. Another interesting special case is

$$[O.A_1B_1.B_1A_1] = (1-2i)[OA_1B_1], \tag{9}$$

and it may be noticed that when the vertices are all complex, but in such a way that A_1B_1, A_2B_2, A_3B_3 are all equal, parallel, and of the same sense, the area $[A_1B_1 . A_2B_2 . A_3B_3]$ reduces to $[A_1A_2A_3]$.



10. Consider the circle $x^2+y^2 = a^2$, and the line $x = c$, where $c > a$. Their intersections are the points $(c, \pm i\sqrt{c^2-a^2})$ represented in the figure by (NQ), (NQ'). The real points Q, Q' are $(c, \pm\sqrt{c^2-a^2})$, which, as c varies from a to $+\infty$, describe one branch of the hyperbola $x^2-y^2 = a^2$. Moreover the complex point (NQ) is converted into the real point Q' by the substitution $x' = x$, $y' = iy$.

Similarly (MP), (MP') represent the points where $P'MP$ cuts the hyperbola.

If we take any point (c, k) on NQ its polar with respect to the circle cuts QQ' in the point (c, k') where $kk' = a^2-c^2 = -NQ^2$. Hence Q, Q' are conjugate points of the involution on QQ' determined by the circle $x^2+y^2 = a^2$. Similarly P, P' are conjugate points of the involution on PP' determined by the hyperbola $x^2-y^2 = a^2$. As will be seen later on, this brings the present theory into direct relation with that of v. Staudt.

Any ellipse and a line which does not cut it in real points may be orthogonally projected into a circle and a line; and similarly any hyperbola and a line may be orthogonally projected into a rectangular hyperbola and a line. Hence we can define and construct the intersections of any ellipse or hyperbola with any straight line. The involution property holds also for a parabola (it is sufficient to consider $y^2 = 4ax$ and $x+h = 0$);

hence we have a complete theory of the intersections of "real" conics with "real" lines. But it must be remembered that the complete theory of conics must now include the intersections of complex conics with complex lines; and this gives rise to innumerable geometrical problems. For instance, a circle is given by means of three complex points upon it, (A_1B_1), (B_2B_2), (A_3B_3), and a line by means of two complex points (C_1D_1), (C_2D_2), it is required to find by construction the intersections of the line with the circle.

11. It is now easy to apply the theory to three dimensions. A point $(a+di, b+ei, c+fi)$ will be represented by AB, where A, B are the real points (a, b, c) and $(a+d, b+e, c+f)$. This point and its conjugate lie on the real line

$$\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}.$$

The formula (2) of Art. 3 for the distance between two points holds good for three dimensions, as may be proved by the use of quaternions or by projecting on to the coordinate planes. So does the construction of Art. 5 for the division of Z_1Z_2 in any ratio; but here we are confronted by a new and remarkable fact. If, in the figure of Art. 5, (AB) and (CD) lie in a real plane, HL is in that plane, whatever the ratio of division may be. But there may be no real plane containing (AB) and (CD); in this case (HL) is not in the same real plane with (AB) or (CD) or with any point on the line $[(AB), (CD)]$. Thus two complex points which lie on two real lines which are askew determine a new kind of line, which corresponds to what Staudt calls an imaginary straight line of the second kind.

We obtain all the points on this line by dividing $[(AB), (CD)]$ in all possible ratios. Now a reference to the figure of Art. 5 shows that, if (HL) is the point corresponding to any assigned ratio, L must lie in the plane BED which passes through BD and is parallel to AC, while H is in the plane through AC which is parallel to BD. H may be any point in its plane, and L is uniquely determined from it by a collineation of the form

$$\begin{vmatrix} x'-x-d, & d'-d, & a-a' \\ x-a, & a'-a, & d'-d \\ y-b, & b'-b, & e'-e \end{vmatrix} = 0,$$

with two other similar equations involving y', z'. This collineation leaves unaltered the real line at infinity in the plane AHC. If H describes any curve in its plane, L will describe the collinear curve in its plane, and so on. In particular, BD is the locus of L which is collinear to AC as the locus of H, and the points which divide $[(AB), (CD)]$ in real ratios are represented by segments joining points of AC to points of BD. Moreover, the real lines on which this chain of points lies form one set of generators of a hyperbolic paraboloid containing AB, CD, AC, BD; and on each of these generators there is an involution determined by the two given points. Thus, if (AB') is conjugate to (AB), the involution on AB has A for its centre and (B, B') a pair of conjugate points.

12. Let (A_1B_1), (A_2B_2), (A_3B_3) be any three complex points Z_1, Z_2, Z_3: it is required to find an expression for the volume $[OZ_1Z_2Z_3]$. By definition

$$6[OZ_1Z_2Z_3] = \begin{vmatrix} a_1+d_1i, & b_1+e_1i, & c_1+f_1i \\ a_2+d_2i, & b_2+e_2i, & c_2+f_2i \\ a_3+d_3i, & b_3+e_3i, & c_3+f_3i \end{vmatrix}$$

$$= (a_1b_2c_3)-(a_1e_2f_3)-(d_1b_2f_3)-(d_1e_2c_3) \\ +i\{(d_1b_2c_3)-(d_1e_2f_3)+(a_1b_2f_3)+(a_1e_2c_3)\}.$$

Now write α_1, α_2, α_3, β_1, β_2, β_3 for the vectors OA_1, OA_2, OA_3, OB_1, OB_2, OB_3 respectively; then

$$-S\alpha_1\alpha_2\alpha_3 = (a_1b_2c_3),$$

$$-S\beta_1\beta_2\beta_3 = (a_1b_2c_3)+(a_1e_2f_3)+(d_1b_2f_3)+(d_1e_2c_3) \\ +(d_1b_2c_3)+(d_1e_2f_3)+(a_1b_2f_3)+(a_1e_2c_3),$$

and so on, Then, if we put

$$L = S(\alpha_2\alpha_3\beta_1+\alpha_3\alpha_1\beta_2+\alpha_1\alpha_2\beta_3),$$

$$M = S(\beta_2\beta_3\alpha_1+\beta_3\beta_1\alpha_2+\beta_1\beta_2\alpha_3),$$

we find that

$$6[OZ_1Z_2Z_3] = (2S\alpha_1\alpha_2\alpha_3-2L+M)+(2S\alpha_1\alpha_2\alpha_3+S\beta_1\beta_2\beta_3-M)\,i. \quad (10)$$

Since every term of L and M is at once interpretable as an actual volume connected with the figure, the above is a formula of the kind we require. In ordinary notation

$$[O.A_1B_1.A_2B_2.A_3B_3]$$

$$= 2\Sigma[OA_2A_3B_1]-\Sigma[OA_1B_2B_3]-2[OA_1A_2A_3] \\ -i\{2[OA_1A_2A_3]+[OB_1B_2B_3]-\Sigma[OA_1A_2B_3]\}. \quad (11)$$

If we have four complex points given by $Z_i = (A_iB_i)$, then

$$[Z_1Z_2Z_3Z_4] = [OZ_2Z_3Z_4]-[OZ_1Z_3Z_4]+[OZ_1Z_2Z_4]-[OZ_1Z_2Z_3].$$

Now the right-hand side being independent of O, we may suppose O at A_1: this brings the expression to the form

$$[Z_1Z_2Z_3Z_4]$$
$$= 2\lambda-\mu-2[A_1A_2A_3A_4]+i\{\mu-[B_1B_2B_3B_4]-2[A_1A_3A_3A_4]\}, \tag{12}$$

where

$$\left.\begin{aligned} \lambda = {} & [A_1A_3A_4B_2]+[A_1A_4A_2B_3]+[A_1A_2A_3B_4]+[A_1A_2A_4B_1] \\ & -[A_1A_2A_3B_1]-[A_1A_3A_4B_1], \\ \mu = {} & [A_1A_2B_3B_4]+[A_1A_3B_4B_2]+[A_1A_4B_2B_3] \\ & +[A_1A_2B_4B_1]+[A_1A_4B_1B_2]-[A_1A_3B_4B_1] \\ & -[A_1A_4B_1B_3]-[A_1A_2B_3B_1]-[A_1A_3B_1B_2] \end{aligned}\right\}. \tag{13}$$

Numerous special formulæ may be derived from this by making particular assumptions. For instance,

$$[A_1.A_2.A_3.A_4B_4] = [A_1A_2A_3A_4]+i\{[A_1A_2A_3B_4]-[A_1A_2A_3A_4]\}, \tag{14}$$

$$[A_1.A_2.A_3.A_3B_4] = i[A_1A_2A_3B_4], \tag{15}$$

$$\begin{aligned} [A_1.A_2.A_3B_3.A_4B_4] = {} & [A_1A_4A_2B_3]+[A_1A_2A_3B_4]-[A_1A_2B_3B_4] \\ & +i\{[A_1A_4A_2B_3]+[A_1A_2A_3B_4]-2[A_1A_2A_3A_4]\}, \end{aligned} \tag{16}$$

$$[AB.CD.BA.DC] = (3+4i)[ABCD] = (2+i)^2[ABCD], \tag{17}$$

which is what we might expect, because, by formula (3),

$$\overline{AB.BA}^2 = -(3+4i)AB^2, \qquad \overline{CD.DC}^2 = -(3+4i)CD^2,$$

and we know that if two opposite edges of a tetrahedron are situated in two given straight lines, the volume of the tetrahedron varies as the product of these edges.

13. The element which has yet to be considered is the complex plane. Consider the locus

$$x = a+bi,$$

where a, b are real constants. This is the aggregate of complex points $(a+bi,\ p+qi,\ r+si)$, where p, q, r, s are any real quantities. Hence the points in question are represented by segments (AB), where A is any point in the plane $x = a$, and B is any point in the plane $x = a+b$. Conversely, any pair of real parallel planes ϖ, ϖ' determine, and may be taken to represent, a complex plane $(\varpi\varpi')$, the points of which are represented by segments drawn from points of ϖ to points of ϖ'. The plane $(\varpi\varpi')$ must be distinguished from $(\varpi'\varpi)$.

The general equation of a plane of this kind is

$$lx+my+nz=p+qi,$$

where l, m, n, p, q are real: it will be called a slice-plane. Two slice-planes intersect in a strip-line, and cut at a real angle: three slice-planes intersect, two by two, in three concurrent strip-lines, and so on. Ordinary cardboard models of regular solids may be considered as made up of portions of slice-planes.

It is not so easy to realise the aggregate of points on a general complex plane. It will contain one real straight line; let this be l, and let (AB) or Z be any complex point in the plane. If we take any point P on l, then PZ is a line in the plane, and the aggregate of points upon it may be determined as in Art. 5, with the proper modifications. By making P describe the whole of l, we obtain the aggregate of points in the plane.

Let α, β denote the vectors PA, PB; then

$$(m+ni)\{\alpha+i(\beta-\alpha)\}=(m+n)\alpha-n\beta+i\{(-m+n)\alpha+m\beta\},$$

so we obtain a point on PZ by joining the end of the vector $(m+n)\alpha-n\beta$ to the end of the vector $2n\alpha+(m-n)\beta$, and by giving m, n all possible real values we thus obtain all points on PZ. As a verification we observe that, if m, n lead to the point (CD), we have, in the notation of Art. 9,

$$\begin{aligned}2\Delta_1&=2[PAC]=V\alpha\{(m+n)\alpha-n\beta\}=-nV\alpha\beta,\\2\Delta_2&=2[PAD]=(m-n)V\alpha\beta,\\2\Delta_3&=2[PBC]=-(m+n)V\alpha\beta,\\2\Delta_4&=2[PBD]=-2nV\alpha\beta,\end{aligned}$$

whence $\Delta_2+\Delta_3-\Delta_4=0=\Delta_2+\Delta_3-2\Delta_1$, which, by formula (7), are the conditions that P, (AB), (CD) may be collinear. If we like, we can change the notation by putting $m+n=p$, $-n=q$; the vectors PC, PD are then $p\alpha+q\beta$ and $-2q\alpha+(p+2q)\beta$, which may be written $p\beta+2q(\beta-\alpha)$. Compare the figure of Art. 5 when B moves up to A; the vector AH becomes, in the notation of that article

$$mAC+t(AD-AC)=(m-t)AC+tAD,$$

and AL becomes

$$mAD+t(AD-AC-AC)=-2tAC+(m+t)AD,$$

which agrees with the above on putting $m-t=p$ and $t=q$.

14. It is rather interesting to verify by quaternions that, if (AB), (CD), (HL) are any three collinear points, the planes ACH, BDL are parallel, and $[LBD]=2[HAC]$.

Take the origin O so that, if α, β, γ, δ represent the vectors OA, OB, OC, OD, then $\alpha+\beta+\gamma+\delta=0$, and let ξ, η stand for the vectors OH, OL. Then, in the notation of Art. 5,

$$\xi+i(\eta-\xi)=(n+ti)\{\alpha+i(\beta-\alpha)\}+(m-ti)\{\gamma+i(\delta-\gamma)\},$$

whence
$$\xi=n\alpha+m\gamma-2t(\beta+\gamma),$$
$$\eta=n\beta+m\delta+t(\alpha-2\beta-3\gamma).$$

Hence we find, remembering that $m+n=1$,

$$\begin{aligned}V(\xi-\alpha)(\xi-\gamma)&=V\{m(\gamma-\alpha)-2t(\beta+\gamma)\}\{n(\alpha-\gamma)-2t(\beta+\gamma)\}\\&=2tV(\alpha-\gamma)(\beta+\gamma)\\&=2tV(\alpha\beta+\alpha\gamma+\beta\gamma),\end{aligned}$$

$$\begin{aligned}V(\eta-\beta)(\eta-\delta)&=V\{m(\delta-\beta)+t(\alpha-2\beta-3\gamma)\}\{n(\beta-\delta)+t(\alpha-2\beta-3\gamma)\}\\&=tV(\delta-\beta)(\alpha-2\beta-3\gamma)\\&=-tV(\alpha+2\beta+\gamma)(\alpha-2\beta-3\gamma)\\&=4tV(\alpha\beta+\alpha\gamma+\beta\gamma)=2V(\xi-\alpha)(\xi-\gamma),\end{aligned}$$

on reduction; and this proves the proposition.

15. By a suitable choice of axes, the coordinates of any two complex points, in a general skew position, may be reduced to

$$(a+di,\ ei,\ fi),\quad (d'i,\ a+e'i,\ fi);$$

let us find the points collinear with them which have their bases on the circle

$$x^2+y^2=a^2,\qquad z=0.$$

Putting
$$(n+ti)(a+di)+(m-ti)d'i=a\cos\phi+ui,$$
$$(n+ti)ei+(m-ti)(a+e'i)=d\sin\phi+vi,$$

we have
$$na+t(d'-d)=a\cos\phi,$$
$$ma+t(e'-e)=a\sin\phi;$$

therefore
$$t(d'+e'-d-e)=a(\cos\phi+\sin\phi-1),$$

$$m=\left[\sin\phi-\frac{(\cos\phi+\sin\phi-1)(e'-e)}{(d'-d+e'-e)}\right]=\frac{(1-\cos\phi)(e'-e)+(d'-d)\sin\phi}{d'-d+e'-e},$$

$$n=\frac{\cos\phi(e'-e)+(1-\sin\phi)(d'-d)}{d'-d+e'-e}.$$

It is clear that u, v will be linear integral functions of $\cos\phi$, $\sin\phi$; in fact, putting

$$d'-d=h, \qquad e'-e=k,$$

we find that

$$u=\frac{(a^2-hk)\cos\phi+(a^2+h^2)\sin\phi-a^2+dh+d'k}{h+k},$$

$$v=\frac{-(a^2+k^2)\cos\phi-(a^2-hk)\sin\phi+a^2+eh+e'k}{h+k}.$$

Now the real lines on which all these points lie are found by making ϕ a parameter in the equations

$$\frac{x-a\cos\phi}{u}=\frac{y-a\sin\phi}{v}=\frac{z}{f},$$

and as ϕ varies, this will describe a quartic surface, the section of which by the plane $z=f$ is the ellipse defined by

$$z=f, \quad x=a\cos\phi+u, \quad y=a\sin\phi+v;$$

and this is the curve homologous to the circle $z=0$, $x^2+y^2=a^2$, in the collineation established by the complex points with which we started.

16. A fairly simple calculation shows that, by a suitable choice of axes, the collineation established by two complex points can be reduced to the standard form

$$\left.\begin{aligned} x' &= (1+\theta)\,x+qy \\ y' &= -\frac{1+\theta^2}{q}x+(1-\theta)\,y \\ z' &= z+f \end{aligned}\right\}. \tag{18}$$

The collineation found in Art. 15 is a case of this with

$$\theta=\frac{a^2-hk}{a(h+k)}, \qquad q=\frac{a^2+h^2}{a(h+k)},$$

except that the origin requires to be shifted in the plane $z=0$.

It will be seen that, with the standard axes, the point O' corresponding to the origin is $(0, 0, f)$; in other words, the axis of z is that one real line which is perpendicular to the slice-plane determined by the two given points, and also contains a point collinear with them.

It has been assumed that neither a nor $(h+k)$ vanishes: this is allowable, because on either hypothesis the two complex points we start with are no longer skew. Thus, if $a = 0$, they are cobasal; while if $h+k = 0$, they are on the strip-line in which the planes

$$x+y = a+(d+e)\,i, \quad z = fi$$

intersect.

From equations (18) we find that

$$\begin{aligned}\xi &= x+i(x'-x) = (1+\theta i)x+qyi,\\ \eta &= y+i(y'-y) = -\frac{1+\theta^2}{q}xi+(1-\theta i)\,y,\\ \zeta &= fi;\end{aligned}$$

whence the complex line corresponding to the collineation is determined by the equations

$$(\theta+i)\xi+q\eta = 0, \quad \zeta = fi. \tag{19}$$

This, then, is an analytical form to which the equations of any complex line of the second kind can be reduced. That it is sufficiently general follows from the foregoing and is confirmed by counting constants. The special position of the origin involves the fixing of three real constants; the direction of the axis of z depends upon two more: and besides this, there are three explicit real constants f, q, θ, making eight in all. Now a complex line corresponds to four complex constants, equivalent to eight real ones; so that the results are consistent.

From equations (18) we deduce

$$(\theta \pm i)\,x'+qy' = (1 \pm i)\{(\theta \pm i)\,x+qy\},$$

whence we see that for all values of c the cylinder

$$x^2+(\theta x+qy)^2 = c^2$$

is converted into

$$x^2+(\theta x+qy)^2 = 2c^2.$$

17. For the sake of completeness, we give the normal form of the collineation established in the plane $z = 0$ by a complex line which is not a strip-line (cf. Art. 8). The line may be supposed that which joins the origin to $(a+ci,\ b+di)$; this means that the origin is taken at the one real point on the line. Thus the collineation is

$$\left.\begin{aligned} x' &= (1+\theta)x+qy\\ y' &= -\frac{1+\theta^2}{q}x+(1-\theta)\,y\\ \text{with}\quad \theta &= (ab+cd)/(ad-bc), \quad q = -(a^2+c^2)/(ad-bc)\end{aligned}\right\}. \tag{20}$$

This is of the same form as the first two of the equations (19), and shows that a complex line of the first kind is, in a way, a limiting case of one of the second, obtained by putting $f = 0$. The equations of the line are

$$(\theta+i)\,\xi+q\eta = 0, \qquad \zeta = 0,$$

so it is, in fact, the projection upon the plane $z = 0$ of the line given by (19); supposing that the values of θ, q are the same.

We can make a further and final simplification by turning the axes of x and y, in their plane, through an angle ψ, found from the equation

$$\tan 2\psi = \frac{2q\theta}{\theta^2+1-q^2}; \qquad (21)$$

if x, y are the new coordinates, the collineation assumes the form

$$\left.\begin{aligned} x' &= x+ky \\ y' &= -\frac{x}{k}+y \end{aligned}\right\}, \qquad (22)$$

where k is the appropriate root of the equation

$$qk^2-(\theta^2+1+q^2)k+q = 0. \qquad (23)$$

The equations of the line are now, according as it is of the first or second kind,

$$ix+ky = 0, \qquad z = 0, \qquad (24)$$

or

$$ix+ky = 0, \qquad z = fi, \qquad (25)$$

the former of which may be considered a special case of the latter.

Thus every complex line which is not a strip-line determines a single set of rectangular axes associated with it, and also two definite real constants k, f, the second of which may vanish, but not the former; moreover, by suitably choosing the positive directions of the axes, we may make $k > 0$ and $f \geqslant 0$. Conversely, a definite set of axes, and two given values of k, f satisfying these last conditions, can be associated with a definite complex line, and a collineation of the form (22).

The equations (22), by themselves, may also be considered as representing a complex plane, the real line of which is the axis of z, and the equation of which is

$$ix+ky = 0.$$

All these results may be obtained more rapidly by purely algebraic work, but their geometrical meaning does not come out so clearly.

18. It remains to bring the present theory into connexion with that of Staudt. Let us take an elliptic involution on the axis of x with its centre at the origin: then, p being a variable parameter, conjugate pairs are defined by the equation

$$x^2+2px-a^2 = 0; \tag{26}$$

so that for the double points $p^2+a^2=0$ and $x = \pm ai$. We represent these by (OA), (OA'), where A, A' are the points $(a, 0)$, $(-a, 0)$ respectively: Staudt represents them by the complete involution, associated with the sense $A'OA$ or AOA' respectively. The same thing holds good for any other imaginary point.

Let the above involution of points be projected from (h, k): this leads to

$$k^2(x-h)^2-2k(h+p)(x-h)(y-k)+(h^2+2hp-a^2)(y-k)^2 = 0. \tag{27}$$

In Staudt's theory this involution, with the appropriate sense, is the complex line joining (h, k) to (OA) or (OA'), and, *by definition*, any section of it by a real line gives a complex point upon it: we have to show that this is consistent with the Cartesian definitions. Let the involution (27) be cut by the line

$$y = tx,$$

and the section orthogonally projected on to the axis of x. The result is

$$\{k^2-2k(h+p)\,t+(h^2+2hp-a^2)\,t^2\}\,x^2+2k\,\{kp-(hp-a^2)\,t\}\,x-k^2a^2 = 0. \tag{28}$$

Now the Cartesian equation of the line joining (h, k) to $(ai, 0)$ is

$$-kx+(h-ai)\,y+aik = 0,$$

and, combined with $y = tx$, this gives

$$(ht-k-ati)\,x+aki = 0. \tag{29}$$

It is found that the value of p which correspond to the double points of (28) are given by $p = \pm ai$, as might have been anticipated; and these reduce (28) to the form

$$\{kai+[-ati\pm(k-ht)]\,x\}^2 = 0,$$

which is in agreement with (29). For the moment, let g denote the complex line we are considering: then we have proved that any real line $y = tx$ which cuts the corresponding involutionary pencil meets it in a line-involution which determines the Cartesian representation of the complex point where g and the line $y = tx$ intersect, in a manner consistent with our initial definitions. If $t = k/h$, then $y = tx$ becomes a ray of the pencil, and equation (29) gives $x = h$, whence $y = k$, and the intersection is (h, k), the real point of g: this again is in agreement with Staudt. That the same thing is true for all lines in the plane, as well as those which pass through the origin, is easily proved geometrically.

The undoubted advantage of Staudt's definitions is that they are projective, and lead to simple constructions for the joins of two complex points or planes, and so on. On the other hand, the projective definition of a straight line of the second kind is difficult to grasp, and the results of Arts. 15–17 may perhaps deserve attention as bringing this element into a more concrete and realisable shape.

ON THE PROOFS OF THE PROPERTIES OF RIEMANN'S SURFACES DISCOVERED BY LÜROTH AND CLEBSCH

By M. J. M. HILL.

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1. The theorem known by Lüroth's name was stated by him in a paper in the fourth volume of the *Mathematische Annalen*. It was very briefly indicated, the proofs of the various steps in the argument for the most part not being given.

In the sixth volume of the same journal there appeared a complete demonstration of the theorem by Clebsch with some additions, including the theorem now known by his name. The proof is long and intricate, and the next attempt to simplify the argument was made by Clifford in a paper in the eighth volume of the *Proceedings of the London Mathematical Society*; and, lastly, there appeared a proof, differing somewhat from Clifford's, in Forsyth's *Theory of Functions*, Cap. XV.

These, I think, are all the distinct proofs.

2. I will first indicate the principal steps in Clebsch's argument.

Denoting by (ik) the loop which interchanges the i-th and k-th branches of a function w of the complex variable z, and by G_{ik} a group of consecutive loops all having the same symbol (ik), Clebsch shows that the loops can be so drawn that they may be expressed in the form

$$G_{12}\,G_{13}\,\ldots\,G_{1n}\,G_{23}\,G_{24}\,\ldots\,G_{2n}\,\ldots\,G_{n-1,\,n},$$

where each symbol G_{ik} represents an *even* number of consecutive loops (ik).* Then he shows that the loops can be so drawn that they can be expressed in the form

$$G_{12}\,G_{23}\,G_{34}\,\ldots\,G_{n-2,\,n-1}\,G_{n-1,\,n},$$

where, as before, each symbol G_{ik} represents an even number of loops.

This is Lüroth's theorem.

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* It is to be understood that some of the groups G_{ik} may be absent.

Then he shows that the loops can be so drawn that each of the first $(n-2)$ groups G_{12}, G_{23}, G_{34}, ..., $G_{n-2,\,n-1}$ consists of a single pair of loops, whilst $G_{n-1,\,n}$ consists of some even number of loops.

This is Clebsch's theorem.

3. Clifford's demonstration of Lüroth's theorem commences in a different way.

Starting from any loop (pq), he shows that the configuration can be replaced by another in which the next loop has the same symbol (pq).

It will be convenient in what follows to describe two consecutive loops having the same symbol as a loop-couple, or, briefly, a couple.

Clifford thus *begins* with the loop-couple.

Clebsch, on the other hand, following Lüroth, does in effect, though not explicitly, introduce the loop-couple *after* he has obtained the arrangement

$$G_{12}\,G_{13}\,\ldots\,G_{1n}\,G_{23}\,\ldots\,G_{2n}\,\ldots\,G_{n-1,\,n},$$

when he shows that each symbol G_{ik} contains an even number of loops with the same symbol (ik).

Clifford having obtained his first loop (pq), which he calls OA, and his second loop (pq), which he calls OB, commences* to describe OB with the p-th branch of the function and describes all the loops from left to right. If the p-th branch is not restored before OA is described, and if also in describing the loops there are loops passed over which do affect the p-th branch, then these loops have to be replaced by others which lie between OA and OB. Thus, after the loop-couple OA, OB has once been obtained, its two loops must be again severed in the case contemplated, and these severed loops have afterwards to be replaced by others to the same singular points which form a couple with the same symbols (pq, pq) as before.

It is this severing of the loop-couple after it has been once obtained that seems to me to account for the difficulty of the argument both in Clifford's and Forsyth's work.

My statement of the matter is given in § 15 below, where I indicate the place at which I deviate from their line of thought.

4. With regard to Clebsch's theorem, the proof given by Forsyth is essentially the same as that of Clebsch, but Forsyth states separately a number of steps which Clebsch puts together (see § 11 below).

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* Forsyth commences to describe the loop OB with the q-th branch of w, but this is not an essential difference, because the symbol (pq) is equivalent to the symbol (qp).

On the other hand, Clifford seems to me to strike out a different line of thought in his demonstration of Clebsch's theorem.

Whilst Clebsch's proof consists in showing how the loops may be drawn in different ways so that their original and final configurations correspond to different Riemann surfaces, Clifford's consists really in successive deformations of a single Riemann surface.

Clifford's argument is on this point very obscure. Whether I have correctly interpreted his meaning or not, I think the deformations of the Riemann surface I give are sufficiently interesting to be preserved.

I proceed now to the statement of my argument. I give proofs of some quite well known theorems in order to make the argument complete in itself.

Definitions.

5. (i) If there be any number of consecutive repetitions of the same loop-couple *ab*, *ab*, then the configuration of loops so formed will be called the multiple loop-couple *ab*, *ab*.

(ii) If there be a succession of any number of loops whose symbols are such as

$$ab,\ bc,\ cd,\ de,\ ef,$$

so that each symbol contains a letter that appears in the symbol of the preceding loop and a letter that appears in the symbol of the following loop, then the configuration of loops so formed will be called a loop-chain or, briefly, a chain.

(iii) If there be a succession of any number of loops of a chain such as

$$ab,\ bc,\ cd,\ de,\ ef,\ fa,$$

so that if we start with the branch w_a at the first loop, and if the complex variable z describe the loops in succession, then the value w_a is restored at the end, then the chain is said to be closed.*

Subsidiary Propositions.—Arts. 6–14, *Props.* I–IX.

6. PROP. I.—*The consecutive loops*

$$ab,\ ac$$

in this order can be replaced by the loops

$$bc,\ ab$$

or by the loops $ac,\ bc.$

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* If there were no other loops present, then the closed chain might be called a cycle; but as in this work it generally happens that there are other loops present, it is convenient to have a distinctive name.

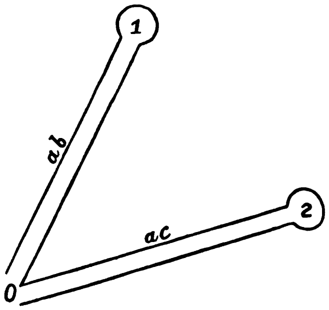


FIG. 1.

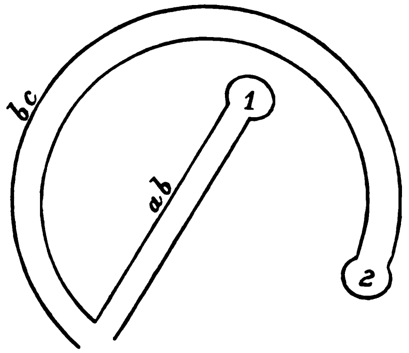


FIG. 2.

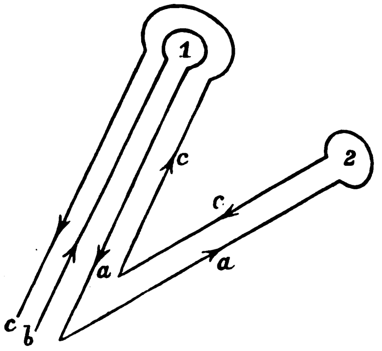


FIG. 3.

Fig. 1 shows the original configuration of the loops.

Fig. 2 shows the change in the loop to the second singular point.

Fig. 3 shows a loop equivalent to the new loop to the second singular point; and, if we commence to describe it with w_b, it shows that it becomes w_a after going round 1, then w_c after going round 2, and that there is no subsequent alteration.

The loop to the first singular point, which we will call 1, has the symbol (ab). We will suppose that looking along this loop, the loop to the second singular point, which we will call 2, lies on the right of that to the first singular point.

Now let the loop to the second singular point be removed, and draw another loop to the second singular point which starts from the origin of the loops to the left of that to 1, goes beyond 1, and then round 2 (see Fig. 2).

This new loop is equivalent to the original loop to 1, then the original loop to 2, then the original loop to 1, and so back to the origin (see Fig. 3).

If we set out to describe this loop with the value w_b, then on going round 1 we get the value w_a, then on going round 2 we get the value w_c, then on going round 1 the value w_c is unaltered.

Hence the new loop changes w_b into w_c, and its symbol is therefore (bc).

Moreover, it starts from the origin to the left of the loop to 1, and so the order of the loops is now (bc), (ab) to the points 2, 1 respectively.

Similarly, if we remove the loop to 1, keep the loop (ac) to 2, then draw a new loop, starting from O on the right of the loop to 2, going beyond 2, and then round 1, its symbol will be (bc).

Then the order of the loops is (ac), (bc) to the points 2, 1 respectively.

7. PROP. II.—*A loop on one side of two loops having the same symbol as each other can be replaced by another loop to the same singular point which starts from the origin on the other side of these two loops without altering its symbol.*

(1) Suppose that the loops in order to the singular points 1, 2, 3 have the symbols ab, ab, cd, where the letters a, b, c, d are all different. Then, if the loop to the point 3 be removed and a new loop be drawn to 3 which starts to the left of the loop to 1 and then passing beyond the loops to 1 and 2 goes to 3, this new loop is equivalent to the original loops to 1, 2, 3 in succession, and then in reverse order the original loops to 2, 1. If we commence this new loop with the value w_c, then the loops to 1 and 2 have no effect on it; but, on describing the loop to 3 it becomes w_d, which is unaltered by describing the loops to 2 and 1.

Hence the new loop interchanges w_c and w_d, and its symbol is therefore still (cd).

(2) Suppose that the loops to 1, 2, 3 have the symbols (ab), (ab), (ac).

Draw a new loop as before. It is equivalent to loops round 1, 2, 3, 2, 1.

If we commence to describe it with the value w_a, this becomes successively

$$w_b,\ w_a,\ w_c,\ w_c,\ w_c.$$

Thus w_a is changed into w_c, and its symbol is (ac) as before.

8. PROP. III.—*The configuration consisting of the two loop-couples*

$$ab,\ ab\ ;\ ac,\ ac$$

can be replaced by a configuration consisting of the two loop-couples

$$ac,\ ac\ ;\ ab,\ ab.$$

By Prop. II we can replace the configuration

$$ab,\ ab\ ;\ ac,\ ac$$

by

$$ac,\ ab,\ ab,\ ac.$$

And again, using Prop. II, we can replace this by

$$ac,\ ac\ ;\ ab,\ ab.$$

9. PROP. IV.—*To prove that a configuration consisting of two loop-couples, whose symbols contain a common letter, can be replaced by a configuration in which one of the loop-couples has the same symbol as before, whilst the symbol of the other loop-couple is obtained from the symbols of the original couples by removing the common letter.*

Thus it is to be proved that the configuration

$$ab,\ ab\ ;\ ac,\ ac$$

can be replaced by any one of the configurations

$$bc,\ bc\ ;\ ab,\ ab, \qquad \text{(I)}$$

$$bc,\ bc\ ;\ ac,\ ac, \qquad \text{(II)}$$

$$ab,\ ab\ ;\ bc,\ bc, \qquad \text{(III)}$$

$$ac,\ ac\ ;\ bc,\ bc. \qquad \text{(IV)}$$

Starting from the configuration

$$ab,\ ab\ ;\ ac,\ ac,$$

we can, by Prop. I, replace this by

$$ab,\ bc,\ ab,\ ac.$$

Using Prop. I again, we can replace this by

$$ab,\ bc,\ bc,\ ab.$$

Now use Prop. II, we can replace this either by

$$ab,\ ab,\ bc,\ bc$$

or by $$bc,\ bc,\ ab,\ ab.$$

Again, if we start from $ab,\ ab\ ;\ ac,\ ac,$ we can, by Prop. I, replace this by

$$ab,\ ac,\ bc,\ ac.$$

Using Prop. I again, we can replace this by

$$ac,\ bc,\ bc,\ ac.$$

Using Prop. II, we can replace this by either

$$bc,\ bc,\ ac,\ ac$$

or by $$ac,\ ac,\ bc,\ bc.$$

10. PROP. V.—*To prove that the configuration*

$$ab,\ ab\ ;\ ab,\ ab\ ;\ ac,\ ac \qquad \text{(I)}$$

can be replaced by the configuration

bc, bc; *ab, ab*; *ab, ab*. (II)

Applying Prop. IV to the 2nd and 3rd couples in (I), it becomes

ab, ab; *bc, bc*; *ab, ab*.

Applying Prop. III to the 1st and 2nd couples in the preceding, it becomes the configuration (II).

COR. I.—It follows immediately, by using Prop. IV once and Prop. III $(p-1)$ times, that the configuration consisting of p couples *ab*, *ab* followed by one couple *ac*, *ac* can be replaced by one couple *bc*, *bc* followed by p couples *ab*, *ab*.

COR. II.—By using the preceding corollary q times, it can be shown that p couples *ab*, *ab* followed by q couples *ac*, *ac* can be replaced by q couples *bc*, *bc* followed by p couples *ab*, *ab*.

[By using Prop. II, the q couples *bc*, *bc* can be made to occupy any place amongst the p couples *ab*, *ab*.]

This proposition is given by Lüroth, *l.c.*, p. 183, and Clebsch, *l.c.*, p. 223. Prop. IV is a particular case of it.

11. PROP. VI.—*To prove that the configuration*

ab, ab; *ab, ab*; *bc, bc* (I)

can be replaced by the configuration

ab, ab; *bc, bc*; *bc, bc*. (II)

This is the essential part of Clebsch's theorem.

Comparing the two configurations, it is seen that they differ only in the 2nd couple.

Starting from (I) and applying Prop. IV to the 2nd and 3rd couples so as to change the 2nd couple, (I) is replaced by

ab, ab; *ac, ac*; *bc, bc*. (III)

This differs from (II) only in the 2nd couple. Hence, apply Prop. IV to the 1st and 2nd couples in (III) so as to change the 2nd couple. This gives

ab, ab; *bc, bc*; *bc, bc*,

which is (II).*

---

* This is, in effect, Clebsch's statement of the matter. Forsyth sets out separately the three steps of Prop. IV in each of the above applications of that proposition, without showing that the six steps he gives fall into two groups of three. The demonstration is made more easy to grasp when it is set out as above.

12. PROP. VII.—*To prove that a closed chain, e.g.,*

$$ab,\ bc,\ cd,\ de,\ ea,$$

can be replaced by $ab,\ ab,\ bc,\ cd,\ de$

which starts by a repetition of the first loop, which takes the place of the last loop.

The singular points to which the loops are drawn may be numbered

1, 2, 3, 4, 5.

We remove the loop (ea) to 5.

Then, starting between the loops to 1 and 2, we draw a new loop to 5 which goes out beyond 2, 3, 4, and then goes to 5. The description of this loop is equivalent to the description of the loops 2, 3, 4, 5, 4, 3, 2 in this order. If we start to describe it with the value w_b, it becomes successively

$$w_c,\ w_d,\ w_e,\ w_a,\ w_a,\ w_a,\ w_a.$$

Thus w_b is changed into w_a. Hence its symbol is (ab).

Thus the loops $ab,\ bc,\ cd,\ de,\ ea,$

to the points 1, 2, 3, 4, 5

respectively, are replaced by the loops

$ab,\ ab,\ bc,\ cd,\ de$

to the points 1, 5, 2, 3, 4

respectively.

13. PROP. VIII.—*If there be interposed amongst the loops of a closed chain other loops which are not effective, e.g., suppose the succession of loops to be*

$$ab,\ bc,\ mn,\ cd,\ af,\ de,\ ea,$$

then it is still possible to replace this by a configuration in which the loop ab is followed immediately by another loop ab.

To do this, it is essential to replace the non-effective loops, whose symbols contain the letter a, by others which are not interposed amongst the loops of the chain; but it is just as convenient to deal with all the non-effective loops in this way. The symbols of the new loops which replace them are not material to the argument, and the proof can be shortened by passing over this determination, although in any actual case of rearrangement of the loops, it would be necessary to find out what these symbols were.

Suppose that the singular points, to which the loops

$$ab,\ bc,\ mn,\ cd,\ af,\ de,\ ea$$

are drawn, are numbered 1, 2, 3, 4, 5, 6, 7
respectively.

There are two non-effective loops, viz., (mn) to the point 3, and (af) to the point 5.

Remove the loop (af) to the point 5, and replace it by a loop beyond the loop (ea) going out beyond 7 and 6 to 5. Call the symbol of this new loop $a'f'$ (which in this case is ef). Then the configuration is

$$ab,\ bc,\ mn,\ cd,\ de,\ ea,\ a'f'$$

to the points 1, 2, 3, 4, 6, 7, 5

respectively. Then remove the loop to 3, and draw a loop between those to 7 and 5, out beyond 7, 6, 4 to 3. Call the symbol of the new loop $m'n'$ (which in the present case is still mn). Then the configuration is

$$ab,\ bc,\ cd,\ de,\ ea,\ m'n',\ a'f'$$

to the points 1, 2, 4, 6, 7, 3, 5
respectively.

Now apply Prop. VII to the closed chain formed by the first five loops, then the configuration is replaced by

$$ab,\ ab,\ bc,\ cd,\ de,\ m'n',\ a'f'$$

to the points 1, 7, 2, 4, 6, 3, 5.

Props. VII and VIII therefore amount to this:—

When a configuration consists of a closed chain, or of what would be a closed chain if non-effective loops were replaced by others branching out from the origin outside its limits, then the configuration can be replaced by another in which the first loop of the chain is followed by another loop having the same symbol.

14. PROP. IX.—*To prove that if the initial configuration consist of a set of couples, each of which affects the a-th branch* w_a, *and if the first couple have the symbol ab, ab, then the configuration can be replaced by another consisting of couples each of which affects the b-th branch* w_b, *but not* w_a, *followed by a single or multiple couple ab, ab.*

Consider, for example, the configuration

$$ab,\ ab;\ ac,\ ac;\ ad,\ ad;\ ae,\ ae.$$

(i) Suppose, first, the letters b, c, d, e all distinct.

Apply Prop. IV to the first and second couples, then the configuration can be replaced by

$$bc,\ bc\ ;\ ab,\ ab\ ;\ ad,\ ad\ ;\ ae,\ ae.$$

Apply Prop. IV to the second and third couples, then the preceding configuration can be replaced by

$$bc,\ bc\ ;\ bd,\ bd\ ;\ ab,\ ab\ ;\ ae,\ ae.$$

Apply Prop. IV to the third and fourth couples, then the preceding configuration can be replaced by

$$bc,\ bc\ ;\ bd,\ bd\ ;\ be,\ be\ ;\ ab,\ ab.$$

(ii) If b, c, d, e are not all distinct so that some of the couples and the couple ab, ab itself may be repeated, then Prop. V or one of its corollaries should be used instead of Prop. IV.

The preceding propositions are all that are necessary for the proof of Lüroth's theorem, to which I now proceed.

Lüroth's Theorem.

15. *To prove that any configuration of the loops drawn from a non-singular point of an algebraic function to its singular points, each of these being of the first order and at a finite distance from the origin of the loops, can be replaced by another configuration which consists of a single or multiple couple ab, ab, followed by a single or multiple couple bc, bc, followed by a single or multiple couple cd, cd, and so on.*

Commencing with any loop whose symbol is ab, let z describe a circuit containing all the branch points of w, then w will have the same value at the end of the circuit as it had at the beginning.

Hence, if we commence at the loop ab with the value w_a and describe the loops clockwise in succession, this value must be restored when we have gone right round at latest, but it may be restored before we have gone right round. Suppose it restored when the loop ae has been described.

Apply Prop. VIII to the loops ab to ae inclusive. This part of the configuration can be replaced by another in which the loop ab is followed by another with the same symbol.*

Commencing from this loop-couple ab, ab and turning round clockwise, look for the first loop whose symbol contains a. Suppose it is ak.

---

* In what follows I proceed differently from Clifford and Forsyth. The step I now take is one which Forsyth takes at a later stage in his argument.

Suppose the loops in order are now

$$ab,\ ab,\ bc,\ cd,\ de,\ m'n',\ a'f',\ pq,\ ak,$$

followed by other loops.

Then, by applying Prop. II, we may replace each of the loops between the couple ab, ab and the loop ak by others preceding the couple ab, ab without altering their symbols.

Thus we get the configuration

$$bc,\ cd,\ de,\ m'n',\ a'f',\ pq,\ ab,\ ab,\ ak$$

followed by other loops.

If now we start with the branch w_a along the loop ak, and go right round the loops, it must be restored at last; but it is to be observed that it will be restored before we reach the loop-couple ab, ab, because, if we commence to describe this loop-couple with any branch w_m, then when we have finished describing it, we shall still have w_m. Hence w_m must be w_a, because w_a is restored when, starting from ak with the value w_a, all the loops have been described, and therefore w_a is restored before we reach the loop-couple ab, ab.

Now apply Prop. VIII to the loops of the configuration beginning with ak.

When this has been done, ak will be followed by another loop ak.

Consequently the couples ab, ab; ak, ak are now consecutive.

Commencing with these couples and turning clockwise, suppose the first loop which affects a is al.

Then replace the loops between the couples ab, ab; ak, ak on the one hand and the loop al on the other by other loops which precede the couples ab, ab; ak, ak. This can be done without altering their symbols.

Then we have ab, ab; ak, ak; al consective.

If we begin to describe the loop al with the branch w_a, it will be restored when we have described all the loops; but, as the description of the couples ab, ab; ak, ak cannot alter any branch of w, we see that w_a will be restored before we reach these couples.

Hence, applying Prop. VIII to the part of the configuration which commences with al, we obtain another configuration with the consecutive couples ab, ab; ak, ak; al, al; and so on.

In this way we can gradually replace every loop, whose symbol contains a, by another loop which belongs to some loop-couple, whose symbol contains a; and then the symbol a will occur only in loops which belong to a set of consecutive loop-couples.

Call these *ab*, *ab* ; *ak*, *ak* ; *al*, *al* ; *am*, *am* ; ...,

followed by loops whose symbols do not contain *a*.

By Prop. IX these can be replaced by

bk, *bk* ; *bl*, *bl* ; *bm*, *bm* ; ... ; *ab*, *ab*,

followed by loops whose symbols do not contain *a*, if all the letters *b*, *k*, *l*, *m*, ... are distinct.

If the letters *b*, *k*, *l*, *m*, ... are not all distinct, then we obtain instead a series of single or multiple couples whose symbols must contain *b*, but do not contain *a*, followed by a single or multiple couple *ab*, *ab*, followed by loops whose symbols do not contain *a*.

The *whole* configuration will now consist of a single or multiple couple *ab*, *ab* and other loop-couples and loops, the symbols of some of which must contain *b*, but none of which contain *a*.

We can now apply to the configuration consisting of the loop-couples and loops which do not belong to the single or multiple couple *ab*, *ab* the same argument. The symbol *b* is known to occur in them. Hence we can reduce them to a single or multiple couple, which we may call *bc*, *bc* and other loop-couples and loops, some of which contain *c*, but none of which contain either *a* or *b*.

These can be treated in like manner.

Thus the configuration of the loops can be reduced to a single or multiple couple *ab*, *ab*, followed by a single or multiple couple *bc*, *bc*, followed by a single or multiple couple *cd*, *cd*, and so on.

This is Lüroth's theorem.

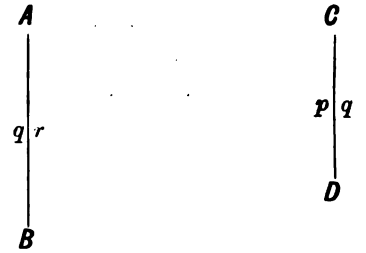
From this Clebsch's theorem follows immediately by applying Prop. VI to Lüroth's result.

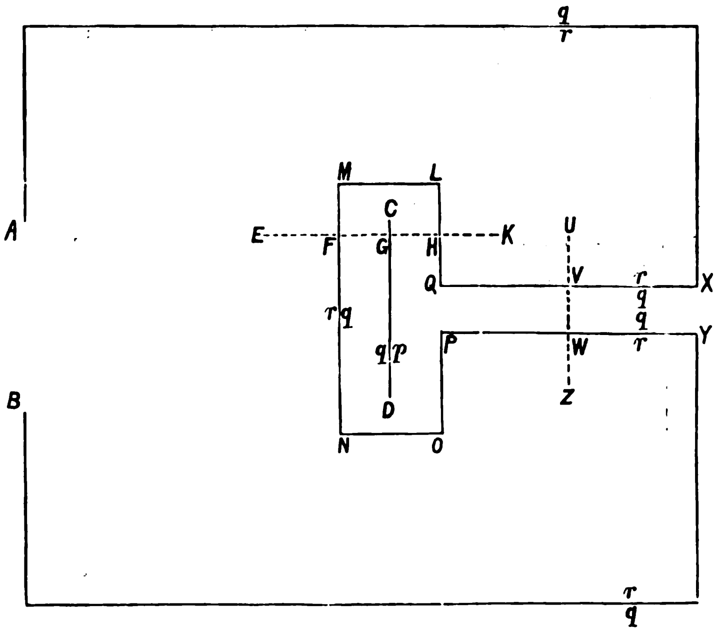
Clebsch's Theorem.

16. It remains only to explain Clifford's proof of Clebsch's theorem (*Collected Works*, p. 249). Clifford speaks of passing the branch line *qr*, keeping its ends fixed, over the branch line *pq*, and says that the branch line *pq* is then changed into a branch line *pr*.

The meaning of these words is obscure, because a branch line in one of the planes of a Riemann's surface cannot be deformed across a singular point in that plane. Nevertheless, a deformation of the Riemann surface exists which gives the result stated by Clifford.

Suppose that the lines *AB*, *CD* represent the initial positions of the branch lines qr and pq in the plane of the q-th sheet. Let the branch line qr be deformed, but without altering the position of its ends, as shown in the next figure, into the broken line $AXVQLMNOPWYB$. Then a point





starting from E in the plane of the r-th sheet would cross at F into the plane of the q-th sheet, and at G into the plane of the p-th sheet. Thence it would traverse the path GHK in the p-th sheet. (This part of the path GHK does not really cross the branch line qr at H, because it is in the p-th sheet.) Now imagine the part of the branch line qr which surrounds the branch line pq, viz., $QHLMFNOP$, to shrink round CD and tend to coincide with it. Then the path $EFGHK$ will be altered by the coincidence of F and H with G, and will denote a passage from E in the r-th sheet to K in the p-th sheet.

Thus the connection between the p-th and q-th sheets along CD will be replaced by a connection between the p-th and r-th sheets along CD.

If we had started from E in the q-th sheet, we should have crossed at F into the r-th sheet, there would have been no crossing at G, at H we should have crossed back into the q-th sheet, and so reached K in the q-th sheet.

Consequently there is no longer any junction along CD from the q-th to any other sheet.

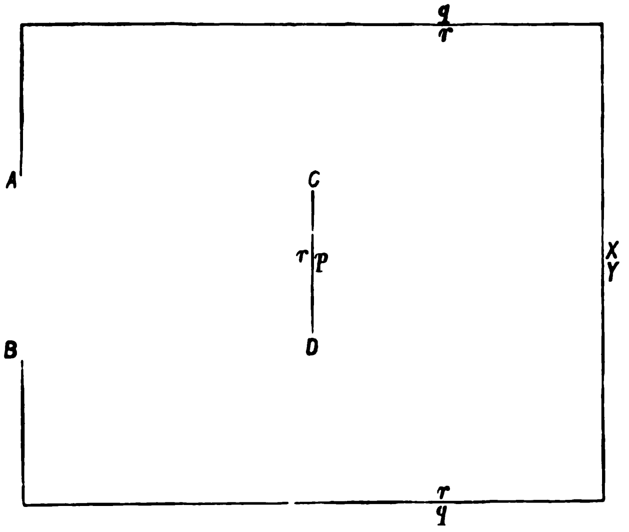
Consider next a point moving from U in the r-th sheet. It crosses at V into the q-th sheet and at W back into the r-th sheet, so that at Z it is in the r-th sheet again.

If we now imagine the parts QVX and PWY of the cross-cut qr to move up to coincide with each other, then V and W coincide, and the path from U to Z may be regarded as being wholly in the r-th sheet. Hence the line with which QVX and PWY coincide need no longer be treated as part of the cross-cut qr, and may be obliterated.

As Q and P coincide, and $QHLMFNOP$ shrinks up to coincide with CD, CD becomes a junction between the r-th and p-th sheets.

As X and Y coincide, $AXYB$ becomes a continuous cross cut qr.

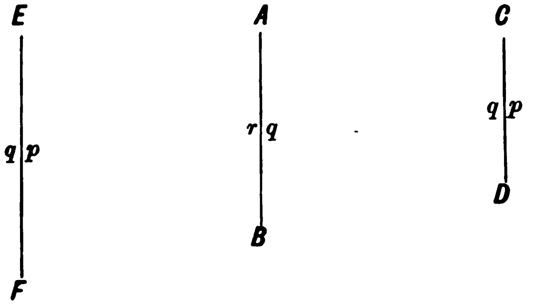
Thus the cross cuts AB (qr) and CD (pq) are now replaced by the cross cuts $AXYB$ (qr) and CD (rp), which now have the following form.



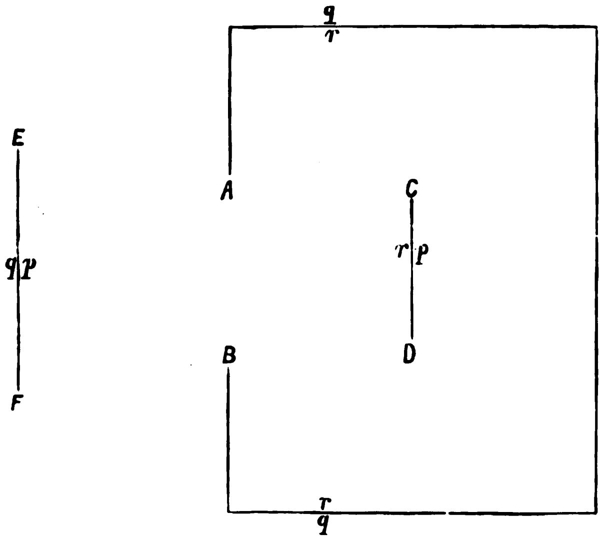
This is equivalent to the figure which Clifford gives.

The rest of his proof, which I give to make the discussion complete in itself, is as follows :—

Commencing with the branch lines



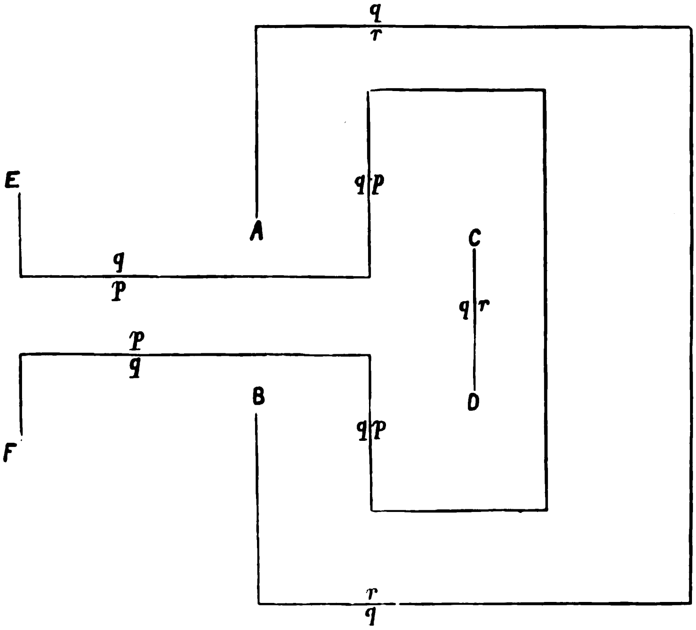
This is first replaced by the process described above by the following :



(We have now effected the process which in this method corresponds to the first application of Prop. IV in the proof of Clebsch's theorem in Art. 11.)

We now apply the same process again to the branch lines *EF* (*pq*) and *CD* (*pr*), so that the connection along *CD* becomes a connection between the *q*-th and *r*-th sheets.

Then we get the figure



(This is equivalent to the second application of Prop. IV in the proof of Clebsch's theorem in Art. 11.)

Thus one branch line qr and two branch lines pq are replaced by two branch lines qr and one branch line pq.

This is equivalent to Prop. VI, the essential part of Clebsch's theorem.

A SYMMETRICAL METHOD OF APOLARLY GENERATING CUBIC CURVES

By W. P. Milne

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1. In the *Proceedings of the London Mathematical Society*, Series 2, Vol. 9, Part 3, I gave a general solution to the following problem:—

Given a cubic curve, to find two inscribed triangles ABC and DEF such that, if P be any point on the cubic curve, the two pencils of lines $P(A, B, C)$ and $P(D, E, F)$ shall be apolar for all positions of P on the curve.

In the former paper, the solution given was unsymmetrical. I have since then found a symmetrical solution, which I now proceed to describe.

2. It easily follows from the above paper that, if the chord DEF of the given cubic curve touch its Cayleyan, and if ABC be an inscribed apolar triangle, the necessary and sufficient condition that A, B, C and D, E, F be co-apolar triads with respect to the given cubic curve is that the common intersection G of the tangents at D, E, F be the pole of the line DEF with respect to the triangle ABC.

For the apolar locus of ABC and DEF passes through A, B, C and touches DG, EG, FG at D, E, F respectively. Hence the given cubic and the apolar locus have the following nine-points in common, A, B, C, D, D, E, E, F, F, and must therefore coincide unless A, B, C be collinear, which is contrary to the hypothesis.

3. It will be convenient for our purpose to regard the given cubic curve as the projection from a point on itself of a skew quartic formed by the intersection of two quadrics. Any one chord of the quartic defines a unique quadric of the above system, and we shall call chords, which are non-intersecting generators of the same quadric, chords of the same system.

Let us now prove that, if x and y be two given chords of the quartic belonging to the same system, and if P be a variable point on the quartic, then a one-to-one algebraic correspondence exists between the planes xP and yP.

Let x and y be each generators of the quadric S of the above system. Then the plane xP will cut the quartic in a fourth point P', such that PP' is a generator of S of the opposite system to x. Thus yP will also cut the quartic in the fourth point P', whence it is evident that, given any plane through x, there exists one, and only one, corresponding plane through y such that their line of intersection is a chord of the quartic, and *vice versa*. Thus a one-to-one algebraic correspondence exists between the planes xP and yP. It is thus plain that, *if A, B, C and D, E, F be two triads of points on the quartic, and if x and y be two chords of the same system, then if $x(A, B, C)$ is apolar to $x(D, E, F)$, therefore $y(A, B, C)$ is apolar to $y(D, E, F)$.*

We thus see that, if A, B, C and D, E, F be two triads of points on the quartic, and if O be one point on the quartic such that the projection of the quartic from O is a cubic curve having the projections of A, B, C and D, E, F as co-apolar triangles, the same property holds wherever O is situated on the quartic. Let, in fact, O' be another point on the quartic. Then, since every chord x of the quartic passing through O is such that $x(A, B, C)$ is apolar to $x(D, E, F)$ by hypothesis, therefore every chord x' passing through O' is such that $x'(A, B, C)$ is apolar to $x'(D, E, F)$ by what we have just proved.

It is plain, therefore, that in order to solve the problem of finding two triads A, B, C and D, E, F on a given quartic such that the quartic projects from a given point O on it into a cubic curve having the projections of A, B, C and D, E, F as co-apolar triangles, we have only to solve the problem for *one particular position of O*.

4. Let A, B, C and D, E, F be two suitable triads on the quartic, and let the planes ABC and DEF cut the quartic again in U and V respectively. Consider the cubic curve which is the projection of the quartic from U as vertex. Then A, B, C project into three collinear points A', B', C'. Now it was proved in my former paper (Art. 5) that, if A', B', C' be an apolar triad and at the same time collinear, the tangents at A', B', C' to the cubic are concurrent. Hence a line can be drawn through U intersecting the tangents at A, B, C to the quartic. This is the property of a suitable triad on the quartic that corresponds to the fact of an apolar triangle inscribed in the plane cubic. (See Art. 2 of my former paper.) We thus have that, *if A, B, C, U be four coplanar*

points on a quartic such that the same line through U meets the tangents at A, B, C, then the quartic projects from any point on itself into a cubic curve having the projection of A, B, C as an apolar triangle. We shall refer henceforth to a triad A, B, C situated on the quartic and possessing the above property as an "apolar triangle on the quartic," and we shall refer to the line through U meeting the tangents at A, B, C as "the axis of the apolar triangle ABC."

5. Let now u be the line through U meeting the tangents at A, B, C, and let v be the line through V meeting the tangents at D, E, F. Let the two planes ABC and DEF intersect in the line l. Then, if we project the quartic from the point U into a cubic lying on the plane DEF, it is plain from Art. 2 that the line u cuts the plane DEF in a point which is the pole of l with respect to the triangle DEF. Now, by Art. 2, the necessary and sufficient conditions that the given quartic be projected from U into a cubic curve having the projections of A, B, C and D, E, F as co-apolar tringles is (1) that ABC and DEF be apolar triangles on the quartic, and (2) that u pass through the pole of l with respect to the triangle DEF. If these conditions be fulfilled, it is plain from Art. 3 that the same property holds with regard to V. We have thus the following theorem:—

If ABC be an apolar triangle of the quartic, whose axis passes through the pole with respect to the apolar triangle DEF of the line of intersection of the two planes ABC and DEF, then the axis of DEF passes through the pole of the same line with respect to ABC.

6. We may now state the symmetrical solution of the problem proposed:—

If ABC and DEF be two apolar triangles on an elliptic skew quartic such that the axis of either passes through the pole with respect to the other of the line of intersection of the planes ABC and DEF, then the quartic projects from any point on itself into a plane cubic having the projections of ABC and DEF as co-apolar triangles.

7. The following is a noteworthy property of an apolar triangle situated on the elliptic skew quartic.

We use the same notation as above:—

The quadric having the tangents at A, B, C as generators touches the quartic at U.

Let the quadric T having the tangents at A, B, C as generators meet the quartic again in U and U'.

Then the quartic would project from U' into a cubic having A', B', C' (the projections of A, B, C) as an apolar triangle. But a certain line through U' intersects the tangents at A, B, C, since U' lies on the quadric T. Hence the tangents at A', B', C' to the cubic meet in a point. But it is easy to shew that, if A', B', C' be an inscribed apolar triangle to the cubic curve (A', B', C' not being collinear), the tangents at whose vertices are concurrent, then the point of concurrency lies on the cubic; in which case the line through U' meeting the tangents at A, B, C must be a chord of the quartic. But it is known that, if the tangents at A, B, C meet a chord of the quartic, they will meet all the chords of the quartic belonging to the same system. This would require that the line through U meeting the tangents at A, B, C should be a chord of the quartic, which is known not to be the case in general, since the tangents at the points of intersection of a tangent to the Cayleyan with the projected cubic curve do not in general intersect in a point on the cubic. To obviate this difficulty, U' must coincide with U.

This is otherwise obvious from using elliptic parameters.

THE SOLUTION OF THE HOMOGENEOUS LINEAR DIFFERENCE EQUATION OF THE SECOND ORDER

(SECOND PAPER.)

By G. N. WATSON.

[Received March 28th, 1911.—Read April 27th, 1911.]

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PART II. $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$.

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1. In a paper with the same title,* I obtained a definite functional form for the solution of the difference equation

$$A(x)f(x+1) - B(x)f(x) + C(x)f(x-1) = 0, \tag{1}$$

where $A(x)$, $B(x)$, $C(x)$ were known uniform functions of the complex variable x, such that, when $|x| > \mu$, the function $\dfrac{A(x)\,C(x+1)}{B(x)\,B(x+1)}$ could be expanded into the convergent series

$$\frac{A(x)\,C(x+1)}{B(x)\,B(x+1)} \equiv v(x) = \lambda_0 + \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \dots, \tag{2}$$

where λ_0 was not a real quantity greater than $\frac{1}{4}$.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 125-161; this will be referred to as the first paper.

It was shewn that a solution $f(x)$, of the equation (1), could be constructed by means of an infinite product of infinite continued fractions, a typical continued fraction being

$$\psi(x) = \underset{m=\infty}{\mathrm{Lt}} \left[\frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \dots - \frac{v(x+m-1)}{1} - w(x+m)\right], \quad (2a)$$

where $w(x)$ was a function of x depending on the first few coefficients of $v(x)$; and the first two terms of the asymptotic expansion of $\psi(x)$ when $|x|$ was large and* $R(x) > -\frac{1}{2}$ were obtained.

The objects of the present paper are two-fold: the first is to shew that if we assume a series for $\psi(x)$, in descending powers of x, and determine the coefficients from the equation

$$\psi(x) = [1 - v(x)\,\psi(x+1)]^{-1},$$

the series with coefficients so obtained does actually give a complete asymptotic expansion† of $\psi(x)$ when $|x|$ is large and $R(x) \geqslant -\frac{1}{2}$; the second is to determine how $\psi(x)$ and the connected solution $f(x)$ behave when $R(x)$ is large and negative.

In Part I of this paper we make the investigation when λ_0 is not equal to $\frac{1}{4}$; we shew that $\psi(x)$ possesses a complete asymptotic expansion,‡ when $|x|$ is large and $R(x) \geqslant -\frac{1}{2}$; if λ_0 is not a real quantity greater than $\frac{1}{4}$, this holds for all large values of $|x|$ except in the vicinity of the negative part of the real axis; this result is true when λ_0 is a real quantity greater than $\frac{1}{4}$, but it has to be obtained by a much more complicated method, and the proof is deferred to § 8 of Part II; the results of Part I are generalisations of Cases (I), (II) and (III) of the first paper.

In Part II we generalise the results of Case (III*a*) of the first paper, to prove theorems similar to those just stated when $\lambda_0 = \frac{1}{4}$ and $\lambda_1 = 0$.

As in the first paper, we need the following theorems:—

(I) (Weierstrass's theorem.)—If $\sum_{n=0}^{\infty} u_n$ be not absolutely convergent, while $\sum_{n=0}^{\infty} (u_n - a_n)$ and $\sum_{n=0}^{\infty} u_n^2$ are absolutely convergent,

$$\prod_{n=0}^{\infty} [(1+u_n)\exp(-a_n)]$$

converges absolutely.

---

* We denote the real part of a complex variable x by $R(x)$; and we put $R(-ix) = I(x)$, so that $x = R(x) + iI(x)$.

† The expansion is asymptotic in the sense of Poincaré.

‡ That is to say, a series which can be expanded to *any assigned finite* number of terms of negative powers of x.

(II) (The fundamental continuity theorem.)—If when $|x| > \mu$,

$$S = \frac{a_1}{x} + \frac{a_2}{x^2} + \dots$$

be a convergent series, a finite quantity μ' can be found such that when $|x| > \mu'$, $|S|$ is less than an assigned positive quantity.

We also need the following Lemma:—

If, when $|x| > \gamma$ and* $R(x) \geqslant -\frac{1}{2}$, ζ_x be such that $|\zeta_x| < K$, then a quantity K_1 can be found such that

$$\left| \prod_{n=0}^{s} \left\{ 1 + \frac{a}{x+n} + \frac{\zeta_{x+n}}{(x+n)^2} \right\} \right| < K_1,$$

where K, K_1 are independent of s and x (s being *any* positive integer), provided that $R(a) \leqslant 0$.

To prove the Lemma we observe that we can find a quantity γ' ($\geqslant \gamma$), such that when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$,

$$\left| \frac{a}{x+n} + \frac{\zeta_{x+n}}{(x+n)^2} \right| < 1,$$

so that $$\log \left\{ 1 + \frac{a}{x+n} + \frac{\zeta_{x+n}}{(x+n)^2} \right\} = \frac{a}{x+n} + \frac{\xi_{x+n}}{(x+n)^2},$$

where $|\xi_{x+n}| < J$, a finite quantity depending on γ'. Hence

$$\left| \prod_{n=0}^{s} \left\{ 1 + \frac{a}{x+n} + \frac{\zeta_{x+n}}{(x+n)^2} \right\} \right| < \exp \left[R \left\{ \sum_{n=0}^{s} \frac{a}{x+n} \right\} + J \sum_{n=0}^{s} |(x+n)|^{-2} \right].$$

It is readily proved† that $\sum_{n=0}^{\infty} |x+n|^{-2} < J_1$, a finite quantity depending on γ'. Also

$$\sum_{n=0}^{s} \frac{1}{x+n} = \frac{\Gamma'(x+s+1)}{\Gamma(x+s+1)} - \frac{\Gamma'(x)}{\Gamma(x)}$$

$$= \log(x+s+1) - \log x + \frac{\eta_{x+s+1}}{x+s+1} - \frac{\eta_x}{x},$$

where $|\eta_x| < J_2$, a finite quantity depending on γ'; this follows from Mellin's asymptotic expansion of the gamma function.

---

* So that $|x|$, $|x+1|$, $|x+2|$, ... form a non-decreasing sequence.

† See, *e.g.*, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 168, footnote.

Therefore

$$R\sum_{n=0}^{s}\frac{a}{x+n} < R[a\{\log(x+s+1)-\log x\}]+\frac{2J_2|a|}{\gamma'}$$

$$< R(a)\log\left|\frac{x+s+1}{x}\right| - I(a)\{\arg(x+s+1)-\arg x\}+\frac{2J_2|a|}{\gamma'}$$

$$< \pi|I(a)|+2J_2|a|(\gamma')^{-1},$$

since $R(a)\leqslant 0$, $|x+s+1|\geqslant|x|$ and $|\arg(x+s+1)-\arg x|<\pi$.

From this inequality, we deduce the required result when $|x|>\gamma'$; to obtain it when $|x|>\gamma$, we observe that, when $|x|>\gamma$, $R(x)\geqslant-\frac{1}{2}$, there are only a finite number of terms at the beginning of the original product, for which $|x|\leqslant\gamma'$, and the modulus of the product of these terms does not exceed a finite quantity independent of x and s.

PART I.—λ_0 is not equal to $\frac{1}{4}$.

2. Let a_1, a_2 be the roots of the equation

$$Z^2-Z+\lambda_0=0,$$

so chosen* that $|a_2|\leqslant|a_1|$; we notice that $a_1\neq a_2$.

We shall shew that it is possible to determine (by purely algebraical processes) a series of constants $k_0, k_1, k_2, \ldots$ (where $k_0=a_1^{-1}$), such that if $\Psi(x)$ be defined by the equation

$$\Psi(x)=\underset{m=\infty}{\mathrm{Lt}}\left[\frac{1}{1}-\frac{v(x)}{1}-\frac{v(x+1)}{1}-\ldots-\frac{v(x+m)}{1}\sum_{n=0}^{r}\frac{k_n}{(x+m+1)^n}\right], \quad (3)$$

then $\Psi(x)$ has a definite limit independent of r (when r is any assigned integer greater than or equal to 2), and it asymptotically approaches the value given by

$$\Psi(x)\sim k_0+\frac{k_1}{x}+\frac{k_2}{x^2}+\ldots+\frac{k_r}{x^r}.$$

If $|a_2|=|a_1|$, it is also necessary to assume that

$$r>R\{\lambda_1\lambda_0^{-1}(a_1-a_2)^{-1}\}.$$

[It may also be shewn that,† when $|a_2|<|a_1|$, if $\psi(x)$ be defined as in § 6 of the first paper, viz.,

---

* If the moduli of the roots are equal, a_1 may be *either* root; this is the case when λ_0 is real and greater than $\frac{1}{4}$.

† It cannot be proved that $\psi(x)$ as defined in the first paper has a definite value when $|a_2|=|a_1|$.

$$\psi(x) = \operatorname*{Lt}_{m=\infty} \left[\frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots - \frac{v(x+m)}{1} \left\{ k_0 + \frac{k_1}{x+m+1} \right\} \right],$$

then

$$\Psi(x) = \psi(x).]$$

Putting $k_0 = a_1^{-1}$, let us consider quantities η, η' connected by the relation

$$\sum_{n=0}^{r} \frac{k_n}{x^n} + \frac{\eta}{x^r} = \frac{1}{1 - v(x) \left\{ \sum_{n=0}^{r} \frac{k_n}{(x+1)^n} + \frac{\eta'}{(x+1)^r} \right\}}. \tag{3a}$$

By the fundamental continuity theorem, we may put

$$1 - v(x) \sum_{n=0}^{r} \frac{k_n}{(x+1)^n} = a_1 \left[1 - \sum_{n=1}^{r+1} \frac{c_n}{x^n} - \frac{\epsilon_x}{x^{r+2}} \right], \tag{3b}$$

where $|\epsilon_x|$ does not exceed a finite quantity independent of x, provided that $|x| > \mu$ and $|x+1| > \mu$; and ϵ_x is independent of η'.

Also, when $n \leqslant r$,

$c_n = a_1^{-1}\lambda_0 k_n +$ terms involving the quantities 'k' with suffixes *less than n*.

[This equation is of the same form no matter what value the integer r may have.]

Thus

$$\sum_{n=0}^{r} \frac{k_n}{x^n} + \frac{\eta}{x^r} = a_1^{-1} \left[1 - \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta' . v(x)}{(x+1)^r} \right\} \right]^{-1}.$$

Using the identity

$$(1-U)^{-1} \equiv 1 + U + U^2 + \ldots + U^{r+1} + U^{r+2}/(1-U),$$

we get

$$\sum_{n=0}^{r} \frac{k_n}{x^n} + \frac{\eta}{x^r}$$

$$= a_1^{-1} + \sum_{m=1}^{r+1} a_1^{-1} \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta' . v(x)}{(x+1)^r} \right\}^m$$

$$+ a_1^{-1} \left\{ \sum_{n-1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta' . v(x)}{(x+1)^r} \right\}^{r+2}$$

$$\times \left[1 - \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta' . v(x)}{(x+1)^r} \right\} \right]^{-1}. \tag{4}$$

On expanding the terms

$$\left\{ \left(\sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} \right) + \frac{a_1^{-1}\eta' . v(x)}{(x+1)^r} \right\}^m$$

by the binomial theorem, we see that

$$\sum_{m=1}^{r+1} a_1^{-1} \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta'.v(x)}{(x+1)^r} \right\}^m$$

may be written in the form

$$\sum_{n=1}^{r+1} \frac{d_n}{x^n} + \frac{\epsilon'_x}{x^{r+2}} + \sum_{n=1}^{r+1} \left\{ \frac{\eta'.v(x)}{(x+1)^r} \right\}^n \{g_n(x) + a_1^{-n-1}\},$$

where $|x.g_n(x)|$ does not exceed a finite quantity independent of x when $|x| > \mu$ and $|x+1| > \mu$; ϵ'_x satisfies the same conditions as ϵ_x; and $g_n(x)$ is independent of η'.

The first few terms in the expansion of $g_1(x)$ are

$$g_1(x) = \frac{2c_1 a_1^{-2}}{x} + \frac{(2c_2 + 3c_1^2)\, a_1^{-2}}{x^2} + \text{terms in } x^{-3}, \;\ldots. \tag{4a}$$

Also, when $n \leqslant r$,

$d_n = a_1^{-2}\lambda_0 k_n +$ terms involving the quantities 'k' with suffixes *less than n*.

[This equation does not involve the quantity r.]

Now let us choose $d_1, d_2, \ldots, d_r$, so that

$$d_1 = k_1, \quad d_2 = k_2, \quad \ldots, \quad d_r = k_r.$$

Since $\lambda_0/a_1^2 \neq 1$, this choice determines finite values for* $k_1, k_2, \ldots, k_n$ in terms of $a_1, a_2, \lambda_1, \lambda_2, \ldots, \lambda_r$ uniquely; the quantities† $c_1, c_2, \ldots, c_{r+1}$ are also determined uniquely; d_{r+1} can then be determined uniquely in terms of these quantities and λ_{r+1}.

With this choice of $k_1, k_2, \ldots, k_r$ we have, from (4),

$$\begin{aligned} \frac{\eta}{x^r} = \sum_{n=1}^{r+1} \left\{ \frac{\eta'.v(x)}{(x+1)^r} \right\}^n \{g_n(x) + a_1^{-n-1}\} + \frac{d_{r+1}}{x^{r+1}} + \frac{\epsilon'_x}{x^{r+2}} \\ + a_1^{-1} \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta'.v(x)}{(x+1)^r} \right\}^{r+2} \\ \times \left[1 - \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1}\eta'.v(x)}{(x+1)^r} \right\} \right]^{-1}. \end{aligned}$$

---

* The value of k_1 (which will be required subsequently) is $\lambda_1 a_1^{-2}(a_1 - a_2)^{-1}$.

† c_{r+1} is determined by equation (3b).

After reduction, this may be written in the form

$$\begin{aligned}\eta = \eta' &\left\{ \frac{a_2}{a_1} + \frac{\lambda_1(a_1-a_2)^{-1}-r\lambda_0}{a_1^2 x} + \frac{\zeta_x}{x^2} \right\} \\ &+ \sum_{n=2}^{r+1} \frac{\{\eta' . v(x)\}^n x^n}{(x+1)^{nr}} \{g_n(x)+a_1^{-n-1}\} + \frac{d_{r+1}}{x} + \frac{\epsilon'_x}{x^2} \\ &+ \frac{1}{a_1 x^2} \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^{n-1}} + \frac{\epsilon_x}{x^{r+1}} + \frac{x a_1^{-1} \eta' . v(x)}{(x+1)^r} \right\}^{r+2} \\ &\times \left[1 - \left\{ \sum_{n=1}^{r+1} \frac{c_n}{x^n} + \frac{\epsilon_x}{x^{r+2}} + \frac{a_1^{-1} \eta' . v(x)}{(x+1)^r} \right\} \right]^{-1}, \qquad (5)\end{aligned}$$

where $|\zeta_x|$ does not exceed a finite quantity independent of x, when $|x| > \mu$ and $|x+1| > \mu$.

The subsequent analysis differs according as $|a_2|$ is or is not equal to $|a_1|$.

3A. If $|a_2| < |a_1|$, we can find a finite quantity β, greater than or equal to μ, such that when $|x| > \beta$ and $|x+1| > \beta$, we have

$$\left| \frac{a_2}{a_1} + \frac{\lambda_1(a_1-a_2)-r\lambda_0}{a_1^2 x} + \frac{\zeta_x}{x^2} \right| < \sigma,$$

where σ is any quantity such that $|a_2 a_1^{-1}| < \sigma < 1$; this follows from the fundamental continuity theorem.

Writing $$\sum_{n=1}^{r+1} \frac{c_n}{x^{n-1}} + \frac{\epsilon_x}{x^{r+1}} + \frac{x a_1^{-1} \eta' . v(x)}{(x+1)^r} = F(x, \eta')$$

for brevity, and observing that when $2 \leqslant n \leqslant r+1$, we have

$$\left| \frac{\{v(x)\}^n x^r}{(x+1)^{nr}} \right| |g_n(x)+a_1^{-n-1}| < \frac{\Theta}{|x|^r}$$

(where Θ is a finite quantity independent of x when $|x| > \beta$ and $|x+1| > \beta$), we deduce, from (5), the following inequality:

$$|\eta| < \sigma|\eta'| + \frac{\Theta \sum\limits_{n=2}^{r+1} |\eta'|^n}{|x|^r} + \frac{\Theta_1}{|x|} + \frac{|F(x, \eta')|^{r+2}}{|a_1 x^2|} |[1-x^{-1}F(x, \eta')]|^{-1}, \qquad (6)$$

where Θ_1 is a finite quantity so chosen that

$$|d_{r+1}+\epsilon'_x x^{-1}| < \Theta_1.$$

Let Θ_2 be any quantity such that $0 < \Theta_2 < 1$; then we can find a finite quantity Θ_3 such that, when $|x| > \beta$ and $|x+1| > \beta$,

$$\left|\sum_{n=1}^{r+1}\frac{c_n}{x^{n-1}}+\frac{\epsilon_r}{x^{r+1}}\right|+\Theta_2\left|\frac{x\,.\,v(x)}{a_1(x+1)^r}\right| < \Theta_3. \tag{7}$$

We can now find a finite quantity γ greater than or equal to β such that when $|x| > \gamma$ and $|x+1| > \gamma$, the following inequalities hold, viz.,

$$|x| > 2\Theta_3, \tag{8}$$

$$\frac{\Theta_1\sum\limits_{n=2}^{r+1}\Theta_2^n}{|x|^r}+\frac{\Theta_1}{|x|}+\frac{2\Theta_3^{r+2}}{|a_1x^2|} < (1-\sigma)\Theta_2. \tag{9}$$

We shall now suppose that $|x| > \gamma$ *and* $|x+1| > \gamma$; *and also that*

$$|\eta'| < \Theta_2. \tag{10}$$

From (7) and (10) it follows at once that $|F(x, \eta')| < \Theta_3$; and therefore

$$|1-x^{-1}F(x, \eta')| > 1-|x|^{-1}\Theta_3 > \tfrac{1}{2},$$

on using (8). Hence, from (6),

$$|\eta| < \sigma|\eta'|+\frac{\Theta\sum\limits_{n=2}^{r+1}\Theta_2^n}{|x|^r}+\frac{\Theta_1}{|x|}+\frac{2\Theta_3^{r+2}}{|a_1x^2|}, \tag{11}$$

so that, on using (9), we get

$$|\eta| < \sigma|\eta'|+(1-\sigma)\Theta_2;$$

and therefore, by (10),
$$|\eta| < \Theta_2. \tag{12}$$

We notice that (11) may be written in form

$$|\eta| < \sigma|\eta'|+\Theta_4|x|^{-1}, \tag{12a}$$

where Θ_4 is finite and independent of x.

That is to say, we have shewn that when η *is connected with* η' *by equation* (3a), *we can find a finite quantity* γ *such that when* $|x| > \gamma$, $|x+1| > \gamma$ *and* $|\eta'| < \Theta_2$ *the inequalities* (12) *and* (12a) *are true.*

Now suppose that x is so restricted that all the quantities

$$|x|,\ |x+1|,\ |x+2|,\ \ldots$$

are greater than γ; and define quantities $\eta_x, \eta_{x+1}, \ldots, \eta_{x+m}$ by the following series of equations

$$\sum_{n=0}^{r}\frac{k_n}{(x+p)^n}+\frac{\eta_{x+p}}{(x+p)^r}=\frac{1}{1-v(x+p)\left\{\sum\limits_{n=0}^{r}\frac{k_n}{(x+p+1)^n}+\frac{\eta_{x+p+1}}{(x+p+1)^r}\right\}}, \tag{13}$$

where p takes the values 0, 1, 2, ..., m and $\eta_{x+m+1}=0$, so that

$$|\eta_{x+m+1}| < \Theta_2.$$

By the same analysis as that by which we deduced (12) and (12a) from (3a), since the necessary conditions are satisfied, we deduce in succession from the series of equations (13) that

$$|\eta_{x+p}| < \Theta_2, \tag{14}$$

$$|\eta_{x+p}| < \sigma |\eta_{x+p+1}| + \Theta_4 |x+p|^{-1}, \tag{14a}$$

where p takes the values $m, m-1, m-2, \ldots, 2, 1, 0$.

Let y be that one of the quantities $x, x+1, x+2, \ldots$ which has the smallest modulus; then, from (14a), we get

$$|\eta_{x+p}| < \sigma |\eta_{x+p+1}| + \Theta_4 |y|^{-1},$$

where $p = 0, 1, 2, \ldots, m$; from this series of inequalities we deduce* that

$$|\eta_x| < \sigma^{m+1} |\eta_{x+m+1}| + \frac{1-\sigma^{m+1}}{1-\sigma} \Theta_4 |y|^{-1}.$$

That is to say, $\quad |\eta_x| < (1-\sigma)^{-1} \Theta_4 |y|^{-1},$

since $\eta_{x+m+1} = 0$ and $\sigma < 1$.

Consequently, if we define $\Psi_m^{(r)}(x)$ by the equation

$$\Psi_m^{(r)}(x) = \frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots - \frac{v(x+m)}{1} \sum_{n=0}^{r} \frac{k_n}{(x+m+1)^n}, \tag{15}$$

and if all the quantities $|x|, |x+1|, |x+2|, \ldots$ are greater than γ, then

$$\Psi_m^{(r)}(x) = k_0 + \frac{k_1}{x} + \ldots + \frac{k_r}{x^r} + \frac{\eta_x}{x^r}, \tag{15a}$$

where $|\eta_x| < (1-\sigma)^{-1} \Theta_4 |y|^{-1}$, and the quantities Θ_4 and σ are independent of m.

We deduce one or other of two conclusions from (15a) according to the value of x; first, if $|x| > \gamma$ and $R(x) \geqslant -\frac{1}{2}$, all the quantities

$$|x|, \; |x+1|, \; |x+2|, \; \ldots$$

are greater than γ, and $y = x$; so that, from (15a), $\Psi_m^{(r)}(x)$ has for its asymptotic expansion

$$\Psi_m^{(r)}(x) \sim k_0 + \frac{k_1}{x} + \ldots + \frac{k_r}{x^r}. \tag{15b}$$

Secondly, if $R(x) \leqslant -\frac{1}{2}$ and $|I(x)| > \gamma$, we have

$$|y| \geqslant |I(x)| \quad \text{and} \quad |k_r + \eta_x| \leqslant |k_r| + (1-\sigma)^{-1} \Theta_4 \gamma^{-1};$$

so that in the regions of the plane defined by $R(x) \leqslant -\frac{1}{2}$ and $|I(x)| > \gamma$,

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* By eliminating $\eta_{x+1}, \eta_{x+2}, \ldots, \eta_{x+m}$.

$\Psi^{(r)}_m(x)$ has for its asymptotic expansion

$$\Psi^{(r)}_m(x) \sim k_0 + \frac{k_1}{x} + \ldots + \frac{k_{r-1}}{x^{r-1}}, \tag{15c}$$

and r is any finite integer greater than or equal to 2.

4A. We now wish to show that $\operatorname*{Lt}_{m=\infty} \Psi^{(r)}_m(x)$ is a definite quantity independent of r.

Let
$$v(x+m) \sum_{n=0}^{r} \frac{k_n}{(x+m+1)^m} = W^{(r)}_m.$$

Also let the n-th convergents of the continued fractions for $\Psi^{(r)}_m(x)$, $\Psi^{(r)}_{m+1}(x)$ be $p^{(r)}_n/q^{(r)}_n$, $P^{(r)}_n/Q^{(r)}_n$ respectively; the last elements of the continued fractions being taken to be $W^{(r)}_m$, $W^{(r)}_{m+1}$ respectively.

Since, when $n \leqslant m+1$, $p^{(r)}_n = p^{(r+1)}_n$, $q^{(r)}_n = q^{(r+1)}_n$, we have

$$\Psi^{(r+1)}_m(x) - \Psi^{(r)}_m(x) = \frac{p^{(r+1)}_{m+1} - W^{(r+1)}_m p^{(r+1)}_m}{q^{(r+1)}_{m+1} - W^{(r+1)}_m q^{(r+1)}_m} - \frac{p^{(r)}_{m+1} - W^{(r)}_m p^{(r)}_m}{q^{(r)}_{m+1} - W^{(r)}_m q^{(r)}_m}$$
$$= \frac{(W^{(r+1)}_m - W^r_m)(p^{(r)}_{m+1} q^{(r)}_m - q^{(r)}_{m+1} p^{(r)}_m)}{q^{(r+1)}_{m+2} q^{(r)}_{m+2}}.$$

Hence
$$\Psi^{(r+1)}_m(x) - \Psi^{(r)}_m(x) = \frac{k_{r+1}}{(x+m+1)^{r+1}} \frac{\prod_{t=0}^{m-1} \{v(x+t)\}}{q^{(r+1)}_{m+2} q^{(r)}_{m+2}},$$

and it is easily proved that

$$q^{(r)}_{m+2} = \prod_{t=0}^{m} \{\Psi^{(r)}_{m-t}(x+t)\}^{-1}.$$

Consequently

$$\Psi^{(r+1)}_m(x) - \Psi^{(r)}_m(x)$$
$$= \frac{k_{r+1}}{(x+m+1)^{r+1}} \prod_{t=0}^{m-1} [v(x+t)] \prod_{t=0}^{m} [\Psi^{(r)}_{m-t}(x+t)] \prod_{t=0}^{m} [\Psi^{(r+1)}_{m-t}(x+t)]. \tag{16a}$$

In a similar manner, we may shew that

$$\Psi^{(r)}_{(m+1)}(x) - \Psi^{(r)}_m(x)$$
$$= U^{(r)}_m \prod_{t=0}^{m} [v(x+t)] \prod_{t=0}^{m+1} [\Psi^{(r)}_{m+1-t}(x+t)] \prod_{t=0}^{m} [\Psi^{(r)}_{m-t}(x+t)], \tag{16b}$$

where

$$U_m^{(r)} = \left[1 - v(x+m+1) \sum_{n=0}^{r} \frac{k_n}{(x+m+2)^n}\right]$$
$$\times \left[\frac{1}{1 - v(x+m+1) \sum_{n=0}^{r} \frac{k_n}{(x+m+2)^n}} - \sum_{n=0}^{r} \frac{k_n}{(x+m+1)^r}\right].$$

By methods similar to those employed on p. 144 of the first paper, we may shew that

$$\left| \prod_{t=0}^{m} \left\{ k_0^{-1} \Psi_{m-t}^{(r)}(x+t) \exp\left(\frac{-k_1}{k_0(x+t)}\right) \right\} \right| < J_1^{(r)}, \tag{17a}$$

$$\left| \prod_{t=0}^{m} \left\{ \lambda_0^{-1} v(x+t) \exp\left(\frac{-\lambda_1}{\lambda_0(x+t)}\right) \right\} \right| < J_2, \tag{17b}$$

where $J_1^{(r)}$ and J_2 are finite quantities* independent of m, when $r \geqslant 2$ and $|x| > \gamma$, $R(x) \geqslant -\frac{1}{2}$. The first of these results follows from (15*b*).

Also, noticing how the coefficients $k_0, k_1, \ldots$ were chosen, by means of (3*a*), it is not difficult to see that when

$$\left[1 - v(x+m+1) \sum_{n=0}^{r} \frac{k_n}{(x+m+2)^n}\right]^{-1}$$

is expanded in powers of $(x+m+1)^{-1}$ the first $r+1$ coefficients are $k_0, k_1, \ldots, k_r$. Hence $|U_m^{(r)}| < J_3^{(r)} |x+m+1|^{-r-1}$, where $J_3^{(r)}$ is finite.

Applying these inequalities to (16*a*) and (16*b*), we obtain the two following results

$$|\Psi_m^{(r+1)}(x) - \Psi_m^{(r)}(x)|$$
$$< |k_{r+1}| J_1^{(r)} J_1^{(r+1)} J_2 |x+m+1|^{-r-1} |k_0^{2m+2} \lambda_0^m|$$
$$\times \left[\exp\left\{\left(\frac{2k_1}{k_0} + \frac{\lambda_1}{\lambda_0}\right)\left(\frac{\Gamma'(x+m+1)}{\Gamma(x+m+1)} - \frac{\Gamma'(x)}{\Gamma(x)}\right) + \frac{2k_1}{k_0(x+m+1)}\right\}\right]$$
$$< J_4 |x+m+1|^{-r-1} \left|\frac{a_2}{a_1}\right|^m \left|\exp\left\{\frac{\lambda_1}{\lambda_0(a_1-a_2)} \log\left(\frac{x+m+1}{x}\right)\right\}\right|$$

(using the asymptotic expansion of the gamma function).

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* The inequality (17*b*) has to be slightly modified when $\lambda_0 = 0$; the conclusions concerning $\mathop{\mathrm{Lt}}\limits_{m=\infty} \Psi_m^{(r)}(x)$ are, however, unaffected.

And similarly

$$|\Psi^{(r)}_{m+1}(x)-\Psi^{(r)}_{m}(x)|$$
$$< J_5|x+m+1|^{-r-1}\left|\frac{a_2}{a_1}\right|^m \left|\exp\left\{\frac{\lambda_1}{\lambda_0(a_1-a_2)}\log\left(\frac{x+m+1}{x}\right)\right\}\right|.$$

We may shew, as on p. 145 of the first paper, that for the range of values of x under consideration

$$\left|\frac{a_2}{a_1}\right|^m \left|\exp\left\{\frac{\lambda_1}{\lambda_0(a_1-a_2)}\log\left(\frac{x+m+1}{x}\right)\right\}\right| < J_6.$$

And consequently

$$|\Psi^{(r+1)}_{m}(x)-\Psi^{(r)}_{m}(x)| < J_7|x+m+1|^{-r-1}, \tag{18a}$$

$$|\Psi^{(r)}_{m+1}(x)-\Psi^{(r)}_{m}(x)| < J_8|x+m+1|^{-r-1}, \tag{18b}$$

where J_7, J_8 are finite and independent of m and x.

From (18*b*), if $n > m$,

$$|\Psi^{(r)}_{n}(x)-\Psi^{(r)}_{m}(x)| < |\Psi^{(r)}_{n}(x)-\Psi^{(r)}_{n-1}(x)|+\ldots+|\Psi^{(r)}_{m+1}(x)-\Psi^{(r)}_{m}(x)|$$
$$< J_8[|x+n|^{-r-1}+|x+n-1|^{-r-1}+\ldots$$
$$+|x+m+1|^{-r-1}].$$

Since $r \geqslant 2$, $\sum_{t=m+1}^{n} |x+t|^{-r-1} \to 0$ as $m \to \infty$, *i.e.*, $\Psi^{(r)}_{m}(x)-\Psi^{(r)}_{n}(x) \to 0$ as m, $n \to \infty$.

That is to say, $\operatorname*{Lt}_{m=\infty} \Psi^{(r)}_{m}(x)$ is a definite quantity, possibly depending on r; and by proceeding to the limit when $m = \infty$ in (18*a*), we finally arrive at the result that

$\operatorname*{Lt}_{m=\infty} \Psi^{(r)}_{m}(x)$ *is a definite quantity independent of* r.

We have thus proved the statement at the beginning of § 2, when $|a_2| < |a_1|$ and $|x| > \gamma$, $R(x) \geqslant -\frac{1}{2}$.

By the recurrence formula

$$\Psi^{(r)}_{m}(x) = [1-v(x)\,\Psi^{(r)}_{m-1}(x+1)]^{-1},$$

we see that the statement is true for all finite values of x with the exception of singularities of $v(x)$, $v(x+1)$, ...

Putting $$\operatorname*{Lt}_{m=\infty} \Psi^{(r)}_{m}(x) = \Psi(x),$$

the asymptotic expansion of $\Psi(x)$ is given by (15*b*) and (15*c*).

5A. From the results at the end of § 4A, we can deduce the form of the asymptotic expansion of a solution of the difference equation (1) when $|a_2| < |a_1|$. The solution is[*]

$$f(x) = u(x) \exp\left[-x \log a_1 + \frac{\lambda_1}{a_1(a_1-a_2)} \frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]$$

$$\times \prod_{n=1}^{\infty} \left[\frac{1}{a_1 \Psi(x+n)} \exp \frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right], \qquad (19a)$$

where $u(x)$ satisfies $\qquad u(x) B(x) = u(x-1) C(x)$.

If $|x+n|$ be sufficiently large, it follows without difficulty, from the results of § 3A, that $-\log\{a_1 \Psi(x+n)\}$ possesses an asymptotic expansion of the form

$$-\frac{k_1}{k_0(x+n)} + \frac{k'_2}{(x+n)^2} + \ldots + \frac{k'_m + \theta_{x+n}}{(x+n)^m},$$

where $|\theta_{x+n}| < J_m |x+n|^{-1}$ and J_m is finite when $|x+n| > \gamma'$ and $R(x+n) \geqslant -\frac{1}{2}$, γ' being finite.

Hence, when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$,

$$\prod_{n=1}^{\infty} \left[\frac{1}{a_1 \Psi(x+n)} \exp \frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right]$$

$$= \exp\left[\sum_{n=1}^{\infty} \left\{\frac{k_2}{(x+n)^2} + \ldots + \frac{k'_m + \theta_{x+n}}{(x+n)^m}\right\}\right].$$

Now $\sum_{n=1}^{\infty} (x+n)^{-m}$ possesses an asymptotic expansion given by the equation

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)^m} = \frac{1}{(m-1)x^{m-1}} - \frac{1}{2x^m} + \sum_{n=1}^{r} \frac{(-)^{n-1} {}_1B_{n+1} (m+n-1)!}{x^{m+n}\, n!\,(m-1)!} + \frac{L_{r,m}}{|x|^{r+m+1}},$$

where $L_{r,m}$ is finite and ${}_1B_{n+1}$ is one of Bernoulli's numbers.[†] Also

$$\left|\sum_{n=1}^{\infty} \frac{\theta_{x+n}}{(x+n)^m}\right| < J_m \sum_{n=1}^{\infty} |x+n|^{-m-1}$$

$$< J_m \sum_{n=1}^{\infty} [|x+\tfrac{1}{2}|^2 + (n-\tfrac{1}{2})^2]^{-\frac{1}{2}(m+1)} \quad [\text{when } R(x) \geqslant -\tfrac{1}{2}].$$

Hence, by Cauchy's integral formula, since $[|x+\frac{1}{2}|^2+(t-\frac{1}{2})^2]^{-\frac{1}{2}(m+1}$ steadily decreases when $t > 1$, we have

$$\left|\sum_{n=1}^{\infty} \frac{\theta_{x+n}}{(x+n)^m}\right| < J_m \left[\{|x+\tfrac{1}{2}|^2+\tfrac{1}{4}\}^{-\frac{1}{2}(m+1)} + \int_1^{\infty} \{|x+\tfrac{1}{2}|^2 + (t-\tfrac{1}{2})^2\}^{-\frac{1}{2}(m+1)} dt\right].$$

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[*] This follows as in § 6 of the first paper.

[†] The notation is that employed in Barnes's memoir on the gamma function, *Messenger of Mathematics*, Vol. XXIX, pp. 64-128.

Putting $t = \frac{1}{2}+\rho|x+\frac{1}{2}|$, we find that, when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$,

$$\left| \sum_{n=1}^{\infty} \frac{\theta_{x+n}}{(x+n)^m} \right| < J|x|^{-m},$$

where J is finite and independent of x.

Using the asymptotic expansion of the gamma function, we see that $f(x)$ possesses an asymptotic expansion of the form

$$f(x) = u(x) \exp\left[-x \log a_1 + \frac{\lambda_1}{a_1(a_1-a_2)} \log x + \sum_{n=1}^{m-1} \frac{l_n}{x^n} + \frac{\theta}{x^m}\right], \tag{19b}$$

where $|\theta| < J'$, and the quantities l_1, l_2, ... may be calculated, with sufficient labour, in terms of a_1, a_2, λ_1, λ_2, ...; and

$$|\arg x| < \pi, \qquad R(x) \geqslant -\tfrac{1}{2}, \qquad |x| > \gamma'.$$

By means of (15*c*) we can determine the form of the asymptotic expansion of $f(x)$ when $|I(x)| > \gamma'$ and $R(x) \leqslant -\frac{1}{2}$; the following alterations have to be made in the work immediately preceding:

In the asymptotic expansion of $\Gamma'(x+1)/\Gamma(x+1)$, $\log x$ has to be replaced by $-\pi \cot x\pi + \log(-x)$, where $|\arg(-x)| < \pi$.

And the appropriate expansion for $\sum_{n=1}^{\infty} (x+n)^{-m}$ is

$$\frac{(-)^m}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} (\pi \cot x\pi) + \frac{1}{(m-1)x^{m-1}} - \frac{1}{2x^m} + \sum_{n=1} \frac{(-)^{n-1}{}_1B_{n+1}(m+n-1)!}{x^{m+n}\, n!\, (m-1)!}.$$

Since, however, we can only prove that

$$\left| \sum_{n=1}^{\infty} \theta_{x+n}(x+n)^{-m} \right| < J''|I(x)|^{-m},$$

where J'' is finite, the asymptotic expansion does not give a good approximation when $|\arg(-x)|$ is small.

The expansion (19*b*) is that of which the dominant term was given by Ford[*] when x is restricted to be a positive integer.

3B. We now return to the point of divergence, at the end of § 2, to discuss the case when $|a_2| = |a_1|$ and $a_1 \neq a_2$; it is easily seen that, since

$$\frac{d_{r+1}}{x} \equiv \frac{a_1 d_{r+1}}{(a_1-a_2)x} - \frac{a_2}{a_1} \frac{a_1 d_{r+1}}{(a_1-a_2)(x+1)} - \frac{a_2 d_{r+1}}{(a_1-a_2)x(x+1)},$$

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[*] *Transactions of the American Mathematical Society*, Vol. x, pp. 319-336.

we may write (5) in the form

$$\left\{\eta-\frac{a_1 d_{r+1}}{(a_1-a_2)x}\right\} = \left\{\eta'-\frac{a_1 d_{r+1}}{(a_1-a_2)(x+1)}\right\}\left\{\frac{a_2}{a_1}+\frac{\lambda_1(a_1-a_2)^{-1}-r\lambda_0}{a_1^2 x}+\frac{\zeta_x}{x^2}\right\}$$

$$+\sum_{n=2}^{r+1}\frac{\{\eta'.v(x)\}^n x^r}{(x+1)^{nr}}\{g_n(x)+a_1^{-n-1}\}$$

$$+\frac{\epsilon_x''}{x^2}+\frac{1}{a_1x^2}\left\{\sum_{n=1}^{r+1}\frac{c_n}{x^{n-1}}+\frac{\epsilon_x}{x^{r+1}}+\frac{xa_1^{-1}\eta'.v(x)}{(x+1)^r}\right\}^{r+2}$$

$$\times\left[1-\left\{\sum_{n=1}^{r+1}\frac{c_n}{x^n}+\frac{\epsilon_x}{x^{r+2}}+\frac{a_1^{-1}\eta'.v(x)}{(x+1)^r}\right\}\right]^{-1}.$$

where ζ_x and ϵ_x'' are functions of the same nature as ϵ_x'.

Putting

$$\sum_{n=1}^{r+1}\frac{c_n}{x^{n-1}}+\frac{\epsilon_x}{x^{r+1}}+\frac{xa_1^{-1}\eta'.v(x)}{(x+1)^r} = F(x,\eta'),$$

we get, by taking the modulus of each side of the above equation,

$$\left|\eta-\frac{a_1 d_{r+1}}{(a_1-a_2)x}\right| \leqslant \left|\eta'-\frac{a_1 d_{r+1}}{(a_1-a_2)(x+1)}\right|\left|\left[1+\frac{\lambda_1\lambda_0^{-1}(a_1-a_2)^{-1}-r}{x}+\frac{a_1\zeta_x}{a_2x^2}\right]\right|$$

$$+\left|\epsilon_x''x^{-2}\right|+\sum_{n=2}^{r+1}\left|\frac{\{\eta'.v(x)\}^n x^r}{(x+1)^{nr}}\{g_n(x)+a_1^{-n-1}\}\right|$$

$$+\frac{1}{|a_1x^2|}\,|F(x,\eta')|^{r+2}\,|\{1-x^{-1}F(x,\eta')\}|^{-1}. \tag{20}$$

Remembering that we take r so large that

$$r > R\{\lambda_1\lambda_0^{-1}(a_1-a_2)^{-1}\}, \tag{20a}$$

we may shew, by the Lemma proved in the introduction, that

$$\left|\prod_{n=0}^{s}\left[1+\frac{\lambda_1\lambda_0^{-1}(a_1-a_2)^{-1}-r}{x+n}+\frac{a_1\zeta_{x+n}}{a_2(x+n)^2}\right]\right| < \Theta_5,$$

where Θ_5 is a finite quantity depending on μ, but not on s, when $|x| > \mu$ and $R(x) \geqslant -\frac{1}{2}$; we may obviously take $\Theta_5 > 1$.

Let

$$\left|\frac{a_1 d_{r+1}}{(a_1-a_2)\mu}\right| = \Theta_6.$$

Suppose that η' is chosen so that

$$|\eta'| < \Theta_6 \tag{21}$$

Now we can find finite quantities β and Θ_7 (β being greater than μ) such that

$$\left|\sum_{n=1}^{r+1}\frac{c_n}{x^{n-1}}+\frac{\epsilon_x}{x^{r+1}}\right|+\left|\frac{xa_1^{-1}v(x)}{(x+1)^r}\right|\Theta_6 < \tfrac{1}{2}\beta$$

[so that $|F(x, \eta')| < \tfrac{1}{2}\beta$] and

$$|\epsilon_r''| + \sum_{n=2}^{r+1} (\Theta_6)^n \left| \frac{\{v(x)\}^n x^{r+2}}{(x+1)^{nr}} \{g_n(x) + a_1^{-n-1}\} \right| + |a_1|^{-1} 2^{-r-1} \beta^{r+2} < \Theta_7,$$

so that Θ_7 and β are independent of η'.

Supposing now that $|x| > \beta$ and $R(x) \geqslant -\tfrac{1}{2}$, we see that

$$|1 - x^{-1} F(x, \eta')| > \tfrac{1}{2},$$

and consequently, from (20), we deduce that

$$\left| \eta - \frac{a_1 d_{r+1}}{(a_1 - a_2) x} \right|$$
$$\leqslant \left| \eta' - \frac{a_1 d_{r+1}}{(a_1 - a_2)(x+1)} \right| \left| \left[1 + \frac{\lambda_1 \lambda_0^{-1} (a_1 - a_2)^{-1} - r}{x} + \frac{a_1 \zeta_r}{x^2} \right] \right| + \frac{\Theta_7}{|x|^2}. \qquad (22)$$

We may find a finite quantity Θ_8 such that,* when

$$|x| > \beta \quad \text{and} \quad R(x) \geqslant -\tfrac{1}{2},$$

we have
$$\sum_{n=0}^{\infty} |x+n|^{-2} < \Theta_8 |x|^{-1}.$$

Now let
$$\left[\left| \frac{a_1 d_{r+1}}{a_1 - a_2} \right| + \Theta_8 \Theta_5 \Theta_7 \right] \Theta_6^{-1} \leqslant \gamma,$$

where γ is finite and $\gamma \geqslant \beta$.

We shall now assume that $|x| > \gamma$ *and* $R(x) \geqslant -\tfrac{1}{2}$.

Put
$$\left| 1 + \frac{\lambda_1 \lambda_0^{-1} (a_1 - a_2)^{-1} - r}{x} + \frac{a_1 \zeta_r}{x^2} \right| = \Omega_x,$$

and define quantities $\eta_x, \eta_{x+1}, \ldots, \eta_{x+m}$ by the system of equations

$$\sum_{n=0}^{r} \frac{k_n}{(x+p)^n} + \frac{\eta_{x+p}}{(x+p)^r} = \left[1 - v(x+p) \left\{ \sum_{n=0}^{r} \frac{k_n}{(x+p+1)^n} + \frac{\eta_{x+p+1}}{(x+p+1)^r} \right\} \right]^{-1},$$

where $p = 0, 1, \ldots, m$ and $\eta_{x+m+1} = a_1 d_{r+1} (a_1 - a_2)^{-1} (x+m+1)^{-1}$, so that

$$|\eta_{x+m+1}| < \Theta_6.$$

Then, if all the quantities $|\eta_{x+p+1}|, |\eta_{x+p+2}|, \ldots, |\eta_{x+m}|$ *are less than* Θ_6, we can deduce the system of inequalities

$$\left| \eta_{x+q} - \frac{a_1 d_{r+1}}{(a_1 - a_2)(x+q)} \right| < \left| \eta_{x+q+1} - \frac{a_1 d_{r+1}}{(a_1 - a_2)(x+q+1)} \right| \Omega_{x+q} + \frac{\Theta_7}{|x+q|^2}$$

(where $q = p, p+1, \ldots, m$), in the same manner as we deduced (22) from

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* This can be proved by Cauchy's integral formula; see § 5A above.

(3). Eliminating $\eta_{x+p+1}, \eta_{x+p+2}, \ldots, \eta_{x+m}$ from this system of inequalities, we get

$$\begin{aligned}\left|\eta_{x+p}-\frac{a_1 d_{r+1}}{(a_1-a_2)(x+p)}\right| &< \left|\eta_{x+m+1}-\frac{a_1 d_{r+1}}{(a_1-a_2)(x+m+1)}\right| \prod_{q=p}^{m} \Omega_{x+q} \\ &\quad + \frac{\Theta_7}{|x+p|^2}+\sum_{q=p+1}^{m} \Theta_7 \frac{1}{|x+q|^2}\left(\prod_{n=p+1}^{q} \Omega_{x+n}\right) \\ &< \Theta_5 \left|\eta_{x+m+1}-\frac{a_1 d_{r+1}}{(a_1-a_2)(x+m+1)}\right| + \Theta_7\Theta_5\Theta_8 |x+p|^{-1}.\end{aligned}$$

Therefore
$$\begin{aligned}|\eta_{x+p}| &< \left\{\left|\frac{a_1 d_{r+1}}{a_1-a_2}\right| + \Theta_7\Theta_5\Theta_8\right\} |x+p|^{-1} \\ &< \left\{\left|\frac{a_1 d_{r+1}}{a_1-a_2}\right| + \Theta_7\Theta_5\Theta_8\right\} \gamma^{-1} < \Theta_6.\end{aligned}$$

Hence, by induction, *all* the quantities $|\eta_{x+m+1}|, |\eta_{x+m}|, \ldots, |\eta_x|$ are less than Θ_6; and also, by the inequality immediately preceding,

$$|\eta_x| < \left\{\left|\frac{a_1 d_{r+1}}{a_1-a_2}\right| + \Theta_7\Theta_5\Theta_8\right\} |x|^{-1} < J|x|^{-1},$$

where J is finite. That is to say, if we define $\Psi_m^{(r)}(x)$ by the equation

$$\Psi_m^{(r)}(x) = \frac{1}{1-}\,\frac{v(x)}{1-}\,\frac{v(x+1)}{1}\,-\ldots-\,\frac{v(x+m)}{1}$$
$$\times\left\{\sum_{n=0}^{r} \frac{k_n}{(x+m+1)^n}+\frac{a_1 d_{r+1}}{(a_1-a_2)(x+m+1)^{r+1}}\right\}, \tag{23}$$

then, when $|x| > \gamma$ and $R(x) \geqslant -\frac{1}{2}$,

$$\Psi_m^{(r)}(x) = k_0 + \frac{k_1}{x} + \ldots + \frac{k_r+\eta_r}{x^r}, \tag{23a}$$

where $$|\eta_r| < J|x|^{-1}.$$

The proof that $\underset{m=\infty}{\mathrm{Lt}}\, \Psi_m^{(r)}(x)$ is a definite quantity, $\Psi(x)$, independent* of r is almost verbatim† the same as in § 4A. And, just as in § 5A, the

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* Provided that r satisfies the inequality (20a).

† It is necessary to make use of the fact that the series

$$\sum^{\infty} |x+m+1|^{-r-1}\left|\exp\left\{\frac{\lambda_1}{\lambda_0(a_1-a_2)} \log\frac{x+m}{x}\right\}\right|$$

is absolutely and uniformly convergent; and equations (18a) and (18b) have to be modified by inserting a factor $\left|\exp\left\{\frac{\lambda_1}{\lambda_0(a_1-a_2)}\log\left(\frac{x+m}{x}\right)\right\}\right|$ on the right-hand side.

expression $f(x)$ given in equation (19a) is a solution of the difference equation, provided that $\Psi(x)$ be defined by means of equation (23) and not by equation (15); and, finally, the asymptotic expansion (19b) of $f(x)$ is still true.

The asymptotic expansion of $f(x)$, when $R(x) < -\frac{1}{2}$, in the case when $|a_1| = |a_2|$ has to be obtained by other methods; this investigation is deferred to § 8 of Part II of the paper.

The solution of the associated transcendental equation

$$1 - v(x)\,\Psi(x)\,\Psi_1(x) = 0,$$

where $$\Psi_1(x) = \frac{1}{1-}\,\frac{v(x-1)}{1\;-}\,\frac{v(x-2)}{1\;-}\ldots$$

may be carried out in precisely the same manner as in Part I of the paper* by the writer on the subject when it was assumed that $|a_2| < |a_1|$.

PART II.—$\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$.

6. Let a, a' be the roots of the equation

$$Z^2 - \tfrac{1}{2}Z + \lambda_2 = 0,$$

it being assumed that $4a$ is not a positive integer† greater than unity.

We shall show that it is possible to determine a series of constants $k_0, k_1, k_2, \ldots$ (where $k_0 = 2$, $k_1 = 4a$) such that, if $\Psi(x)$ be defined by the equation

$$\Psi(x) = \underset{m=\infty}{\mathrm{Lt}}\left[\frac{1}{1-}\,\frac{v(x)}{1\;-}\,\frac{v(x+1)}{1\;-}\ldots-\frac{v(x+m)}{1}\left\{\sum_{n=0}^{r}\frac{k_n}{(x+m+1)^n}\right\}\right],$$

then $\Psi(x)$ is a definite quantity, independent of r, which asymptotically approaches the value given by

$$\Psi(x) \sim k_0 + \frac{k_1}{x} + \ldots + \frac{k_r}{x^r}$$

when $|x| > \gamma$, $R(x) \geqslant -\frac{1}{2}$, where γ is some finite quantity and r is any assigned integer greater than unity such that $r > R(4a)$.

Taking $k_0 = 2$, $k_1 = 4a$, let us consider quantities η, η' connected by the relation

$$\sum_{n=0}^{r}\frac{k_n}{x^n} + \frac{\eta}{x^r} = \left[1 - v(x)\left\{\sum_{n=0}^{r}\frac{k_n}{(x+1)^n} + \frac{\eta'}{(x+1)^r}\right\}\right]^{-1}. \quad .$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 162–177.

† a may be *either* root, except in so far as this restriction is concerned.

By the fundamental continuity theorem, we may put

$$1-v(x)\sum_{n=0}^{r}\frac{k_n}{(x+1)^n}=\tfrac{1}{2}\left[1-\sum_{n=1}^{r}\frac{c_n}{x^n}-\frac{\gamma_{r+1}}{x^{r+1}}-\frac{\epsilon_x}{x^{r+2}}\right],$$

where $|\epsilon_x|$ does not exceed a finite quantity independent of x when $|x|>\mu$ and $R(x)\geqslant -\tfrac{1}{2}$; and ϵ_x is independent of η'.

Also

$c_n=\tfrac{1}{2}k_n-\tfrac{1}{2}(n-1)k_{n-1}+$terms involving the quantities 'k' with suffixes *less* than $n-1$;

and*

$\gamma_{r+1}=\quad -\tfrac{1}{2}rk_r\quad +$terms involving the quantities 'k' with suffixes *less* than r, the term involving k_0 being $2\lambda_{r+1}k_0$.

We thus get

$$\sum_{n=0}^{r}\frac{k_n}{x^n}+\frac{\eta}{x^r}=2\left[1-\left\{\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}+\frac{2\eta'.v(x)}{(x+1)^r}\right\}\right]^{-1}.$$

Using the identity

$$(1-U)^{-1}\equiv 1+U+U^2+\ldots+U^{r+1}+U^{r+2}/(1-U),$$

we obtain the result that

$$\begin{aligned}\sum_{n=0}^{r}\frac{k_n}{x^n}+\frac{\eta}{x^r}=2+2\sum_{m=1}^{r+1}&\left\{\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}+\frac{2\eta'.v(x)}{(x+1)^r}\right\}^m\\&+2\left\{\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}+\frac{2\eta'.v(x)}{(x+1)^r}\right\}^{r+2}\\&\times\left[1-\left\{\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}+\frac{2\eta'.v(x)}{(x+1)^r}\right\}\right]^{-1}.\qquad(24a)\end{aligned}$$

On expanding the terms

$$\left\{\left(\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}\right)+\frac{2\eta'.v(x)}{(x+1)^r}\right\}^m$$

by the binomial theorem, we obtain an equation of the form

$$\begin{aligned}2\sum_{m=1}^{r+1}&\left\{\sum_{n=1}^{r}\frac{c_n}{x^n}+\frac{\gamma_{r+1}}{x^{r+1}}+\frac{\epsilon_x}{x^{r+2}}+\frac{2\eta'.v(x)}{(x+1)^r}\right\}^m\\&=\sum_{n=1}^{r}\frac{d_n}{x^n}+\frac{\delta_{r+1}}{x^{r+1}}+\frac{\epsilon'_x}{x^{r+2}}+\sum_{n=1}^{r+1}\left\{\frac{\eta'.v(x)}{(x+1)^r}\right\}^n\{2^{n+1}+g_n(x)\},\end{aligned}$$

where $|\epsilon'_x|$ and $|xg_n(x)|$ are subject to conditions similar to those postulated for $|\epsilon_x|$.

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* Apart from the first term, the formulæ for c_n $(n\leqslant r)$ and for γ_{r+1} are of precisely the same form.

Also*

$$d_n = 2c_n + 4c_1 c_{n-1} + \text{terms involving the quantities '}c\text{' with suffixes less than } n-1,$$

$$= k_n - (n-1)\,k_{n-1} + k_1 k_{n-1} + \text{terms involving the quantities '}k\text{' with suffixes less than } n-1, \text{ and the quantities } a, \lambda_2, \ldots, \lambda_n.$$

δ_{r+1} is defined by a formula of precisely the same form with the first term omitted, since k_{r+1} is absent from our analysis.

The quantities k_2, k_3, ... are at our disposal; let us choose them, if possible, so that

$$d_1 = k_1,\ d_2 = k_2,\ \ldots,\ d_r = k_r,\ \delta_{r+1} = 0. \qquad (24b)$$

[These equations will all be of *the same form* when we substitute the values given above for $d_1, \ldots, d_r, \delta_{r+1}$, *whatever be the value of r.*]

The equations $d_1 = k_1$, $d_2 = k_2$ are satisfied, the first obviously, the second because

$$d_2 = 2c_2 + 2c_1^2 = k_2 - k_1 + 8\lambda_2 + \tfrac{1}{2}k_1^2 = k_2;$$

and, since $k_1 - n + 1 \neq 0$, *ex hypothesi*, when $n = 3, 4, \ldots$, the series of equations

$$d_3 = k_3,\ d_4 = k_4,\ \ldots,\ d_r = k_r,\ \delta_{r+1} = 0$$

determine $k_2, k_3, \ldots, k_r$ uniquely in terms of $a, \lambda_2, \lambda_3, \ldots, \lambda_{r+1}$; the quantities so determined are finite, and we get the *same* values for k_2, k_3, ... whatever integer r may be.

Having determined $k_2, k_3, \ldots, k_r$ in this manner, we may write equation ($24a$) in the form

$$\eta = \sum_{n=1}^{r+1} \left\{ \frac{\eta'.v(x)}{(x+1)^r} \right\}^n x^r \{2^{n+1} + g_n(x)\} + \epsilon_x' x^{-2}$$

$$+ 2x^r \left\{ \sum_{n=1}^{r} \frac{c_n}{x^n} + \frac{\gamma_{r+1}}{x^{r+1}} + \frac{\epsilon_x}{x^{r+2}} + \frac{2\eta'.v(x)}{(x+1)^r} \right\}^{r+2}$$

$$\times \left[1 - \left\{ \sum_{n=1}^{r} \frac{c_n}{x^n} + \frac{\gamma_{r+1}}{x^{r+1}} + \frac{\epsilon_x}{x^{r+2}} + \frac{2\eta'.v(x)}{(x+1)^r} \right\} \right]^{-1}.$$

And, from the definition of $g_n(x)$ it is easy to shew that,

$$\frac{v(x)\,x^r}{(x+1)^r} \{4 + g_1(x)\} = 1 + \frac{4a - r}{x} + \frac{\theta_x}{x^2},$$

where θ_x is subject to the same restrictions as ϵ_x.

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[*] When $n = 2$, this takes the slightly different form $d_2 = 2c_2 + 2c_1^2$.

Putting

$$\sum_{n=1}^{r} \frac{c_n}{x^{n-1}} + \frac{\gamma_{r+1}}{x^r} + \frac{\epsilon_r}{x^{r+1}} + \frac{2\eta'.v(x).x}{(x+1)^r} = F(x, \eta'),$$

and taking the modulus of each side of the above equation, we get

$$|\eta| \leqslant |\eta'| \left|\left[1+\frac{4a-r}{x}+\frac{\theta_x}{x^2}\right]\right| + \sum_{n=2}^{r+1} \left| \left\{\frac{\eta'.v(x)}{(x+1)^r}\right\}^n x^r \{2^{n+1}+g_n(x)\} \right| + |\epsilon'_x.x^{-2}|$$

$$+2|x|^{-2}|\{F(x, \eta')\}|^{r+2}|[1-x^{-1}F(x, \eta')]|^{-1}. \quad (25)$$

Let Θ be a finite positive quantity, and suppose that

$$|\eta'| < \Theta.$$

Since $R(4a-r)$ is negative, it follows by the Lemma proved in the introduction that when $|x| > \mu$ and $R(x) \geqslant -\frac{1}{2}$,

$$\prod_{n=0}^{s} \left|\left[1+\frac{4a-r}{x+n}+\frac{\theta_{x+n}}{(x+n)^2}\right]\right| < \Theta_1,$$

where s is any integer and Θ_1 is independent of s and x, and $\Theta_1 > 1$.

Now we can find finite quantities β and Θ_2 (β being greater than μ), such that

$$\left|\sum_{n=1}^{r} \frac{c_n}{x^{n-1}} + \frac{\gamma_{r+1}}{x^r} + \frac{\epsilon_r}{x^{r+1}}\right| + \Theta \left|\frac{2x.v(x)}{(x+1)^r}\right| < \tfrac{1}{2}\beta$$

[so that $|F(x, \eta')| < \frac{1}{2}\beta$], and

$$|\epsilon'_x| + \sum_{n=2}^{r+1} \Theta^n |v(x)|^n \frac{|x|^{r+2}}{|x+1|^{nr}} |\{2^{n+1}+g_n(x)\}| + 2^{-r}\beta^{r+2} < \Theta_2,$$

so that Θ_2 and β are independent of η'.

Supposing now that $|x| > \beta$ and $R(x) \geqslant -\frac{1}{2}$, we see that

$$|1-x^{-1}F(x, \eta')| > \tfrac{1}{2},$$

and consequently from (25) we deduce that

$$|\eta| < |\eta'| \left|\left[1+\frac{4a-r}{x}+\frac{\theta_r}{x^2}\right]\right| + \frac{\Theta_2}{|x|^2}. \quad (26)$$

As in § 3B we can find a finite quantity Θ_3 such that when $|x| > \beta$ and $R(x) \geqslant -\frac{1}{2}$,

$$\sum_{n=0}^{\infty} |x+n|^{-2} < \Theta_3 |x|^{-1}.$$

Now, let $\Theta_3\Theta_1\Theta_2.\Theta^{-1} \leqslant \gamma$, where γ is finite and $\gamma \geqslant \beta$.

We shall now assume that $|x| > \gamma$ *and* $R(x) \geqslant -\frac{1}{2}$.

Put $$\left|1+\frac{4a-r}{x}+\frac{\theta_x}{x^2}\right| = \Omega_x,$$

and define quantities $\eta_x, \eta_{x+1}, \dots, \eta_{x+m}$ by the system of equations

$$\sum_{n=0}^{r}\frac{k_n}{(x+p)^n}+\frac{\eta_{x+p}}{(x+p)^r} = \left[1-v(x+p)\left\{\sum_{n=0}^{r}\frac{k_n}{(x+p+1)^n}+\frac{\eta_{x+p+1}}{(x+p+1)^r}\right\}\right]^{-1},$$

where $p = 0, 1, 2, \dots, m$ and $\eta_{x+m+1} = 0$.

Then if all the quantities $|\eta_{x+p+1}|, |\eta_{x+p+2}|, \dots, |\eta_{x+m}|$ *are less than* Θ (and $|\eta_{x+m+1}|$ is less than Θ), we can deduce the system of inequalities

$$|\eta_{x+q}| < |\eta_{x+q+1}|\,\Omega_{x+q}+\frac{\Theta_2}{|x+q|^2}$$

(where $q = p, p+1, \dots, m$), in the same manner as we deduce (26) from (24).

Eliminating $|\eta_{x+p+1}|, |\eta_{x+p+2}|, \dots, |\eta_{x+m}|$ from this system of inequalities, we get

$$|\eta_{x+p}| < |\eta_{x+m+1}|\prod_{q=p}^{m}\Omega_{x+q}+\frac{\Theta_2}{|x+p|^2}+\sum_{q=p+1}^{m}\Theta_2\frac{1}{|x+q|^2}\left(\prod_{n=p+1}^{q}\Omega_{x+n}\right),$$

i.e., $|\eta_{x+p}| < |\eta_{x+m+1}|\,\Theta_1+\Theta_1\Theta_2\sum_{q=p}^{m}|x+q|^{-2}$,

i.e., $|\eta_{x+p}| < \Theta_1\Theta_2\Theta_3\,|x+p|^{-1}$

$< \Theta$ (since $|x+p| > \gamma$).

We infer, by induction, that *all* the quantities $|\eta_{x+m}|, |\eta_{x+m-1}|, \dots, |\eta_x|$ are less than Θ, and
$$|\eta_x| < \Theta_1\Theta_2\Theta_3\,|x|^{-1}.$$

Consequently, if we define $\Psi_m^{(r)}(x)$ by the equation

$$\Psi_m^{(r)}(x) = \frac{1}{1-}\,\frac{v(x)}{1-}\,\frac{v(x+1)}{1-}\cdots-\frac{v(x+m)}{1}\left\{\sum_{n=0}^{r}\frac{k_n}{(x+m+1)^n}\right\}, \tag{27}$$

then when $|x| > \gamma$ and $R(x) \geqslant -\tfrac{1}{2}$,

$$\Psi_m^{(r)}(x) = k_0+\frac{k_1}{x}+\dots+\frac{k_r+\eta_x}{x^r}, \tag{27a}$$

where $|\eta_x| < J|x|^{-1}$ and J is finite and independent of x.

In the same way as we obtained (16a) and (16b) in § 4A, we may shew that with this definition of $\Psi_m^{(r)}(x)$,

$$|\Psi_m^{(r+1)}(x)-\Psi_m^{(r)}(x)|$$

$$=\left|\frac{k_{r+1}}{(x+m+1)^{r+1}}\right|\prod_{t=0}^{m-1}|[v(x+t)]|\prod_{t=0}^{m}|[\Psi_{m-t}^{(r)}(x+t)]|\prod_{t=0}^{m}|[\Psi_{m-t}^{(r+1)}(x+t)]|,$$

$$|\Psi_{m+1}^{(r)}(x)-\Psi_m^{(r)}(x)|$$

$$<\frac{J'}{|x+m+1|^{r+1}}\prod_{t=0}^{m}|[v(x+t)]|\prod_{t=0}^{m+1}|[\Psi_{m+1-t}^{(r)}(x+t)]|\prod_{t=0}^{m}|[\Psi_{m-t}^{(r)}(x+t)]|.$$

And, from (27a), $\left|\prod_{t=0}^{m}\left[\tfrac{1}{2}\Psi_{m-t}^{(r)}(x+t)\exp\left(\frac{2a}{x+t}\right)\right]\right|<J^{(r)}$,

$$\prod_{t=0}^{m}|[4v(x+t)]|<J_1,$$

where J_1, $J^{(r)}$, J' are independent of m and x.

Hence

$$|\Psi_m^{(r+1)}(x)-\Psi_m^{(r)}(x)|<J_2\exp\{R(4a-r-1)\log|x+m|\},$$

$$|\Psi_{m+1}^{(r)}(x)-\Psi_m^{(r)}(x)|<J_3\exp\{R(4a-r-1)\log|x+m|\}$$

(by Stirling's formula), where J_2, J_3 are independent of m and x; and since $\sum_{m=0}^{\infty}\exp\{R(4a-r-1)\log|x+m|\}$ is absolutely and uniformly convergent [since $R(4a)<r$], we deduce, as in § 4A, that $\operatorname{Lt}_{m=\infty}\Psi_m^{(r)}(x)$ is a definite quantity, $\Psi(x)$, independent of r.

As in § 7 of the first paper, a solution of the difference equation is

$$f(x)=u(x)\exp\left[x\log 2+2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]\prod_{n=1}^{\infty}\left[\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right],$$

and by the methods of § 5A we can shew that when $R(x)\geqslant-\frac{1}{2}$ and $|x|$ is large, the asymptotic expansion of $f(x)$ is

$$f(x)=u(x)\exp\left[x\log 2+2a\log x+\sum_{n=1}^{m-1}\frac{l_n}{x^n}+\frac{\theta}{x^m}\right],$$

where $|\theta|$ is finite and the quantities l_1, l_2, ... may be calculated with sufficient labour.

The investigation of the associated transcendental equation*

$$1-v(x)\Psi(x+1)\Psi_1(x)=0$$

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* Where $\Psi_1(x)=\frac{1}{1-}\,\frac{v(x-1)}{1-}\,\frac{v(x-2)}{1-}\dots$.

is a little more difficult than in Part II of the paper* by the writer on the subject in which it was assumed that $R(a) < \frac{1}{4}$. The work is identical except that in proving that

$$[1-v(x)\,\Psi_1(x)\,\Psi(x+1)]\exp\left\{2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}+2a\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}\to 0,$$

as $|x|\to\infty$ when $-\frac{1}{2}\leqslant R(x)\leqslant\frac{1}{2}$, we have to make use of the following arguments:

$\Psi(x)$ satisfies the difference equation $\Psi(x) = [1-v(x)\,\Psi(x+1)]^{-1}$.

If $$[v(x)\,\Psi_1(x)]^{-1} = \chi(x),$$

since $$\Psi_1(x) = [1-v(x-1)\,\Psi_1(x-1)]^{-1},$$

we have $$\chi(x-1) = [1-v(x)\,\chi(x)]^{-1},$$

so that $\Psi(x)$ and $\chi(x-1)$ satisfy the same difference equation, and it may be proved that both $\chi(x-1)$ and $\Psi(x)$ possess complete asymptotic expansions in the strip when $|x|$ is large with the coefficients of the first two terms the same in each expansion; hence, by the work at the beginning of this section, all the corresponding coefficients in the two expansions are equal; therefore $1-\Psi(x+1)\{\chi(x)\}^{-1}$ tends to zero *more rapidly than any finite negative power of* x as $|x|\to\infty$ in the strip; hence since

$$\left|\exp\left\{2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}+2a\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}\right| < J\exp\{R(4a)\log|x|\},$$

where J is finite when $|x|>\gamma$,

$$\{1-v(x)\,\Psi(x+1)\,\Psi_1(x)\}\exp\left\{2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}+2a\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}$$

tends to zero in the strip. The conclusions in Part II of the paper referred to are therefore unimpaired by the introduction of the additional generality.

7. We now wish to determine the behaviour of $\Psi(x)$ when $R(x)$ is large and negative in the two cases (i) when λ_0 is a real quantity greater than $\frac{1}{4}$, (ii) when $\lambda_0=\frac{1}{4}$, $\lambda_1=0$. We shall investigate the second of these first.

It is not possible to carry out analysis similar to that of §§ 3A and 5A, for the reason that, if $R(x)\leqslant-\frac{1}{2}$ and $-\frac{1}{2}<R(y)\leqslant\frac{1}{2}$, $y-x$ being a

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 162–177.

positive integer, $R\left\{(4a-r)\left(\frac{1}{x+1}+\frac{1}{x+2}+\ldots+\frac{1}{y}\right)\right\}$ diverges to $+\infty$ as $|x| \to \infty$; so that $\left|\prod_{n=1}^{y-x}\left\{1+\frac{4a-r}{x+n}+\frac{\theta_{x+n}}{(x+n)^2}\right\}\right|$ diverges to ∞ (and not to 0) as $|x| \to \infty$.

Our method of procedure has to be as follows: we obtain a formula for the quotient of two solutions of the equation

$$f(x-1) = f(x) - v(x) f(x+1),$$

the asymptotic formula for the first of them being valid when $R(x) \geqslant -\frac{1}{2}$, and we thence deduce the asymptotic formula for $\Psi_1(x)$ defined by the continued fraction $\frac{1}{1-}\,\frac{v(x-1)}{1\;-}\,\frac{v(x-2)}{1\;-\ldots}$, when $R(x) \geqslant -\frac{1}{2}$; by an easy transformation we deduce the asymptotic expansion for $\Psi(x)$ when $R(x) \leqslant -\frac{1}{2}$.

It is, unfortunately, necessary to assume that $v(x)$ is the quotient of two polynomials; remembering that

$$v(x) = \frac{A(x)\,C(x+1)}{B(x)\,B(x+1)},$$

we put
$$v(x) = \frac{1}{4}\,\frac{\prod_{t=1}^{2p}(x+\beta_t)}{\prod_{t=1}^{p}(x+\epsilon_t)\prod_{t=1}^{p}(x+\epsilon_t+1)},$$

common factors of numerator and denominator not being cancelled.

We may deduce from the results of § 7 of the first paper that two solutions of

$$f(x-1) = f(x) - v(x) f(x+1)$$

are
$$f_2(x) = \exp\left[\quad x\log 2 + 2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]\prod_{n=1}^{\infty}\left[\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right],$$

$$f_3(x) = \exp\left[-x\log 2 + 2a\frac{\Gamma'(1-x)}{\Gamma(1-x)}\right]\prod_{n=1}^{\infty}\left[\frac{2}{\Psi_1(x-n)}\exp\frac{-2a}{x-n}\right]$$
$$\times 2^{2x}\prod_{n=0}^{\infty}\{4v(x+n)\}.$$

From these definitions we may shew without difficulty that

$$\frac{f_3(x+1)}{f_2(x+1)} - \frac{f_3(x)}{f_2(x)} = \frac{\Theta_0(x)\exp\left\{-2a\frac{\Gamma'(x+1)}{\Gamma(x+1)} - 2a\frac{\Gamma'(x+2)}{\Gamma(x+2)}\right\}\prod_{n=1}^{\infty}\{4v(x+n)\}}{\prod_{n=1}^{\infty}\left\{\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right\}\prod_{n=2}^{\infty}\left\{\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right\}},$$

where

$$\Theta_0(x) = \{1-v(x)\,\Psi(x+1)\,\Psi_1(x)\}\exp\left\{2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}+2a\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}$$
$$\times\prod_{n=1}^{\infty}\left\{\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right\}\prod_{n=0}^{\infty}\left\{\frac{2}{\Psi_1(x-n)}\exp\frac{-2a}{x-n}\right\};$$

and it has been shewn* that $\Theta_0(x)$ is expressible in the form

$$\Theta_0(x) = \frac{\operatorname{cosec}^2\pi x\,P_{p-2}(\cot\pi x)}{\prod\limits_{n=1}^{p}(\cot\pi x+\cot\pi\epsilon_n)},$$

where P_{p-2} denotes a certain polynomial of degree $p-2$.

Accordingly we may write

$$\frac{f_3(x+1)}{f_2(x+1)}-\frac{f_3(x)}{f_2(x)} = \Theta_0(x)\,F_0(x), \tag{28}$$

and by a use of the equation

$$[\Psi(x)]^{-1} = 1-v(x)\,\Psi(x+1),$$

we may shew that

$$F_0(x) = \exp\left\{-4a\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right\}\prod_{n=1}^{\infty}\left[\{\Psi(x+n)-1\}\exp\frac{-4a}{x+n}\right].$$

From this result we can deduce the asymptotic expansion of $F_0(x)$ when $|x|$ is large and $R(x)\geqslant-\frac{1}{2}$; for when $|x|>\gamma$ and $R(x)\geqslant-\frac{1}{2}$, $\Psi(x)-1$ possesses the asymptotic expansion

$$1+\frac{4a}{x}+\frac{k_2}{x^2}+\ldots+\frac{k_r+\eta_x}{x^r};$$

hence by the methods of § 5A, we may shew that when $|x|>\gamma'$ and $R(x)\geqslant-\frac{1}{2}$,

$$F_0(x) = x^{-4a}\left[1+\frac{\mu_1}{x}+\frac{\mu_2}{x^2}+\ldots+\frac{\mu_r+\xi_x}{x^r}\right], \tag{29}$$

where $\mu_1, \mu_2, \ldots$ are determinate constants, $|x\xi_x|$ does not exceed a finite quantity K depending on γ', and $|\arg x|<\pi$.

From this result it follows that:

(A) When $R(-4a)<-1$, the series $\sum\limits_{n=0}^{\infty}F_0(x+n)$ is absolutely convergent for all finite values of x, unless one of the quantities $F_0(x+n)$ happens to be infinite.

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* See § 6 above, and *Proceedings*, Ser. 2, Vol. 8, pp. 172–177.

(B) If $R(-4a) \geqslant -1$, and N be the integer (positive or zero) such that $0 < R(4a)+N \leqslant 1$, then when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$,

$$\left| F_0(x) - x^{-4a}\left(1+\frac{\mu_1}{x}+\dots+\frac{\mu_N}{x^N}\right)\right| < \frac{K}{|x^{4a+N+1}|},$$

where K is independent of x (it is obviously permissible to take $r > N$), and hence the series

$$\sum_{n=0}^{\infty}\left\{F_0(x+n)-(x+n)^{-4a}\left(1+\frac{\mu_1}{x+n}+\dots+\frac{\mu_N}{(x+n)^N}\right)\right\}$$

is absolutely convergent.

Now (28) may be regarded as a difference equation to determine $f_3(x)/f_2(x)$. The general solution of this difference equation is, in the two cases,

$$\frac{f_3(x)}{f_2(x)} = \varpi(x) - \Theta_0(x)\sum_{n=0}^{\infty} F_0(x+n), \tag{A}$$

$$\frac{f_3(x)}{f_2(x)} = \varpi(x) - \Theta_0(x)\left[G_0(x) + \sum_{n=0}^{\infty}\left\{F_0(x+n)-(x+n)^{-4a}\right.\right.$$
$$\left.\left.\times\left(1+\frac{\mu_1}{x+n}+\dots+\frac{\mu_N}{(x+n)^N}\right)\right\}\right], \tag{B}$$

where $\varpi(x)$ is some periodic function of x with period unity and $G_0(x)$ is any particular solution of the difference equation

$$G_0(x+1) - G_0(x) = -x^{-4a}\left(1+\frac{\mu_1}{x}+\dots+\frac{\mu_N}{x^N}\right). \tag{30}$$

If it happens that $N+4a = 1$, we take

$$G_0(x) = -\left[S_{N-1}(x)+\mu_1 S_{N-2}(x)+\dots+\mu_{N-1}S_0(x)+\mu_N\frac{\Gamma'(x)}{\Gamma(x)}\right], \tag{30a}$$

where $S_n(x)$ is the Bernoullian polynomial* satisfying the relation

$$S_n(x+1) - S_n(x) = x^n.$$

If a is not such that $N+4a = 1$, we take (assuming $|\arg x| < \frac{1}{2}\pi$)

$$G_0(x) = -\frac{1}{2\pi i}\int (-t)^{4a-1}\{\Gamma(1-4a)+\mu_1(-t)\,\Gamma(-4a)+\dots$$
$$+\mu_N(-t)^N\Gamma(1-4a-N)\}\,\frac{e^{-xt}}{1-e^{-t}}\,dt, \tag{30b}$$

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* This is Hermite's notation; *Crelle*, Bd. cxv, pp. 201 *et seq.* Or see Barnes, *Messenger of Mathematics*, Vol. xxix, pp. 74–84.

along a contour starting from infinity encircling the origin in a positive direction, and returning to infinity; the contour contains no singularities of the integrand save the origin, and is such that when $|t|$ is large, $R(xt) > 0$, $R(t) > 0$; and $\arg(-t)$ is rendered definite by taking it zero when t is real and negative; the ordinary contour integral for the gamma function shews that the value of $G_0(x)$ satisfies (30).

Further, Barnes has shewn* that when $|\arg x| < \frac{1}{2}\pi$, $G_0(x)$ possesses an asymptotic expansion of the form

$$G_0(x) = \sum_{n=0}^{r} \frac{\kappa_n}{\Gamma(2-4a-n)\, x^{4a-1+n}} + \frac{w_r}{x^{4a+r}},$$

where $|w_r|$ is finite, and $\kappa_0, \kappa_1, \ldots$ are determined from the expansion

$$\frac{-t}{1-e^{-t}} \left\{ \Gamma(1-4a) + \mu_1(-t)\Gamma(-4a) + \ldots + \mu_N(-t)^N \Gamma(1-4a-N) \right\}$$
$$= \sum_{n=0}^{\infty} \kappa_n (-t)^n.$$

Noticing that we can find a finite positive integer n_1 such that

$$|\arg(x+n_1)| < \tfrac{1}{2}\pi,$$

we have, either

$$\frac{f_3(x)}{f_2(x)} = \varpi(x) - \Theta_0(x) \sum_{n=0}^{\infty} F_0(x+n), \tag{31A}$$

or

$$\frac{f_3(x)}{f_2(x)} = \varpi(x) - \Theta_0(x)\Big[G_0(x+n_1) + F_0(x) + F_0(x+1) + \ldots + F_0(x+n_1-1)$$
$$+ \sum_{n=n_1}^{\infty} \left\{ F_0(x+n) - (x+n)^{-4a} \left(1 + \frac{\mu_1}{x+n} + \ldots + \frac{\mu_N}{(x+n)^N} \right) \right\} \Big], \tag{31B}$$

and since $F_0(x)$ possesses a complete asymptotic expansion in powers of x^{-1}, it follows from the asymptotic expansion† of $\sum_{n=0}^{\infty} (x+n)^{-m}$, that when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$, either of these formulæ may be written in the form

$$\frac{f_3(x)}{f_2(x)} = \varpi(x) - \Theta_0(x)\, x^{1-4a} \left\{ \sum_{n=0}^{r} \nu_n x^{-n} + w_r' x^{-r-1} \right\}, \tag{32}$$

where the quantities $\nu_0, \nu_1, \ldots$ are determinate and $|w_r'|$ is finite; if $N+4a = 1$, we see from the last term of (30a) that a term $-\mu_N \log x$ will have to be inserted in the coefficient of $\Theta_0(x)$ in equation (32).

Now let us investigate equations (31A) and (31B) with a view to obtaining precise information concerning the periodic function $\varpi(x)$ which occurs therein.

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* *Philosophical Transactions*, 206A, "The Asymptotic Expansion of Integral Functions," § 5.

† Quoted in § 5A above.

By the difference relation,

$$f_2(x-1) = f_2(x) - v(x)\, f_2(x+1),$$

noting that, from the asymptotic expansion of $\Psi(x)$, $f_2(x)$ has no poles or zeros in that portion of the plane of the complex variable in which $R(x)$ exceeds some finite quantity γ_1, we see that the only singularities of $f_2(x)$ in the finite part of the plane are at the poles of

$$v(x+1),\ v(x+2),\ v(x+3),\ \dots;$$

that is to say, at the points

$$-\epsilon_t-1,\ -\epsilon_t-2,\ -\epsilon_t-3,\ \dots \quad (t = 1, 2, \dots, p).$$

If no two of these points coincide, it readily follows from the difference relation that the poles of $f_2(x)$ at these points are simple;* more generally, the degree of multiplicity of any pole is, *at most*, equal to the number of times that that pole occurs in the list of possible poles

$$\left.\begin{matrix} -\epsilon_1-1, & -\epsilon_1-2, & -\epsilon_1-3, & \dots \\ \dots & \dots & \dots & \dots \\ -\epsilon_p-1, & -\epsilon_p-2, & -\epsilon_p-3, & \dots \end{matrix}\right\}. \tag{C}$$

And from the asymptotic expansion of $\Psi(x)$ we see that the zeros of $f_2(x)$ which lie on the right of $R(x) = -\frac{1}{2}$ lie within the rectangle bounded by

$$R(x) = -\tfrac{1}{2},\quad R(x) = \gamma,\quad I(x) = \pm\gamma'' \quad \text{(where } \gamma'' \text{ is finite).}$$

Putting $$f_3(x) = f_3^{(1)}(x)\, 2^{2x} \prod_{n=0}^{\infty} \{4v(x+n)\},$$

we see that $f_3^{(1)}(x)$ satisfies the difference relation

$$f_3^{(1)}(x+1) = f_3^{(1)}(x) - v(x-1) f_3^{(1)}(x-1),$$

and also that

$$f_3^{(1)}(x) = \exp\left\{-x\log 2 + 2a\frac{\Gamma'(1-x)}{\Gamma(1-x)}\right\} \prod_{n=1}^{\infty}\left[\frac{2}{\Psi_1(x-n)}\exp\frac{-2a}{x-n}\right].$$

From the asymptotic expansion of $\Psi_1(x)$ valid when $R(x) < \frac{1}{2}$ and $|x| > \gamma'$, it follows that $f_3^{(1)}(x)$ has no zero or poles in that portion of the plane in which $R(x) < -\gamma'''$, where γ''' is some finite quantity. And hence the only singularities of $f_4(x)$ in the finite part of the plane are at the points

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* It may happen that $f_2(x)$ is regular at some of the points.

$$\left.\begin{array}{cccc} 1-\epsilon_1, & 2-\epsilon_1, & 3-\epsilon_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1-\epsilon_p, & 2-\epsilon_p, & 3-\epsilon_p, & \ldots \end{array}\right\}, \tag{D}$$

the multiplicity of any pole being *at most* equal to the number of times it occurs in the list (D).

Now consider the singularities of the periodic function $\varpi(x)$, given by one or other of the formulæ:

$$\varpi(x) = \frac{f_3(x)}{f_2(x)} + \Theta_0(x) \sum_{n=0}^{\infty} F_0(x+n), \tag{33A}$$

$$\varpi(x) = \frac{f_3(x)}{f_2(x)} + \Theta_0(x) \Bigg[G_0(x+n_1) + \sum_{n=0}^{n_1-1} F_0(x+n)$$
$$+ \sum_{n=n_1}^{\infty} \left\{ F_0(x+n) - (x+n)^{-4a} \left(1 + \frac{\mu_1}{x+n} + \ldots + \frac{\mu_N}{(x+n)^N}\right) \right\} \Bigg], \tag{33B}$$

in the strips of the plane of unit breadth defined by the following inequalities

(i) when $|I(x)| \leqslant \gamma_1 + 1, \quad \gamma_1 + 1 \leqslant R(x) \leqslant \gamma_1 + 2$;

(ii) when $|I(x)| \geqslant \gamma_1 + 1, \quad -\frac{1}{2} \leqslant R(x) \leqslant \frac{1}{2}$;

where γ_1 is a finite quantity so chosen that all the zeros and poles of $f_2(x)$ are in the strip $|I(x)| \leqslant \gamma_1$ and to the left of the line $R(x) = \gamma_1$; and all the poles of $v(x)$ are inside the circle $|x| = \gamma_1$.

Then the only poles of $f_3(x)/f_2(x)$ which lie in the rectangle defined by $|I(x)| \leqslant \gamma_1 + 1, \; \gamma_1 + 1 \leqslant R(x) \leqslant \gamma_1 + 2$, are at those of the set of points (D) which lie within the rectangle.

In this rectangle, also, neither the function

$$\sum_{n=0}^{\infty} F_0(x+n)$$

nor the function

$$G_0(x+n_1) + \sum_{n=0}^{n_1-1} F_0(x+n)$$
$$+ \sum_{n=n_1}^{\infty} \left\{ F_0(x+n) - (x+n)^{-4a} \left(1 + \frac{\mu_1}{x+n} + \ldots + \frac{\mu_N}{(x+n)^N}\right) \right\}$$

(whichever we have to consider), has any singularities; while the singularities of $\Theta_0(x)$ in the rectangle are poles, at those of the points (D) which lie in the rectangle, each being of multiplicity equal to the number of times it occurs in the list.

Suppose that we divide the quantities $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ into sets

$$\epsilon_{1,1},\ \epsilon_{1,2},\ \ldots,\ \epsilon_{1,a};\quad \epsilon_{2,1},\ \epsilon_{2,2},\ \ldots,\ \epsilon_{2,b};\ \ldots,$$

such that any pair of members of the same set differ by an integer, and no pair of members of different sets differ by an integer.

Then we can theoretically determine a quantity $\varpi_0(x)$, such that

$$\begin{aligned}\varpi_0(x) = {} & A_{1,1}\cot\pi(x+\epsilon_{1,1})+A_{1,2}\cot^2\pi(x+\epsilon_{1,1})+\ldots+A_{1,a}\cot^a\pi(x+\epsilon_{1,1})\\ & +A_{2,1}\cot\pi(x+\epsilon_{2,1})+A_{2,2}\cot^2\pi(x+\epsilon_{2,1})+\ldots+A_{2,b}\cot^b\pi(x+\epsilon_{2,1})\\ & +\ldots,\end{aligned}$$

and such that $\varpi_0(x)-F_2(x)$ has no poles in the rectangle $|I(x)| \leqslant \gamma_1+1$, $\gamma_1+1 \leqslant R(x) \leqslant \gamma_1+2$; where $F_2(x)$ denotes the expression on the right-hand side either of equation (33A) or of equation (33B), whichever is under consideration.

Consequently, since $\varpi(x)-\varpi_0(x)$ is a periodic function with no poles in the rectangle, $\varpi(x)-\varpi_0(x)$ has no poles in the strip $|I(x)| \leqslant \gamma_1+1$.

Also, neither $\varpi_0(x)$ nor $F_2(x)$ has any poles in the finite part of the regions defined by $|I(x)| \geqslant \gamma_1+1$, $-\frac{1}{2} \leqslant R(x) \leqslant \frac{1}{2}$; hence $\varpi(x)-\varpi_0(x)$ is a periodic function with no singularities in the finite part of the plane.

Now consider the behaviour of $F_2(x)$ when

$$|x| \to \infty \quad \text{and} \quad -\tfrac{1}{2} \leqslant R(x) \leqslant \tfrac{1}{2}.$$

By the asymptotic expansions of Ψ and Ψ_1 it may be shewn that

$$\prod_{n=1}^{\infty}\left[\frac{2}{\Psi(x+n)}\exp\frac{2a}{x+n}\right] \to 1, \qquad \prod_{n=1}^{\infty}\left[\frac{2}{\Psi_1(x-n)}\exp\frac{-2a}{x-n}\right] \to 1,$$

and, since $\lambda_1 = 0$, $\qquad \prod_{n=0}^{\infty}\left[4v(x+n)\right] \to 1.$

Hence
$$\begin{aligned}\frac{f_3(x)}{f_2(x)} & \to \exp\left\{2a\frac{\Gamma'(1-x)}{\Gamma(1-x)}-2a\frac{\Gamma'(1+x)}{\Gamma(1+x)}\right\}\\ & \to \exp(\pm 2a\pi i),\end{aligned}$$

the upper or lower sign being taken according as $I(x) \to -\infty$ or $+\infty$.

Also
$$\Theta_0(x) = \frac{\operatorname{cosec}^2\pi x\, P_{p-2}(\cot\pi x)}{\prod\limits_{n=1}^{p}(\cot\pi x+\cot\pi\epsilon_n)},$$

and $\operatorname{cosec}^2\pi x$ tends exponentially to zero as $I(x) \to \pm\infty$.

Hence, from (32), $\varpi(x) \to \exp(\pm 2a\pi i)$; and $\varpi_0(x)$ tends to a finite limit. Therefore, since $\varpi(x)-\varpi_0(x)$ is a uniform periodic function whose modulus never exceeds some finite quantity in the strip $-\frac{1}{2} \leqslant R(x) \leqslant \frac{1}{2}$, $\varpi(x)-\varpi_0(x)$ is never infinite for *any* value of x; therefore, by Liouville's theorem, $\varpi(x)-\varpi_0(x)$ is a constant, A, say.

The formula for $\varpi_0(x)$ as a sum of cotangents shews that we can write $A+\varpi_0(x)$ in the form

$$P_p^{(1)}(\cot \pi x) \prod_{n=1}^{p} (\cot \pi x+\cot \pi\epsilon_n)^{-1},$$

where $P_p^{(1)}$ denotes a certain polynomial of degree p.

We have therefore obtained the result

$$\frac{f_3(x)}{f_2(x)} = \frac{P_p^{(1)}(\cot \pi x)}{\prod\limits_{n=1}^{p} (\cot \pi x+\cot \pi\epsilon_n)} - \Theta_0(x) \sum_{n=0}^{\infty} F_0(x+n) \tag{A}$$

or
$$\frac{f_3(x)}{f_2(x)} = \frac{P_p^{(1)}(\cot \pi x)}{\prod\limits_{n=1}^{p} (\cot \pi x+\cot \pi\epsilon_n)} - \Theta_0(x) \Bigg[G_0(x+n_1)+ \sum_{n=0}^{n_1-1} F_0(x+n)$$
$$+ \sum_{n=n_1}^{\infty} \left\{ F_0(x+n)-(x+n)^{-4a} \left(1+\frac{\mu_1}{x+n}+\ldots+\frac{\mu_N}{(x+n)^N}\right) \right\} \Bigg], \tag{B}$$

according to the value of a.

This result is true for all values of x.

Further, from (32), we may obtain the equation

$$\frac{f_3(x+1)}{f_2(x+1)} = \varpi(x)-\Theta_0(x)\, x^{1-4a} \left\{ \sum_{n=0}^{r} \nu_n' x^{-n}+w_x'' x^{-r-1} \right\},$$

when $|x| > \gamma'$ and $R(x) \geqslant -\frac{1}{2}$; the quantities ν_0', ν_1', ... being determinate. But

$$\frac{f_3(x)}{f_3(x+1)} \, \frac{f_2(x+1)}{f_2(x)} = v(x)\, \Psi_1(x)\, \Psi(x+1).$$

And therefore we have the result

$$\Psi_1(x) = \frac{1}{v(x)\,\Psi(x+1)} \, \frac{\varpi(x)-\Theta_0(x) \sum\limits_{n=0}^{\infty} F_0(x+n)}{\varpi(x)-\Theta_0(x) \sum\limits_{n=1}^{\infty} F_0(x+n)}$$

[this has to be modified as usual when $R(4a) \geqslant 1$] for all values of x. And, when $|x| > \gamma'$, $R(x) \geqslant -\frac{1}{2}$,

$$\Psi_1(x) = \frac{1}{v(x)\,\Psi(x+1)} \, \frac{\varpi(x)-\Theta_0(x)\, x^{1-4a} \left\{ \sum\limits_{n=0}^{r} \nu_n x^{-n}+w_x' x^{-r-1} \right\}}{\varpi(x)-\Theta_0(x)\, x^{1-4a} \left\{ \sum\limits_{n=0}^{r} \nu_n' x^{-n}+w_x'' x^{-r-1} \right\}}.$$

Since, when $|I(x)| \to \infty$, $\varpi(x) \to \exp(\pm 2a\pi i)$, and $\Theta_0(x)$ tends exponentially to zero, we can find finite quantities γ_2 (greater than or equal to γ') and δ such that, when $|I(x)| > \gamma_2$, $|\arg x| > \delta$, and $R(x) \geqslant -\tfrac{1}{2}$,

$$\Psi_1(x) = \frac{1}{v(x)\,\Psi(x+1)}\left\{1+\frac{\zeta_x}{x^s}\right\},$$

where $|\zeta_x| < J$, J being a finite quantity depending on γ_2 and s being any assigned finite integer. Hence, by the asymptotic expansion of $\Psi(x+1)$ valid in this region, we see that $\Psi_1(x)$ possesses an asymptotic expansion in negative powers of x, of which the first two terms are $2-4a/x$. But, if $\Psi_1(x)$ has an asymptotic expansion of this form with the first two terms given, the expansion can be *uniquely* determined by the equation

$$\Psi_1(x) = \{1-v(x-1)\,\Psi_1(x-1)\}^{-1}.$$

Thus the asymptotic expansion of $\Psi_1(x)$, which, by the methods of § 6 of this paper, holds when $|x| > \gamma$, $R(x) \leqslant \tfrac{1}{2}$, holds over the extended region $|I(x)| > \gamma_2$, $|\arg x| > \delta$, $R(x) \geqslant \tfrac{1}{2}$.

Now $\Psi_1(x) = \Psi'(y)$, and $\Psi(x) = \Psi_1'(y)$, where Ψ', Ψ_1' are the functions connected with $v(-y-1)$ in the same way as Ψ, Ψ_1 are connected with $v(x)$; and $x = -y$. Hence it follows without difficulty that *the asymptotic expansion of* $\Psi(x)$ *given by* (27*a*) *holds over the extended region*

$$|I(x)| > \gamma_2, \quad |\arg x| < \pi-\delta, \quad R(x) \leqslant -\tfrac{1}{2}.$$

8. The analysis of § 7 suggests the method to be adopted in dealing with the investigation which was deferred at the end of § 5, namely, the determination of the asymptotic expansion of $\Psi(x)$ for values of x such that $R(x) \leqslant -\tfrac{1}{2}$, when $|a_1| = |a_2|$, but $a_1 \neq a_2$. As the analysis does not widely differ from that of § 7, it will be given as briefly as possible.

Two solutions of the difference equation $f(x-1) = f(x) - v(x)\,f(x+1)$ in the case under consideration (viz., λ_0 real and greater than $\tfrac{1}{4}$) are*

$$f_4(x) = \exp\left[-x\log a_1 + \frac{\lambda_1}{a_1(a_1-a_2)}\,\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]$$
$$\times \prod_{n=1}^{\infty}\left[\frac{1}{a_1\,\Psi(x+n)}\exp\frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right],$$

$$f_5(x) = \exp\left[x\log a_1 - \frac{\lambda_1}{a_1(a_1-a_2)}\,\frac{\Gamma'(1-x)}{\Gamma(1-x)}\right]$$
$$\times \prod_{n=1}^{\infty}\left[\frac{1}{a_1\,\Psi_1(x-n)}\exp\frac{\lambda_1}{a_1(a_1-a_2)(x-n)}\right]$$
$$\times \exp\left[-x\log\lambda_0 - \frac{\lambda_1}{\lambda_0}\,\frac{\Gamma'(x)}{\Gamma(x)}\right]\prod_{n=0}^{\infty}\left[\frac{v(x+n)}{\lambda_0}\exp\frac{-\lambda_1}{\lambda_0(x+n)}\right],$$

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* Since $R(a_1) = R(a_2) = \tfrac{1}{2}$, we may take $\arg a_1 = -\arg a_2$ and $|\arg a_1| < \tfrac{1}{2}\pi$, so that $\arg\lambda_0 = \arg a_1 + \arg a_2 = 0$.

and we find that
$$\frac{f_5(x+1)}{f_4(x+1)} - \frac{f_5(x)}{f_4(x)} = \Theta(x)\, F(x), \tag{34}$$

where
$$\Theta(x) = \{1 - v(x)\,\Psi(x+1))\,\Psi_1(x)\} \exp\left\{\frac{-\lambda_1}{a_1(a_1-a_2)}\,\pi \cot x\pi\right\}$$
$$\times \prod_{n=1}^{\infty}\left[\frac{1}{a_1\,\Psi(x+n)} \exp\frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right]$$
$$\times \prod_{n=0}^{\infty}\left[\frac{1}{a_1\,\Psi_1(x-n)} \exp\frac{\lambda_1}{a_1(a_1-a_2)(x-n)}\right]$$
$$= \frac{P_p(\cot \pi x)}{\prod\limits_{n=1}^{p} (\cot \pi x + \cot \pi \epsilon_n)}$$

(where P_p is a polynomial of degree p; see the end of § 5); and

$$F(x) = \left(\frac{a_1}{a_2}\right)^{x+1} \exp\left[-\frac{\lambda_1}{\lambda_0(a_1-a_2)}\,\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]$$
$$\times \prod_{n=1}^{\infty}\left[\frac{a_1}{a_2}\,\{\Psi(x+n)-1\} \exp\frac{-\lambda_1}{\lambda_0(a_1-a_2)(x+n)}\right].$$

Putting $-\dfrac{\lambda_1}{\lambda_0(a_1-a_2)} = \lambda$ for brevity, we may deduce that when $|x| > \gamma_2$, $R(x) \geqslant -\frac{1}{2}$, $|\arg x| < \pi$, $F(x)$ possesses a complete asymptotic expansion of the form

$$F(x) = \left(\frac{a_1}{a_2}\right)^{x+1} x^{\lambda}\left[1 + \frac{\sigma_1}{x} + \frac{\sigma_2}{x^2} + \ldots + \frac{\sigma_r + w_r}{x^r}\right],$$

where r is any assigned finite integer; so that $\sum\limits_{n=0}^{\infty} F(x+n)$ converges absolutely when $R(\lambda) < -1$ unless one of the quantities $F(x)$ happens to be infinite.

Regarding (34) as a first order difference equation determining $f_5(x)/f_4(x)$, we see that, either

$$\frac{f_5(x)}{f_4(x)} = \varpi(x) - \Theta(x) \sum_{n=0}^{\infty} F(x+n), \tag{35A}$$

or
$$\frac{f_5(x)}{f_4(x)} = \varpi(x) - \Theta(x)\left[G(x+n_1) + \sum_{n=0}^{n_1-1} F(x+n)\right.$$
$$\left. + \sum_{n=n_1}^{\infty}\left\{F(x+n) - \left(\frac{a_1}{a_2}\right)^{x+n+1} (x+n)^{\lambda}\left(1 + \frac{\sigma_1}{x+n} + \ldots + \frac{\sigma_N}{(x+n)^N}\right)\right\}\right], \tag{35B}$$

according as $R(\lambda) < -1$ or $\geqslant -1$; where $\varpi(x)$ is a periodic function of x with period unity, n_1 is an integer such that $R(x+n_1) > 0$, N is the integer such that $-1 \leqslant R(\lambda)-N < 0$, and $G(x)$ is defined by the integral

$$G(x) = -\frac{1}{2\pi i}\left(\frac{a_1}{a_2}\right)^x \int (-t)^{-\lambda-1}\{\Gamma(1+\lambda)+(-t)\sigma_1\Gamma(\lambda)+\dots$$
$$+(-t)^N\sigma_N\Gamma(1+\lambda-N)\}\,\frac{e^{-xt}}{(a_2/a_1)-e^{-t}}\,dt.$$

In the special case when $\lambda+1$ is zero or a positive integer ($=N$), $G(x)$ is defined by the equation

$$G(x) = -\frac{1}{2\pi i}\left(\frac{a_1}{a_2}\right)^x \int (-t)^{-\lambda-1}\{\Gamma(1+\lambda)+(-t)\sigma_1\Gamma(\lambda)+\dots$$
$$+(-t)^{N-1}\sigma_{N-1}\Gamma(2+\lambda-N)\}\,\frac{e^{-xt}}{(a_2/a_1)-e^{-t}}\,dt$$
$$+\left(\frac{a_1}{a_2}\right)^x \sigma_N \int_0^\infty \frac{e^{-xt}}{(a_2/a_1)-e^{-t}}\,dt.$$

The contour integrals are taken round a contour of the type defined in § 7, not including any singularities of the integrand save the origin; the zeros of $(a_2/a_1)-e^{-t}$ are all on the imaginary axis, and $R(x)$ is supposed positive.

From these integrals,[*] we deduce, as in § 7, that when $R(x) \geqslant -\frac{1}{2}$ and $|x|$ is greater than a finite quantity γ', the quantity

$$\sum_{n=0}^{\infty} F(x+n),$$

or the quantity

$$G(x+n_1)+\sum_{n=0}^{n_1-1} F(x+n)+\sum_{n=n_1}^{\infty}\left\{F(x+n)-\left(\frac{a_1}{a_2}\right)^{x+n+1}(x+n)^\lambda\right.$$
$$\left.\times\left(1+\frac{\sigma_1}{x+n}+\dots+\frac{\sigma_N}{(x+n)^N}\right)\right\}$$

(whichever has to be employed) possesses a complete asymptotic expansion which we shall write in the form

$$F_3(x) = \left(\frac{a_1}{a_2}\right)^x x^\lambda\left[\tau_0+\frac{\tau_1}{x}+\dots+\frac{\tau_r+w'_x}{x^r}\right],$$

where τ_0, τ_1, ... are definite constants, and $w'_x \rightarrow 0$ as $x \rightarrow \infty$.

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[*] The definite integral is of Borel's type; see Bromwich, *Theory of Infinite Series*, §§ 136, 137.

Now $\dfrac{f_5(x)}{f_4(x)}+\Theta(x)\,F_3(x)$ is a periodic function of x; its singularities in the finite part of the plane are poles which lie in the strip $|I(x)| \leqslant \gamma_2$, where γ_2 is finite; as in § 7, we can construct a function

$$\varpi_1(x) = \frac{P_p^{(2)}(\cot \pi x)}{\prod\limits_{n=1}^{p} (\cot \pi x + \cot \pi \epsilon_n)},$$

where $P_p^{(2)}$ denotes a polynomial of degree p, such that

$$\frac{f_5(x)}{f_4(x)}+\Theta(x)\,F_3(x)-\varpi_1(x)$$

has no singularities in the finite part of the plane; since it is periodic [it is equal to $\varpi(x)-\varpi_1(x)$], it is a uniform* function of z where $z = e^{\pm 2\pi i x}$; and, *qua* function of z, its only singularities are at $z = 0$ and $z = \infty$.

If $0 < \arg(a_1/a_2) < \pi$, take $z = e^{2\pi i x}$; if $0 > \arg(a_1/a_2) > -\pi$, take $z = e^{-2\pi i x}$.

When $|I(x)| \to \infty$, in such a way that $-\frac{1}{2} \leqslant R(x) \leqslant \frac{1}{2}$, since

$$\prod_{n=1}^{\infty} \left[\frac{1}{a_1 \Psi(x+n)} \exp \frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right] \to 1,$$

$$\prod_{n=1}^{\infty} \left[\frac{1}{a_1 \Psi_1(x-n)} \exp \frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right] \to 1,$$

and
$$\prod_{n=0}^{\infty} \left[\frac{v(x+n)}{\lambda_0} \exp \frac{-\lambda_1}{\lambda_0(x+n)}\right] \to 1,$$

we may shew that

$$\frac{f_5(x)}{f_4(x)} \to \left(\frac{a_1}{a_2}\right)^x \exp\left[\frac{-\lambda_1}{a_1(a_1-a_2)}\{\log(x+1)+\log(1-x)\} - \frac{\lambda_1}{\lambda_0}\log x\right],$$

i.e.,
$$\left|\frac{f_5(x)}{f_4(x)}\right| < K \left|\left(\frac{a_1}{a_2}\right)^x\right| |x^\lambda|,$$

where K is finite.

Hence we deduce that

$$\left|\frac{f_5(x)}{f_4(x)}+\Theta(x)\,F_3(x)-\varpi_1(x)\right| < K' \left|\left(\frac{a_1}{a_2}\right)^x\right| |x^\lambda| + K'',$$

where K', K'' are finite and independent of x when $|I(x)| > \gamma_2$.

Putting
$$\frac{f_5(x)}{f_4(x)}+\Theta(x)\,F_3(x)-\varpi_1(x) = \Omega(z),$$

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* Forsyth, *Theory of Functions*, § 112.

so that Ω is a uniform function of z, we see that

$$\operatorname*{Lt}_{z=0} |\Omega(z)| < K'', \tag{36}$$

since $$\{I(x) \arg(a_1/a_2)\} - R(\lambda \log x) \to +\infty \text{ as } z \to 0.$$

The only possible singularities of $\Omega(z)$ are at 0 and ∞; and hence from this result $\Omega(z)$ is an integral function.*

Let $$\operatorname*{Lt}_{z=0} \Omega(z) = a_0.$$

Then $\dfrac{\Omega(z)-a_0}{z}$ is an integral function of z which, when $|z| > \exp(2\pi\gamma_2)$, is such that
$$\left|\frac{\Omega(z)-a_0}{z}\right| < \frac{1}{|z|}\left\{K'\left|\left(\frac{a_1}{a_2}\right)^x\right| |x^\lambda| + K'''\right\},$$

where K''' is finite.

Now $$\left|z^{-1}\left(\frac{a_1}{a_2}\right)^x x^\lambda\right| = \left|x^\lambda \exp\left\{x\left(\log\frac{a_1}{a_2} \mp 2\pi i\right)\right\}\right|,$$

upper or lower sign being taken as $\arg(a_1/a_2)$ is + or −.

Hence, as $z \to \infty$, $\left|z^{-1}\left(\frac{a_1}{a_2}\right)^x x^\lambda\right|$ tends to zero more rapidly than $|x^\lambda \exp\{-\pi|I(x)|\}|$, since $|\arg(a_1/a_2)| < \pi$.

Therefore $\dfrac{\Omega(z)-a_0}{z}$ is an integral function of z, whose modulus is always less than a finite quantity K_1, and which $\to 0$ as $z \to \infty$. Hence, by Liouville's theorem,
$$z^{-1}[\Omega(z)-a_0] = 0.$$

We have thus shewn that $f_5(x)/f_4(x)$ can be expressed in one or other of the following forms:

$$\frac{f_5(x)}{f_4(x)} = \frac{P_p^{(2)}(\cot\pi x)}{\prod\limits_{n=0}^{p}(\cot\pi x+\cot\pi\epsilon_n)} - a_0 - \Theta(x)\sum_{n=0}^{\infty} F(x+n), \tag{37A}$$

or $$\frac{f_5(x)}{f_4(x)} = \frac{P_p^{(2)}(\cot\pi x)}{\prod\limits_{n=0}^{p}(\cot\pi x+\cot\pi\epsilon_n)} - a_0 - \Theta(x)\Bigg[G(x+n_1) + \sum_{n=0}^{n_1-1} F(x+n)$$
$$+ \sum_{n=n_1}^{\infty}\left\{F(x+n) - \left(\frac{a_1}{a_2}\right)^{x+n+1}(x+n)^\lambda\left(1+\frac{\sigma_1}{x+n}+\dots+\frac{\sigma_N}{(x+n)^N}\right)\right\}\Bigg], \tag{37B}$$

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* The inequality (36) shews that $\Omega(z)$ has neither a pole nor an essential singularity at $z = 0$.

according to the value of λ; and a_0 is such that

$$\frac{P_p^{(2)}(\cot \pi x)}{\prod\limits_{n=0}^{p}(\cot \pi x + \cot \pi \epsilon_n)} \to a_0 \quad \text{as} \quad I(x) \to +\infty \quad (\arg a_1 > 0),$$
$$\text{or as} \quad I(x) \to -\infty \quad (\arg a_1 < 0).$$

And when $R(x) \geqslant -\frac{1}{2}$, $|x| > \gamma_2$, we have the complete asymptotic expansion

$$\frac{f_5(x)}{f_4(x)} = \frac{P_p^{(2)}(\cot \pi x)}{\prod\limits_{n=0}^{p}(\cot \pi x + \cot \pi \epsilon_n)} - a_0 - \Theta(x)\left(\frac{a_1}{a_2}\right)^x x^\lambda \left[\tau_0 + \frac{\tau_1}{x} + \ldots + \frac{\tau_r + w'_x}{x^r}\right].$$

From this we can deduce by the methods of § 7 that when

$$|\arg(-x)| < \pi - \delta, \quad |I(x)| > \gamma_2, \quad R(x) \geqslant -\tfrac{1}{2},$$

$\Psi_1(x)$ has the same asymptotic expansion as when $R(x) < \frac{1}{2}$, $|x| > \gamma$; and that when $|\arg x| < \pi - \delta$, $|I(x)| > \gamma_2$ and $R(x) \leqslant \frac{1}{2}$, $\Psi(x)$ possesses an asymptotic expansion of the form given by equation (23*a*).

This result concerning the asymptotic expansion of $\Psi(x)$ is true when $v(x)$ is *not* the quotient of two polynomials; it can be deduced from equation (22) in much the same way as (23*a*) is deduced from (22), by making use of asymptotic properties of the gamma function;* it is not, however, possible to obtain *any* information concerning $\Psi(x)$ when $|\arg(-x)| < \delta$, whereas when $v(x)$ is the quotient of two polynomials we can get a convenient explicit formula for $\Psi(x)$ in this region of the plane by applying either (37A) or (37B) to the formula

$$\frac{f_5(x-1)}{f_4(x-1)} \Big/ \frac{f_5(x)}{f_4(x)} = v(x-1)\,\Psi_1(x-1)\,\Psi(x).$$

It does not appear to be possible to modify this analysis to cover the case when† $\left|\dfrac{a_1}{a_2}\right| > 1$, so as to obtain an asymptotic expansion of $\Psi(x)$ when λ_0 is not a real quantity greater than $\frac{1}{4}$, and $R(x)$ is large and negative and $|I(x)|$ is not large; it seems that § 10 of the first paper and § 5A contain all that can be determined.

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* We can modify the lemma proved in the introduction to shew that the quantity called Θ_5 in § 3B is finite when $|\arg x| \leqslant \pi - \delta$ and $|I(x)| > \mu$, provided $\delta > 0$.

† For $\sum\limits_{n=0}^{\infty}\left(\dfrac{a_1}{a_2}\right)^n \dfrac{1}{(x+n)^r}$ does not converge for any finite value of r.

NUMBER OF PRIMES OF GIVEN LINEAR FORMS

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[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for reading the proof sheets of this paper.]

1. *Introduction.*—The object of this paper is to give the results of actually counting the numbers of primes of certain linear forms, and to compare the numbers counted with a certain formula for the same.

Let N = total number of primes (p) within a given "range" (say R), of the natural numbers.

Let p denote a prime of the linear form $p = n\varpi + a$ $(a < n)$: n is hereinafter styled the "modulus" of the prime-form.

Let M_a = the number of primes (p) of that form within that range (R).

Let $\phi(n)$ denote the totient of n.

2. *Approximate Formula for* M_a.—Considering the form of p with respect to the "modulus" n, it is seen that a must be prime to n (in order that p may be *prime*): so that the set of values possible for a is the whole set of integers prime to n, and $< n$; and the number of such values (of a) is therefore $= \phi(n)$, the totient of n.

Hence, unless there be some reason for certain of the $\phi(n)$ values of a yielding more primes than other values of a yield, it would follow that the numbers (M_a) of primes of the forms

$$p = n\varpi + a_1, \quad n\varpi + a_2, \quad \dots, \quad n\varpi + a_r \quad (a \text{ prime to } n, \text{ and } < n)$$

in any one *large* range of the natural numbers should be nearly equal; so that

$$M = \frac{1}{\phi(n)} N \text{ approximately (for each value of } a). \tag{1}$$

This rule has now been tested by the author by actually counting* the numbers (M) of primes p of a considerable number of forms ($n\varpi+a$) within various ranges.

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* The labour of such counts is very great: the author had, however, a number of printed and MS. tables ready to hand, specially suited to this work. The risk of error (especially of missing one or two primes in counting) is considerable; all the counts have been done by two independent counters.

3. *Form* $p = n\varpi+1$. *Count of* M_1.—The particular form

$$p = (n\varpi+1)$$

(in which $a = 1$, always) seems the most interesting on account of its connexion with Fermat's theorem for bases a (prime to p), viz., whether

$$a^{(p-1)\div n} \equiv \mp 1 \text{ or } \not\equiv \mp 1 \pmod{p}, \quad [p = n\varpi+1]. \tag{2}$$

The main Table (on p. 251) shows the number (M) of primes of form $p = (n\varpi+1)$ actually connected with certain ranges ($R = 1$ to 10^4, 10^5, or 5.10^5) for a number (61) of values of the modulus (n). The short abstract below shows the total* number (N) of primes in each range (R), and the number of cases (*i.e.*, of different values of n) available for discussion in each range.

Values of n.	Cases.	Total number (N) of primes in Range (R)		
		$R = 1$ to 10^4	$R = 1$ to 10^5	$R = 1$ to 5.10^5
All *even* numbers, 2 to 60	30	$N = 1228$	$N = 9591$	
Certain *even* numbers, 64 to 210	15	$N = 1228$	$N = 9591$	
Numbers $n = 8q$ (q prime $= 101$ to 241)	16		$N = 9591$	$N = 41537$

The Table (on p. 251) shows also the totient $\phi(n)$ of each value of n, and the (computed) value of $N \div \phi(n)$, for comparison with the (counted) M.

An examination of the table shows at once that the formula (1) really is a *very close approximation* to the counted number (M), but also discloses the remarkable result that—

The counted number (M) of primes of form $p = (n\varpi+1)$ is (almost always) *less than*, and in many cases *markedly less than*, the computed average, $N \div \phi(n)$. (3)

The exceptions to this rule, *i.e.*, the number of cases of $M > N \div \phi(n)$, are very few (see detail in the abstract below): and the excess of M over $N \div \phi(n)$ is, in all those cases, *trifling* compared with the relatively large deficiency in many of the other cases.

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* The primes 1 and 2 are excluded—throughout this paper—from the totals denoted by N.

Range.	Values of n yielding $M > N+\phi(n)$.	Cases.
1 to 10^4	22, 28, 34, 44, 50, 54	6 in 45
1 to 10^5	34, 50, 54, 58, 100, 130, 720, 768	8 in 61
1 to 5.10^5	432, 552, 720, 768	4 in 16

This renders it probable that the deficiency of the number (M) of primes of the forms $p = (n\varpi + 1)$ below the average number $N \div \phi(n)$ above disclosed really is a property of primes of that form, when taken through a *large range* of the natural numbers. The cause of this property has yet to be sought.

p	$n =$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
	$\phi(n) =$	1	2	2	4	4	4	6	8	6	8	10	8	12	12	8
$\ngtr 10^4$	$N+\phi =$	1228	614	614	307	307	307	204	153	204	153	123	153	102	102	153
	$M =$	1228	609	611	295	306	300	203	144	203	152	126	143	99	103	152
$\ngtr 10^5$	$N+\phi =$	9591	4795	4795	2398	2398	2398	1598	1199	1598	1199	959	1199	799	799	1199
	$M =$	9591	4783	4784	2384	2387	2374	1593	1188	1592	1181	945	1181	798	787	1189

p	$n =$	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
	$\phi(n) =$	16	16	12	18	16	12	20	22	16	20	24	18	24	28	16
$\ngtr 10^4$	$N+\phi =$	77	77	102	68	77	102	61	56	77	61	51	68	51	44	77
	$M =$	73	78	101	64	71	98	63	55	69	63	50	70	50	43	76
$\ngtr 10^5$	$N+\phi =$	600	599	799	533	599	799	480	436	599	480	399	533	399	342	599
	$M =$	599	603	797	525	582	787	466	429	581	486	392	535	385	345	585

p	$n =$	64	70	72	80	88	90	96	100	104	110	120	130	140	150	210
	$\phi(n) =$	32	24	24	32	40	24	32	40	48	40	32	48	48	40	48
$\ngtr 10^4$	$N+\phi =$	38	51	51	38	30	51	38	31	26	31	38	25	25	30	26
	$M =$	38	48	47	35	29	50	35	30	22	29	35	25	24	29	23
$\ngtr 10^5$	$N+\phi =$	300	399	399	299	240	399	299	240	200	300	299	200	200	300	200
	$M =$	300	392	397	293	228	388	292	243	194	231	286	202	183	235	192

p	$n =$	2	8.101	8.103	8.107	8.109	8.137	8.139	8.149	8.157	8.167	8.173	8.181	8.193	8.199	8.227	8.229	8.241
	$\phi(n) =$	1	400	408	424	432	544	552	592	624	664	688	720	768	792	904	912	960
$\ngtr 10^5$	$N+\phi =$	9591	24	24	23	22	18	17	16	15	15	14	13	13	12	11	10	10
	$M =$	9591	21	18	18	16	17	17	16	11	15	14	15	14	6	11	8	10
$\ngtr 5.10^5$	$N+\phi =$	41537	104	102	98	96	76	75	70	67	63	60	58	54	57	46	46	43
	$M =$	41537	104	99	96	101	75	77	69	64	63	60	61	58	48	44	43	41

4. *Form* $p = n\varpi + a$. *Count of* M_a.—The above result carries with it the property that the number (M_a) of primes of the form $p = (n\varpi + a)$ (with $a > 1$) must *exceed the average number* of the formula for some

values of a (> 1); and that, in particular, some M_a must be $> M_1$; all the M_a being, of course, within the same (large) range (R).

To test this further property, the numbers (M_a) of primes of each of the following forms

$$p = 4\varpi \pm 1;\ 6\varpi \pm 1;\ 8\varpi \pm 1, 3;\ 10\varpi \pm 1, 3;\ 12\varpi \pm 1, 5$$

have now been counted within the same range $R = 1$ to 10^5.

The table below—drawn up very similarly to the preceding—shows under each modulus ($n = 4, 6, 8, 10, 12$) the totient $\phi(n)$ of n, the computed average number $N \div \phi(n)$ of primes of each form $p = (n\varpi + a)$, and lastly the actual counted number (M) of primes of that form.

$n =$	4		6		8				10				12			
$\phi(n) =$	2		2		4				4				4			
$N^* =$	9591		9590		9591				9590				9590			
$N \div \phi(n) =$	4795		4795		2398				2398				2398			
$p =$	$4\varpi+1$	$4\varpi+3$	$6\varpi+1$	$6\varpi+5$	$8\varpi+1$	$8\varpi+3$	$8\varpi+5$	$8\varpi+7$	$10\varpi+1$	$10\varpi+3$	$10\varpi+7$	$10\varpi+9$	$12\varpi+1$	$12\varpi+5$	$12\varpi+7$	$12\varpi+11$
$M_a =$	4783	4808	4784	4806	2384	2409	2399	2399	2387	2402	2411	2391	2374	2409	2410	2397

An examination of this table shows the following somewhat remarkable relations between the numbers (M_a) of primes of the $\phi(n)$ forms $p = (n\varpi + a)$ *with the same modulus* (n), all taken, of course, through the same range R ($= 1$ to 10^5 in this case):—

The numbers M_a are *approximately equal*, so that formula (1) is a *good approximation*. (4)

The number M_1 is the *least of all* the M_a. (5)

The number M_{n-1} is the *next least* of all the M_a. (6)

[It should be stated that the particular result for the modulus $n = 4$, viz., that the number M_1 is $< M_3$, has been *proved generally true* by† Tchebycheff.]

The two results (5), (6) together involve the following:—

The number of primes (within a given large range), of which 2, 3, 5 are 2-ic residues, is *markedly less*‡ *than* the number of which these bases are 2-ic non-residues. (7)

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* The primes 1 and 2 are excluded from all these totals (N): the prime 3 is excluded when $n = 6$ and 12, and 5 is excluded when $n = 10$.

† In a letter to M. Fuss, published in the *Bull. de l'Acad. de St. Pétersbourg*, 1853; quoted in Glaisher's *Factor Table for the Fourth Million*, London, 1879, Introduction. p. 33.

‡ This property as to 2-ic residuacity is believed to be true for other bases; and a similar property is believed to be true for higher orders of residuacity. It is hoped to make these properties the subject of a further communication: the data are well in hand.

5. *Author's Previous Work.*—A certain number of the counts (M_a) now reported had been previously published by the author on p. xx of the Introduction to his *Tables of Quadratic Partitions*, London, 1904; viz.,

8 cases of M_1 (*i.e.*, for $p = n\varpi + 1$), viz., for $n = 4, 6, 8, 10, 12, 16, 24, 30$.

2 cases of M_{n-1}, viz., for $p = (4\varpi - 1)$ and $(6\varpi - 1)$.

Slight errors in those counts have been found, viz.,

Under $p =$	$(6\varpi + 1)$;	$(6\varpi - 1)$;	$(12\varpi + 1)$;	$24\varpi + 1$;
Read	611 (not 612);	4806 (not 4807);	2374 (not 2373);	1181 (not 1180).

ON THE CONVERGENCE OF A FOURIER SERIES AND OF ITS ALLIED SERIES

By W. H. YOUNG, Sc.D., F.R.S.

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1. In a recent paper* published in the *Proceedings* of the Royal Society, I had occasion to make use of the Fourier series of the function $\{f(x+u)-f(x-u)\}\operatorname{cosec} u$. In the present paper, denoting by $F(x)$ the indefinite integral $\int_0^x f(x)\,dx$, I consider the Fourier series and allied series of

$$I = \{F(x+2u)-F(x-2u)\}\operatorname{cosec} u,$$

and
$$J = \{F(x+2u)+F(x-2u)-2F(x)\}\operatorname{cosec} u,$$

and their relations to the Fourier series and allied series of $f(x)$. I am thus enabled to deduce tests for the ordinary and Cesàro convergence of these latter series, which include as particular cases all hitherto known tests. Among such included tests may be specially mentioned the very important one for the convergence, in the ordinary sense, of a Fourier series, given by de la Vallée Poussin, in a recent paper,† a test which itself includes all the usual ones as particular cases.

I begin by obtaining two fundamental properties required in the sequel which themselves relate to the convergence of a Fourier series. These are

> (1) If $xf(x)$ be in some interval containing the origin, the integral of a function, say $g(x)$, which is either continuous, or such that $\int_0^x |g(x)-g(0)|\,dx$ has at the origin a differential coefficient which is zero, then the series of the Fourier constants of $f(x)$, supposing this to be an even function, converges, and has $f(+0)$ for its sum; while, if it be an odd function, the series converges to
>
> $$\operatorname*{Lt}_{e=0} \frac{1}{\pi}\int_e^\pi f(u)\cot\tfrac{1}{2}u\,.\,du,$$
>
> provided only this limit exists.
>
> (2) If $f(x)$ in every interval not containing the origin, be an integral, and have bounded variation everywhere, then the coefficients of the derived series of the Fourier series converge to zero if $f(x)$ be

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* "On the Fourier Constants of a Function," § 10, presented December 10th, 1910.

† "Un nouveau cas de convergence des séries de Fourier," *Rend. di Pal.*, Vol. XXXI, 1911.

an even function; while, if it be an odd function, the odd terms and the even terms have equal limits with opposite signs which coincide if $f(\pi)$ and $f(-\pi)$ are both zero.

I then pass to the consideration of I, and shew that, if s_n and t_n denote the ordinary n-th partial summation and that of Cesàro (index unity) of the Fourier series of $f(x)$, and S_n and T_n the corresponding summations of the series $\sum_{r=1}^{\infty} A_r$, where $A_1, A_2, \ldots$ denote the odd coefficients of the Fourier series of I, then

$$4s_n = S_n - nA_n, \tag{I}$$

$$4t_n = T_n - S_n, \tag{II}$$

whence if we please also $4t_n^{(r+1)} = T_n^{(r+1)} - T_n^{(r)}$, $\quad$ (II$^{(r)}$)

the notation referring to a suitably defined Cesàro summation of index r.

It will be remarked that the success of the method depends essentially on the fact that for the consideration of $f(x)$ is substituted that of a function whose behaviour at the point $u = 0$ alone is, so to speak, abnormal. Elsewhere than at this point, in fact, the functions I and J are integrals. This is a great advantage. The statement, sometimes made, that the character of the convergence of a Fourier series depends only on the nature of the function in the neighbourhood of the point considered is indeed only true, as the property (2) given above suggests, when properly interpreted and under suitable conditions.

From the equations (I), (II), and (II$^{(r)}$) the whole theory follows. For example, the sufficient and, provided only the Fourier series of I is, at $u = 0$, capable of Cesàro summation of some index, necessary condition that the Fourier series of $f(x)$ should converge, is that the coefficients of the derived series of the Fourier series of I should converge to a unique limit, while the necessary and sufficient condition that the Fourier series of $f(x)$ should converge when summed in the Cesàro way (index 1) is that the Fourier series of I should at the point $u = 0$ converge in the ordinary way.

Similar statements are true for the allied series of $f(x)$, if we consider the allied series of J instead of the Fourier series of I. Moreover, these statements may be generalised.

If, on the other hand, we choose to add to the elegance of our results at a slight loss of scope, we may substitute for the convergence of the Fourier series of I, for example, any one of the known conditions, *e.g.*, that of de la Vallée Poussin above referred to. Or we may with less loss of generality use the fundamental properties (1) and (2) stated above. We then obtain, among other interesting results,* the theorem that it is a sufficient condition for the convergence of the Fourier series of $f(x)$ when summed in the Cesàro manner (index 2) that, for a certain value of k, the function of u

$$\int_0^u \left| \frac{F(x+t)-F(x-t)}{t} - k \right| dt,$$

has at the origin a differential coefficient which is zero. Moreover, the Cesàro sum of the Fourier series of $f(x)$ is the ordinary sum at the point $u = 0$ of the Fourier series of

$$\frac{1}{2u} \int_0^u \frac{F(x+t)-F(x-t)}{t} dt.$$

On the other hand, the use of the equation ($\mathrm{II}^{(r)}$) alone, gives us, when coupled with a theorem of Marcel Riesz, the result that the Fourier series of $f(x)$ converges, when summed in any Cesàro manner of index greater than unity, integral or fractional, at any point at which the differential coefficient of its integral exists and is finite, and that its sum is equal to this differential coefficient. Moreover this result may, if we please, be still further generalised. It thus appears, too, that the behaviour of the function of u represented by the p-fold integral

$$\int_0^1 \int_0^1 \cdots \int_0^1 \{f(x+ut_1 t_2 \ldots t_p) + f(x-ut_1 t_2 \ldots t_p)\} dt_1 dt_2 \ldots dt_p,$$

at the point $u = 0$, determines the character of the Fourier series when summed in the corresponding Cesàro manner.

2. We begin with the second of the properties stated in the introduction and consider first the case of an even function.

THEOREM.—*If an even continuous function of bounded variation be an integral in every interval not containing the origin, the coefficients of the derived series of its Fourier series converge to zero.*

Let $g(u)$ be the function got from the given function by subtracting its value at the origin, then, $\frac{1}{2}a_0, a_1, a_2, \ldots, a_n, \ldots$ being the Fourier constants of the given function,

$$\pi n a_n = \int_{-\pi}^{\pi} g(u) \cos nu . n du = \int_0^{\pi} \{g(u)+g(-u)\} \cos nu . n du$$

$$= \int_0^e \{g(u)+g(-u)\} \cos nu . n du + \int_e^{\pi} \{g(u)+g(-u)\} \cos nu . n du, \quad (1)$$

where e is any fixed quantity between 0 and π.

Now $g(u)+g(-u)$ is a function which is continuous and of bounded variation and is zero at the origin. It can therefore be written in the form $P(x)-N(x)$, where $P(x)$ and $-N(x)$ are respectively the positive and negative variations measured from the origin, so that $P(x)$ and $N(x)$ are monotone increasing continuous functions and vanish at the origin. Hence we may write the numerical value of the first integral on the right in (1) in the form

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* See, in particular, Lebesgue's theorem in § 9 and its analogue in § 13.

$$\left|\int_0^e \{P(u)-N(u)\}\cos nu\,.\,ndu\right| \leqslant \{P(e)+N(e)\}\left|\int_{e'}^e \cos nu\,.\,ndu\right|$$
$$\leqslant 2\{P(e)+N(e)\},$$

using the Second Theorem of the Mean. By suitably choosing E, and restraining e to be less than E, this integral can therefore be made numerically as small as we please.

Again, since $g(u)$ is an integral in the interval (e, π) we may integrate the second integral on the right of (1) by parts, and get

$$\Big[\{g(u)+g(-u)\}\sin nu\Big]_e^\pi - \int_e^\pi \{g'(u)-g'(-u)\}\sin nu\,.\,du.$$

Now, since $g(u)+g(-u)$ is an integral, $g'(u)-g'(-u)$ is a summable function, the value zero, or any other convenient value or values, being assigned to $g'(u)-g'(-u)$, wherever $g(u)+g(-u)$ does not possess a differential coefficient, these points forming in fact a set of content zero. Hence, by the theorem of Riemann-Lebesgue, the integral last written down has the unique limit zero when n is indefinitely increased. It follows that the second integral on the right of (1) has a limit, or limits, numerically not greater than

$$|\,g(e)+g(-e)\,|,$$

which, by a suitable choice of E and restraining e to be less than E, is as small as we please.

Hence, also by (1), $\pi n a_n$ has limits when n increases indefinitely, which are all numerically as small as we please, and therefore all coincide and have the value zero. This proves the theorem.

We next consider the case of an odd function.

THEOREM.—*If an odd continuous function of bounded variation be an integral in every interval not containing the origin, then, if* b_r *denote its typical Fourier constant,* $(2m+1)\,b_{2m+1}$ *and* $2mb_{2m}$ *both have unique limits when m increases indefinitely, and these limits are in general equal with opposite sign : they are both zero if the function has equal and therefore zero values at* π *and* $-\pi$.

Let $h(u)$ be the odd function. Then

$$\pi n b_n = \int_{-\pi}^{\pi} h(u) \sin nu . n du = \int_0^{\pi} \{h(u) - h(-u)\} \sin nu . n du.$$

Now

$$\int_{\epsilon}^{\pi} \{h(u) - h(-u)\} \sin nu . n du$$

$$= \Big[- \{h(u) - h(-u)\} \cos nu \Big]_{\epsilon}^{\pi} + \int_{\epsilon}^{\pi} [h'(u) + h'(-u)] \cos nu . du$$

$$= - \{h(\pi) - h(-\pi)\} \cos n\pi + \{h(\epsilon) - h(-\epsilon)\} \cos n\epsilon$$

$$+ \int_{\epsilon}^{\pi} [h'(u) + h'(-u)] \cos nu . du.$$

The last integral has the unique limit zero by the theorem of Riemann-Lebesgue, and the second term on the right can, by suitably restricting ϵ be made as small as we please. Again, as in the preceding proof,

$$\int_0^{\epsilon} \{h(u) - h(-u)\} \sin nu . n du \leqslant 2[P(\epsilon) + N(\epsilon)],$$

where $P(u)$ and $N(u)$ are the positive and negative variations of $\{h(u) - h(-u)\}$ measured from the origin; hence this integral is also as small as we please, when ϵ is suitably restricted.

Hence the limits of $\pi n b_n$ differ by as little as we please from those of $- \{h(\pi) - h(-\pi)\} \cos n\pi$. But the latter expression has only two limits which are equal and opposite, if they are not both zero. Thus

$$\underset{n=\infty}{\mathrm{Lt}}\ \pi n b_n = - h(\pi) + h(-\pi), \text{ if } n \text{ be even,}$$

$$= \quad h(\pi) - h(-\pi), \text{ if } n \text{ be odd.}$$

This proves the lemma.

3. We now come to the first of the properties referred to in the introduction.

THEOREM.—*If $f(u)$ be an even function of u, having the property that $uf(u)$ has throughout some interval containing the origin, a continuous differential coefficient, or, more generally, is expressible in the form*

$$uf(u) = \int_0^u g(u) du,$$

where $g(u)$ is such that $\int_0^u |g(u) - g(0)| du$

possesses at the point $u = 0$ *a differential coefficient which is zero, then the series of the Fourier constants of* $f(x)$ *converges to the value* $f(+0)$, *so that we have*

$$f(+0) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n.$$

Since $\int_0^u |g(u) - g(0)|\, du$ possesses a differential coefficient at the origin which has the value zero, it follows that, if we write

$$\theta(u) = \frac{1}{u}\int_0^u |g(u) - g(0)|\, du,$$

then $\theta(u)$ is a continuous function of u which vanishes at the origin. The same is accordingly true of

$$f_1(u) = \frac{1}{u}\int_0^u \{g(u) - g(0)\}\, du = f(u) - g(0).$$

Hence $f(u)$ has a unique limit as u approaches zero from the right, that is

$$f(+0) = g(0):$$

for brevity we shall denote this limit by $f(0)$, and $g(u) - g(0)$ by $g_1(u)$, $f(u) - f(0)$ being accordingly

$$f_1(u) = \frac{1}{u}\int_0^u g_1(u)\, du.$$

Denote by s_n the partial summation $\tfrac{1}{2}a_0 + \sum_{r=1}^{n-1} a_r$, then

$$\begin{aligned} s_n - f(0) &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \{f(2u) - f(0)\} \frac{\sin(2n-1)u}{\sin u}\, du \\ &= \frac{2}{\pi}\int_0^{\pi} \{f(u) - f(0)\} \frac{\sin ku}{\sin \frac{1}{2}u}\, \tfrac{1}{2}du \quad [k = \tfrac{1}{2}(2n-1)] \\ &= \frac{2}{\pi}\int_0^{\pi} f_1(u) \frac{\sin ku}{u}\, du - \frac{2}{\pi}\int_0^{\pi} f_1(u) \left\{\frac{1}{u} - \frac{1}{2\sin \frac{1}{2}u}\right\} \sin ku\, du. \end{aligned}$$

Now by the theorem of Riemann-Lebesgue, the last integral has, when k increases indefinitely, the unique limit zero, since the factor multiplying $\sin ku$ in the integrand is summable. For the same reason the preceding integral vanishes with $1/k$, if we change the inferior limit of integration from zero to any fixed positive quantity μ. Thus

$$\underset{n\to\infty}{\mathrm{Lt}}\, [s_n - f(0)] = \underset{k\to\infty}{\mathrm{Lt}}\, \frac{2}{\pi}\int_0^{\mu} f_1(u) \frac{\sin ku}{u}\, du. \tag{1}$$

Let e be any selected small positive quantity, and let μ be fixed so that for all values of $u \leqslant \mu$,

$$\theta(u) < e; \tag{2}$$

and therefore

$$|f_1(u)| < e. \tag{3}$$

Then, k being taken greater than π/μ,

$$\int_0^{\pi/k} f_1(u) \frac{\sin ku}{u} du \leqslant \int_0^{\pi/k} |f_1(u)| \frac{\sin ku}{u} du \leqslant e \int_0^{\pi/k} \frac{\sin ku}{u} du \leqslant \pi e, \tag{4}$$

and

$$\int_{\pi/k}^{\mu} f_1(u) \frac{\sin ku}{u} du = \left[-\frac{1}{k} \cos ku \frac{f_1(u)}{u} \right]_{\pi/k}^{\mu} + \frac{1}{k} \int_{\pi/k}^{\mu} \cos ku \frac{d}{du} \left\{ \frac{f_1(u)}{u} \right\} du, \tag{5}$$

Here the quantity in square brackets vanishes with $1/k$, since $f_1(\pi/k)$ does so.

Also the integral on the right is numerically less than

$$\frac{1}{k} \int_{\pi/k}^{\mu} \left| \frac{d}{du} \frac{f_1(u)}{u} \right| du = \frac{1}{k} \int_{\pi/k}^{\mu} \left| \frac{g_1(u)}{u^2} - \frac{2}{u^3} \int_0^u g_1(t)\, dt \right| du$$

$$\leqslant \frac{1}{k} \int_{\pi/k}^{\mu} \frac{|g_1(u)|}{u^2} du + \frac{2}{k} \int_{\pi/k}^{\mu} \frac{|f_1(u)|}{u^2} du.$$

Integrating the former integral by parts, the sum of the two integrals becomes, by (2) and (3),

$$\leqslant \frac{1}{k} \left[u^{-2} \int_0^u |g_1(u)|\, du \right]_{\pi/k}^{\mu} + \frac{4}{k} \int_{\pi/k}^{\mu} u^{-2}\, \theta(u)\, du$$

$$\leqslant \frac{1}{k} \left[\frac{\theta(u)}{u} \right]_{\pi/k}^{\mu} + \frac{4e}{k} \left\{ \frac{k}{\pi} + \frac{1}{\mu} \right\}.$$

Now $\theta(\mu)/k\mu$ and the term $4e/k\mu$ vanish with $1/k$, whence it follows that the integral on the right of (5), and therefore also that on the left, has all its limits numerically less than $5e/\pi$. Hence, remembering that $0 < \pi/k < \mu$, we get by (1) and (4),

$$\left| \operatorname*{Llt}_{n\to\infty} \{s_n - f(0)\} \right| < \left\{ \pi + \frac{5}{\pi} \right\} e.$$

Since e is at our disposal, this shews that the limits on the left all coincide and have the value zero, that is

$$\operatorname*{Lt}_{n\to\infty} s_n = f(0),$$

which proves the theorem,

We now state and prove the corresponding theorem for an odd function.

THEOREM.—*If $f(u)$ is an odd function of u having the property that $uf(u)$ has throughout some interval containing the origin a continuous differential coefficient, or, more generally, is expressible in the form*

$$uf(u) = \int_0^u g(u)\,du,$$

where $g(u)$ is such that $$\int_0^u |g(u)|\,du,$$

possesses at the point $u = 0$ a differential coefficient which is zero, then the series of the Fourier constants of $f(u)$ converges to the value

$$\mathop{\mathrm{Lt}}_{\epsilon=0} \frac{1}{\pi}\int_\epsilon^\pi f(u)\cot\tfrac{1}{2}u\,du,$$

provided only this limit exists.

Since $\int_0^u |g(u)|\,du$ possesses at the point $u = 0$ a differential coefficient which is zero, it follows that, if we write

$$\theta(u) = \frac{1}{u}\int_0^u |g(u)|\,du,$$

then $\theta(u)$ is a continuous function of u, which vanishes at the origin. The same is accordingly true of

$$f(u) = \frac{1}{u}\int_0^u g(u)\,du.$$

We now choose μ so small that $\theta(u)$ and $f(u)$ are numerically less than e for all values of u in the interval $(0, \mu)$.

Denote by s_n the partial summation $\sum\limits_1^{n-1} b_r$, then

$$s_n = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} f(2t)\,\{\cot t - \operatorname{cosec} t\cos 2kt\}\,dt \qquad \left(k = \frac{2n-1}{2}\right).$$

Hence also the limits of s_n as n increases are identical with those of

$$\frac{2}{\pi}\int_\mu^{\frac{1}{2}\pi} f(2t)\,\{\cot t - \operatorname{cosec} t\cos 2kt\}\,dt$$

$$+\frac{2}{\pi}\int_0^\mu f(2t)\,\{\cot t - \operatorname{cosec} t\cos 2kt\}\,dt.$$

The former of these two integrals has the unique limit

$$\frac{2}{\pi}\int_{\mu}^{\frac{1}{2}\pi} f(2t)\cot t\,dt, \tag{1}$$

by the theorem of Riemann-Lebesgue. The second of these two integrals is

$$\frac{2}{\pi}\int_{0}^{\pi/q} f(2t)\,\{\cos t-\cos 2kt\}\ \operatorname{cosec} t\,.\,dt$$

$$+\frac{2}{\pi}\int_{\pi/q}^{\mu} f(2t)\,\{\cot t-\operatorname{cosec} t\cos 2kt\}\,dt.$$

In the former of these two integrals we may write $1/t$ for cosec t without affecting the limits sought, since $\int_{0}^{\pi/q}\left|\,2\left\{\frac{1}{t}-\frac{1}{\sin t}\right\}f(2t)\right|dt$ vanishes with π/q. Hence, using (1), the limits of s_n are identical with those of

$$\frac{2}{\pi}\int_{0}^{\pi/q}\frac{f(2t)}{t}\,2\sin nt\sin(n+1)\,t\,.\,dt+\frac{2}{\pi}\int_{\pi/q}^{\frac{1}{2}\pi} f(2t)\cot t\,.\,dt$$

$$-\frac{2}{\pi}\int_{\pi/q}^{\mu} f(2t)\operatorname{cosec} t\cos 2kt\,dt.$$

Now the first of these integrals is

$$\leqslant \frac{2e}{\pi}\int_{0}^{\pi/q}\frac{2\sin nt}{t}\,dt\leqslant 4e,$$

provided only $q \geqslant n$, and therefore becomes infinite with n. And the third of these integrals, precisely in the same way as in the treatment of the corresponding integral on page 260, is found to be $\leqslant \frac{5e}{\pi}$.

Hence, precisely as on page 260, we see that s_n has a unique limit, namely that of the second of these integrals, provided only that limit exists.

4. Let $f(x)$ be a summable function, rendered if necessary periodic, and for convenience, supposed such that its Fourier series has no constant term, so that we may write

$$f(x) \sim \sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1}$$

Denoting $\int_{0}^{x} f(x)\,dx$ by $F(x)$, we have

$$F(x)=C+\sum_{n=1}^{\infty} n^{-1}(a_n\sin nx-b_n\cos nx), \tag{2}$$

so that $F(x)$ is periodic with period 2π. The same is therefore true of the even function

$$g(u) = \{F(x+2u)-F(x-2u)\} \operatorname{cosec} u, \tag{3}$$

here supposed to be summable.

Evidently $g(\pi+u) = -g(u)$, so that the Fourier series of $g(u)$ will contain only odd terms, and we may write

$$g(u) \sim \sum_{n=1}^{\infty} A_n \cos(2n-1)\, u. \tag{4}$$

By (2) and (3),

$$\begin{aligned} A_n - A_{n+1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\{\cos(2n-1)\,u - \cos(2n+1)\,u\}\, du \\ &= \frac{2}{\pi}\int_{-\pi}^{\pi} g(u) \sin u \sin 2nu\, du \\ &= \frac{4}{n}(a_n \cos nx + b_n \sin nx). \end{aligned} \tag{5}$$

Hence remembering that though a_0 is zero, we must regard it as present, and denoting by s_n the partial summation $\sum\limits_{r=1}^{n-1}(a_r \cos rx + b_r \sin rx)$, and by S_n the partial summation $\sum\limits_{r=1}^{n} A_r$, we have, summing the equations (5) after clearing of fractions,

$$\begin{aligned} 4s_n &= \sum_{r=1}^{n-1} r(A_r - A_{r+1}) \\ &= A_1 + A_2 + \ldots + A_n - nA_n \\ &= S_n - nA_n. \end{aligned} \tag{I}$$

Hence also denoting by t_n and T_n the Cesàro partial summations

$$(s_1+s_2+\ldots+s_n)/(n+1) \quad \text{and} \quad (S_1+S_2+\ldots+S_n)/(n+1),$$

we have

$$\begin{aligned} 4t_n &= T_n - \{A_1 + 2A_2 + \ldots + nA_n\}/(n+1) \\ &= T_n - \{S_1 + 2(S_2 - S_1) + \ldots + n(S_n - S_{n-1})\}/(n+1) \\ &= 2T_n - S_n. \end{aligned} \tag{II}$$

Hence also

$$4t_n^{(r+1)} = 2T_n^{(r+1)} - T_n^{(r)}, \tag{II$^{(r)}$}$$

where $T_n^{(r)} = S_n$, when $r = 0$, and otherwise denotes the Cesàro sum of index r.*

5. As will have been noticed, the consideration of $g(u)$ presents itself naturally. We may, however, replace it by the function

$$\phi(u) = \{F(x+2u)-F(x-2u)\}/4u,$$

without interfering with the substantial truth of our equations. In fact,

$$\phi(u)-\tfrac{1}{4}g(u) = \tfrac{1}{4}\{F(x+2u)-F(x-2u)\}\left\{\frac{1}{u}-\frac{1}{\sin u}\right\},$$

whence the second factor, like the first, being an integral—as is easily seen by forming its differential coefficient and remarking that this is continuous—it is evident that the left-hand side of the equation is so too. Moreover, $\phi(u)-\frac{1}{4}g(u)$ has as $u \to 0$ the limit zero. Hence (1) its Fourier series converges and has zero for its sum at the point $u = 0$, (2) the coefficients of its derived series converge to zero.

Thus, though our equations (I) and (II) no longer hold as they stand, they do hold when we regard them as abbreviations for corresponding statements as to limits as n increases.

In fact, S_n, nA_n and T_n referring now to the function ϕ, and therefore, by the above, differing from the same quantities referring to the function $\frac{1}{4}g$ by additive terms whose limit is zero, we have, by (I) and (II) of § 2,

$$s_n = S_n - nA_n + p_n, \qquad (\text{I}\phi)$$

$$t_n = 2T_r - S_n + q_n, \qquad (\text{II}\phi)$$

where p_n and q_n have the unique limit zero.

6. From these equations we can deduce at once the following theorem:

THEOREM.—*If* $\phi(u) = \dfrac{1}{2u}\displaystyle\int_0^u \{f(x+2t)+f(x-2t)\}\,dt$ *be a summable function of u, having the property that the coefficients of the derived series of its Fourier series converge to zero, then, if it is known that either the Fourier series of* $f(x)$, *or that of* $\phi(u)$ *for* $u = 0$,

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* This is, in the first instance, only true for positive integral values of r. It is, however, obvious that we may in a variety of ways so define the Cesàro summation of fractional index that the theorem is true always. Thus, if $r = p+k$, where p is the greatest integer in r, we may suppose the r-th sum to be formed by first finding the p-th sum in the ordinary way, and then operating on the partial summations so obtained by any linear process of index k which has the property of coinciding with Cesàro's process when k is made unity.

converges in any Cesàro manner, it follows that both series converge in the ordinary manner, and have the same sum.

It will evidently be sufficient to prove the theorem for the function $\frac{1}{2}g(u)$ instead of $\phi(u)$. We then have, by (I),

$$s_n = S_n - nA_n,$$

and therefore, t_n and T_n and v_n denoting any Cesàro partial summations of the same order of the series whose partial summations are s_n and S_n and nA_n respectively,

$$t_n = T_n - v_n,$$

where v_n has the unique limit zero, since nA_n has. The latter of these equations shews that, if either of the quantities t_n and T_n has a unique limit, the other has the same unique limit. But the equation (II), viz.,

$$t_n = 2T_n - S'_n,$$

S'_n denoting the Cesàro summation of index one lower than T_n, shews that S'_n has the same limit. Hence, by the above argument, s'_n has the same limit. Proceeding back step by step by means of the various equations (II), we see that S_n has the same limit, and therefore, by (I), s_n has the same limit, which proves the theorem.

COR. 1.—*If the coefficients nA_n converge to some finite limit other than zero, the above is still true, except that the two Fourier series have not then the same sum, the difference being* $\underset{n\to\infty}{\text{Lt}}\, nA_n$.*

COR. 2.—*If the coefficients nA_n converge to a finite unique limit, then both the Fourier series of f and that of ϕ at the point $u = 0$ converge for every value of x for which the integral of f has a finite differential coefficient, and their sums differ by that unique limit.*

In fact, at each such point x, the former series converges in the Cesàro manner.†

COR. 3.—*If $f(x)$ is a finite and summable differential coefficient, the necessary and sufficient condition that the Fourier series of $f(x)$ should converge everywhere to $f(x)$ is that* $\underset{n\to\infty}{\text{Lt}}\, nA_n = 0$.

COR. 4.—*In order that the convergence of the Fourier series of $f(x)$*

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* A referee has kindly called my attention to an interesting theorem of Hardy's to the effect that a series Σu_n converges in the ordinary manner, if at all, provided nu_n is bounded. G. H. Hardy, "On Slowly Oscillating Series," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 308.

† H. Lebesgue, "Sur les Intégrales singulières," § 35, p. 90 top.

in any Cesàro manner and that of the Fourier series of $\phi(u)$ at the point $u = 0$ in the same manner should be the consequence one of the other, it is clearly necessary and sufficient that the series whose partial summations are the coefficients of the derived series of the Fourier series of $\phi(u)$ should converge to zero when summed in the same manner.

7. That nA_n has the unique limit zero if $\phi(u)$ has bounded variation, appears from the first theorem of § 2. Bearing in mind that the Fourier series of a function of bounded variation is convergent, we obtain, by $(\mathrm{I}\phi)$, de la Vallée Poussin's theorem:—

THEOREM.—*If* $\phi(u) = \dfrac{1}{2u}\displaystyle\int_0^u \{f(x+2t)+f(x-2t)\}\,dt$

be a function of u of bounded variation up to, and including, the point $u = 0$, the Fourier series of $f(x)$ converges to the value $\phi(+0)$.

Note.—In this case we may, of course, write

$$\frac{1}{2u}\int_0^u \{f(x+t)+f(x-t)\}\,dt,$$

for $\phi(u)$. It is not, however, convenient and, indeed, possible to do this in general.

This theorem relates to the Fourier series of $f(x)$. The analogous theorem relating to the allied series of $f(x)$ will be proved below. It will be found to be:—

THEOREM.—*If* $\chi(u) = \dfrac{1}{2\sin u}\displaystyle\int_0^u \{f(x+2t)-f(x-2t)\}\,dt$

be a continuous function of u of bounded variation in an interval including the point $u = 0$, the allied series of $f(x)$ converges (or diverges) if that of $\chi(u)$ converges (or diverges) at the point $u = 0$. Moreover, the sums of the two allied series are the same.

8. In § 7 we have only used equation (I) of § 4. If we employ equation (II) alone and its generalisations, we have at once the following result:—

THEOREM.—*If the Fourier series of $\phi(u)$ converge in the ordinary way at the point $u = 0$, then the Fourier series of $f(x)$ converges when summed in the Cesàro way (index 1). More generally, if that of $\phi(u)$ converges in the Cesàro way (index r), that of $f(x)$ converges certainly in the Cesàro way [index $(r+1)$]. Moreover, the sums are the same.*

If we then make use of de la Vallée Poussin's theorem and remark that $\frac{F(x+t)-F(x-t)}{t}$ is an even function, we get, by repeated applications, the theorem :—

THEOREM.—*The Fourier series of $f(x)$ will converge at the point x considered when summed in the Cesàro way (index p) if the $p+1$-fold repeated integral*

$$\phi_{p+1}(u) = \tfrac{1}{2}\int_0^1\int_0^1\dots\int_0^1 \{f(x+ut_1t_2\dots t_{p+1})+f(x-ut_1t_2\dots t_{p+1})\}\,dt_1dt_2\dots dt_{p+1}$$

has bounded variation in an interval containing $u=0$. Moreover, it will converge to the limit of this integral as $u\to 0$.

In particular, in the case $p=1$, we may replace the integral by

$$\frac{1}{2u}\int_0^u \frac{F(x+t)-F(x-t)}{t}\,dt.$$

If instead of using de la Vallée Poussin's test we use the theorem of § 3, we see that

The Fourier series of $f(x)$ will converge at the point x considered when summed in the Cesàro way, index p, if the p-fold repeated integral

$$\phi_p(u) = \tfrac{1}{2}\int_0^1\int_0^1\dots\int_0^1 \{f(x+ut_1t_2\dots t_p)+f(x-ut_1t_2\dots t_p)\}\,dt_1dt_2\dots dt_p$$

is a continuous function of u, or more generally, has the property that

$$\int_0^u |\,\phi_p(t)-\phi_p(0)\,|\,dt$$

has at the point $u=0$ a differential coefficient, which is zero. Moreover, it will converge to $\operatorname*{Lt}_{u=0}\frac{1}{u}\int_0^u \phi_p(u)\,du$.

9. We can now deduce Lebesgue's theorem* that *if at the point $u=0$, $\int_0^u |f(x+u)+f(x-u)-2f(x)|\,du$ has a differential coefficient which is zero, then the Fourier series of $f(x)$ converges when summed in the Cesàro way (index unity) to $f(x)$.*

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* Lebesgue's proof of this result is given on pp. 80, 81, 82, 83, 84, 89, 90 of his paper entitled "Sur les Intégrales singulières," *Fac. de T.*, 3e S., I, 1910; and in a slightly different form in the *Math. Ann.*, Bd. LXI, 1905, p. 274.

In fact in this case $\phi(u) = \frac{1}{2u}\int^{u} g(u)\,du$,

where $\int_0^u |g(u)-g(0)|\,du$ has zero for differential coefficient at the origin. Hence, by the first theorem of § 3, the Fourier series of $\phi(u)$ converges at the point $u = 0$. This proves the theorem.

We can also generalise another of Lebesgue's theorems, and prove the following result:—

THEOREM.—*At a point at which $f(x)$ is the differential coefficient of its integral, or more generally at a point at which the indefinite integral of $f(x)$ has a finite differential coefficient, or finally at a point at which $\{F(x+t)-F(x-t)\}/t$ has a unique limit as t approaches zero, the Fourier series of $f(x)$ will converge when summed in any Cesàro manner, index integral or fractional greater than unity,* moreover its sum is*

$$\operatorname*{Lt}_{t=0} \{F(x+t)-F(x-t)\}/2t.$$

This follows from $(\mathrm{II}^{(r)})$ coupled with Marcel Riesz' theorem that the Fourier series of a continuous function converges to the function when summed in any Cesàro way, whether the index be fractional or not.

10. The question naturally presents itself as to the necessity of the sufficient conditions of § 6 and of § 8. We have this result:—

A necessary condition for the convergence of the Fourier series of $f(x)$ at the point x is either that the Fourier series of

$$\frac{1}{\sin u}\int_{-u}^{u} f(x+2t)\,dt = g(u)$$

should be such that the coefficients of its derived series converge towards a finite limit, or that the Fourier series of $g(u)$ should not at $u=0$ be summable by a Cesàro method of any index.

For suppose the Fourier series of $f(x)$ to converge,

$$4s_n = S_n - nA_n,$$

and

$$4t_n = 2T_n - S_n,$$

$$8u_n = 4U_n - 2T_n,$$

$$\ldots, \qquad \ldots .$$

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* See footnote, p. 264 In Lebesgue's statement the Cesàro order is 2.

Adding up these equations to any finite number of terms, and bearing in mind that the left-hand side of the equation so obtained has a definite unique limit as n increases, it follows that the same is true of the new right-hand side. Hence if some Cesàro method gives a convergence of the Fourier series of $g(u)$ at $u = 0$, nA_n must have a unique limit.

In just the same way we shew that *a necessary condition for the convergence of the Fourier series, when summed in the Cesàro way, index unity, is either that the Fourier series of* $g(u)$ *should converge in the ordinary way, or else it should not converge for any Cesàro integral index of summation.*

11. From the former theorem of § 9 we deduce the following consequence :—

THEOREM.—*The necessary and sufficient condition that the Fourier series of* $f(x)$ *should converge at a point at which*

$$\int_0^u |f(x+t)+f(x-t)-2f(x)|\,dt$$

has a differential coefficient which is zero, is that the coefficients of the derived series of the Fourier series of $\phi(u)$ *should converge to a unique limit.*

In fact $\phi(u)$ is a continuous function of u at the point $u = 0$ for such a value of x. Hence T_n has a finite limit; also, as we have seen in § 9, t_n has a finite limit. But from equations (I) and (II) of § 3, we have

$$4s_n+4t_n = 2T_n-nA_n.$$

Hence s_n and nA_n have either of them a unique limit if the other has.

12. So far we have confined our attention to the Fourier series of $f(x)$. It will now be shewn briefly that similar results hold for the allied series of the Fourier series of $f(x)$ obtained by interchanging the coefficients of the sine and cosine terms, and then changing the sign of the sine terms.

We have now to consider the function

$$\psi(u) = \frac{1}{2u}\int_0^u \{f(x+2t)-f(x-2t)\}\,dt,$$

or, more naturally, the function

$$h(u) = \{F(x+2u)+F(x-2u)-2F(x)\}\operatorname{cosec} u,$$

We proceed as in § 4. Since

$$h(\pi+u) = -h(u),$$

the coefficients of the even terms of the Fourier series of this odd function are all zero, and we write accordingly

$$h(u) \sim \sum_{n=1}^{\infty} B_n \sin(2n-1)u,$$

where $$B_{n+1}-B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(u)\{\sin(2n+1)u-\sin(2n-1)u\}\,du$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} h(u)\sin u \cos 2nu\,du$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} \{F(x+2u)+F(x-2u)-2F(x)\}\cos 2nu\,du$$

$$= \frac{4}{n}\{b_n \cos nx - a_n \sin nx\},$$

since $F(x+2u)+F(x-2u)-2F(x)$

$$= 2\sum_{n=1}^{\infty}\{a_n \sin nx - b_n \cos nx\}(1-\cos 2nu)/n.$$

Hence, summing $r(B_{r+1}-B_r)$ for all values of r from 1 to $n-1$, and denoting by s_n the partial summation $\sum_{r=1}^{n-1}(b_n \cos nx - a_n \sin nx)$, and by S_n the partial summation $\sum_{r=1}^{n} B_r$, we have

$$4s_n = S_n - nB_n, \tag{III}$$

corresponding exactly to (I) of § 4. Hence the equation (II) of § 4 follows, with the meaning of the present article.

Again,

$$\psi(u)-\tfrac{1}{4}h(u) = \tfrac{1}{4}\{F(x+2u)+F(x-2u)-2F(x)\}\left\{\frac{1}{u}-\frac{1}{\sin u}\right\},$$

so that, as in § 5, $\psi(u)$ is equal to $\frac{1}{4}h(u)$ plus an integral. Hence, the letters referring now to the function ψ instead of $h(u)$, we have the equation

$$s_n = S_n - nB_n + p_n, \tag{IIIψ}$$

where p_n has a unique finite limit which is, however, not zero, as it was in

§ 4, since the allied series of the Fourier series of $\psi(u)-\frac{1}{4}h(u)$ converges at $u=0$, but not in general to zero; the sum is, in fact,

$$\frac{1}{\pi}\int_0^\pi \{\psi(u)-\tfrac{1}{4}h(u)\}\cot\tfrac{1}{2}u\,.\,du.$$

From (III) we deduce, as in § 6, the following theorem and corollary:—

THEOREM.—*If* $\chi(u)=\dfrac{1}{2\sin u}\displaystyle\int_0^u \{f(x+2t)-f(x-2t)\}\,dt$ *be a summable function of u, having the property that the coefficients of the derived series of its Fourier series converge to zero, then, if it is known that either the allied series of the Fourier series of* $f(u)$, *or that of* $\chi(u)$, *for* $u=0$, *converges in any Cesàro manner, it follows that both series converge in the ordinary manner, and have the same sum.*

COR.—*If the coefficients* nB_n *converge to some finite limit other than zero, the above is still true, except that the two allied series have not the same sum, the difference being* $\operatorname*{Lt}\limits_{n\to\infty} nB_n$.

If, instead of using the equation (III), we use the equation (IIIψ), we have a theorem more closely analogous to the theorem of § 6, but having the disadvantage, as compared with that just stated, that the sums of the two series compared are no longer the same.

13. From equation (III) we also obtain, in a manner precisely similar to that which we have already employed in proving de la Vallée Poussin's theorem, the corresponding theorem for the allied series, already stated in § 7. We also obtain without difficulty results for the allied series analogous to those obtained in §§ 8 *et seq.* for the Fourier series. In particular we have the following theorem, precisely analogous to that of Lebesgue, stated and proved at the beginning of § 9, viz.,

THEOREM.—*If at the point* $u=0$, $\displaystyle\int_0^u |f(x+u)-f(x-u)|\,du$ *has a differential coefficient which is zero, then the allied series of the Fourier series of* $f(x)$* *when summed in the Cesàro way, index unity, converges to*

$$\operatorname*{Lt}_{\epsilon=0}\int_\epsilon^\infty \frac{f(x+u)-f(x-u)}{u}\,du,$$

provided only this limit exists.

Write $g(u)$ for $\{f(x+2u)-f(x-2u)\}$. Then evidently $g(u)$ is a function which satisfies the conditions of the second of the theorems of § 3. Hence we know that the allied series of the Fourier series of $\psi(u)$ converges at $u = 0$ to

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* $f(x)$ must be supposed made periodic in the usual way.

$$\underset{e=0}{\mathrm{Lt}}\ \frac{1}{\pi}\int_e^\pi \psi(u)\cot\tfrac{1}{2}u\,du.$$

Using the result deduced from (IIIψ), it follows that the allied series of the Fourier series of $\tfrac{1}{4}h(u)$, and accordingly that of $f(x)$, converges to

$$\underset{e=0}{\mathrm{Lt}}\ \frac{1}{\pi}\int_e^\pi \tfrac{1}{4}h(u)\cot\tfrac{1}{2}u\,du,$$

provided this limit exists. Writing for $h(u)$ its value, and substituting for $\sin u$ the expression $2\sin\tfrac{1}{2}u\cos\tfrac{1}{2}u$, this last expression becomes

$$\underset{e=0}{\mathrm{Lt}}\ \frac{1}{8\pi}\int_e^\pi \operatorname{cosec}^2\tfrac{1}{2}u\ \{F(x+2u)+F(x-2u)-2F(x)\}\,du,$$

which, by a simple trigonometrical transformation, and a subsequent change of the independent variable from $2u$ to u, assumes the form

$$\underset{e=0}{\mathrm{Lt}}\int_e^\infty u^{-2}\,\{F(x+u)+F(x-u)-2F(x)\}\,du.$$

Remarking finally that

$$\{F(x+u)+F(x-u)-2F(x)\}/u$$

has the unique limit zero as u approaches zero, since the existence of a zero differential coefficient at $u = 0$ of $\int_0^u g(u)\,du$ follows from the existence of a zero differential coefficient at $u = 0$ of $\int_0^u |g(u)|\,du$, and making use of the method of integration by parts, we at once obtain the required result.*

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[*] [*Note added September* 23*rd*, 1911.—In the paper as originally presented the theorem just proved was not explicitly stated. For the sake of brevity it was merely pointed out that, in consequence of the parallelism of the theorems proved in §§ 2 and 3 for odd and even functions respectively, results for the allied series existed which corresponded precisely to those for Fourier series, and this fact was regarded as sufficiently illustrated by the theorem proved in § 12 and that stated in § 7. The subject appears, however, to gain considerably in interest by the explicit formulation of the theorem of the present article. I should like also to call the attention of my readers to the results of a paper presented in the interim to the königliche baierische Akademie der Wissenschaften. In this paper, which has already been published, I shew *inter alios* that, if $\psi(u)$, in § 12 *supra*, has bounded variation, then the allied series of the Fourier series of $f(x)$ converges to the same sum as that obtained in the present article, provided only the limit representing the sum exists.]

ON PROPERTIES OF CERTAIN LINEAR HOMOGENEOUS SUBSTITUTIONS

By HAROLD HILTON.

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1. Suppose m, k are given positive integers such that $m \geqslant k \geqslant 0$; and i, j are positive integers both $\leqslant m$. Then ϵ_{ij} will denote $+1$ if i, j are both $> k$ or both $\leqslant k$; and -1 if one of i and j is $> k$ and the other $\leqslant k$.

We prove readily $\quad \epsilon_{ih}\epsilon_{jh} = \epsilon_{ij} = \epsilon_{ji}$.

Let M be the linear homogeneous substitution of degree m,

$$x_i' = \epsilon_{ik} x_i \quad (i = 1, 2, \dots, m).$$

Let A be the substitution

$$x_i' = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m \quad (i = 1, 2, \dots, m),$$

let A' be the transposed substitution

$$x_i' = a_{1i}x_1 + a_{2i}x_2 + \dots + a_{mi}x_m.$$

and $\bar{A}$ the conjugate substitution

$$x_i' = \bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 + \dots + \bar{a}_{im}x_m.^*$$

When $a_{ij} = \epsilon_{ij}a_{ji}$, A will be called *quasi-symmetric*. As a particular case, if $k = m$, A is "symmetric."

When $a_{ij} = \epsilon_{ij}\bar{a}_{ji}$, A will be called *quasi-Hermitian*.

If $$\sum_{t=1}^{m} \epsilon_{tk} x_t^2 \equiv x_1^2 + x_2^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_m^2$$

is an invariant of A, A will be called *quasi-orthogonal*.† We readily prove that in this case

$$\sum_{t=1}^{m} \epsilon_{tk} a_{ti}^2 = \epsilon_{ik}, \qquad \sum_{t=1}^{m} \epsilon_{tk} a_{ti} a_{tj} = 0 \quad (i \neq j), \tag{a}$$

to which

$$\sum_{t=1}^{m} \epsilon_{tk} a_{it}^2 = \epsilon_{ik}, \qquad \sum_{t=1}^{m} \epsilon_{tk} a_{it} a_{jt} = 0 \quad (i \neq j) \tag{a'}$$

are algebraically equivalent.

If s is any integer, A^s and A^{-s} are quasi-orthogonal. The product of any number of quasi-orthogonal substitutions is quasi-orthogonal.

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* $\bar{a}$ denotes the complex quantity conjugate to a.

† When $k = m$, A will be orthogonal.

We have, using (α), $AMA' = M$.* From this we readily deduce that A' is quasi-orthogonal.†

Again, if $A^2 = 1$, $A' = M^{-1}A^{-1}M = MAM$, and therefore $a_{ij} = \epsilon_{ij}a_{ji}$.* Hence A is quasi-symmetric as well as quasi-orthogonal, and conversely.

If $$\sum_{t=1}^{m} \epsilon_{tk} x_t \bar{x}_t \equiv x_1\bar{x}_1 + \ldots + x_k\bar{x}_k - x_{k+1}\bar{x}_{k+1} - \ldots - x_m\bar{x}_m$$

is an invariant of A, A will be called *quasi-unitary*.‡ We readily prove that in this case

$$\sum_{t=1}^{m} \epsilon_{tk} a_{ti}\bar{a}_{ti} = \epsilon_{ik}, \qquad \sum_{t=1}^{m} \epsilon_{tk} a_{ti}\bar{a}_{tj} = 0 \quad (i \neq j), \qquad (\beta)$$

to which

$$\sum_{t=1}^{m} \epsilon_{tk} a_{it}\bar{a}_{it} = \epsilon_{ik}, \qquad \sum_{t=1}^{m} \epsilon_{tk} a_{it}\bar{a}_{jt} = 0 \quad (i \neq j) \qquad (\beta')$$

are algebraically equivalent.

If s is any integer, A^s and A^{-s} are quasi-unitary. The product of any number of quasi-unitary substitutions is quasi-unitary.

We have also, using (β), $AM\bar{A}' = M$.* From this we readily deduce that $\bar{A}'$ is quasi-unitary§; and that, if $A^2 = 1$, A is quasi-Hermitian as well as quasi-unitary, and conversely.

2. Suppose that A^{-1} is the substitution

$$x_i' = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m \quad (i = 1, 2, \ldots, m),$$

and that $A^{-1}BA = C$; where A is quasi-orthogonal and B is quasi-symmetric.

Now $AMA' = M$ gives

$$A^{-1} = MA'M^{-1} \quad \text{or} \quad A^{-1} = MA'M.$$

Therefore $a_{\tau j} = \epsilon_{\tau j}a_{j\tau}$,* while $b_{\sigma\tau} = \epsilon_{\sigma\tau}b_{\tau\sigma}$. Hence

$$\begin{aligned} \epsilon_{ji}c_{ji} &= \sum_{\sigma,\tau} \epsilon_{ji}a_{j\sigma}b_{\sigma\tau}a_{\tau i} = \sum_{\sigma,\tau} \epsilon_{\sigma\tau}\epsilon_{\tau i}\epsilon_{ji}b_{\tau\sigma}a_{j\sigma}a_{i\tau} \\ &= \sum_{\sigma,\tau} \epsilon_{\sigma j}b_{\tau\sigma}a_{j\sigma}a_{i\tau} = \sum_{\sigma,\tau} \epsilon_{\tau j}b_{\sigma\tau}a_{j\tau}a_{i\sigma} \\ &= \sum_{\sigma,\tau} a_{i\sigma}b_{\sigma\tau}a_{\tau j} = c_{ij}. \end{aligned}$$

Therefore C is quasi-symmetric, *i.e.*, "The transform of a quasi-symmetric substitution by a quasi-orthogonal substitution is quasi-symmetric."

Similarly, "The transform of a quasi-Hermitian substitution by a quasi-unitary substitution is quasi-Hermitian."

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* We remember that the formula giving the product S of three substitutions P, Q, R ($S = PQR$) is, with an obvious notation, $s_{ij} = \Sigma r_{i\sigma}q_{\sigma\tau}p_{\tau j}$ ($\sigma, \tau = 1, 2, \ldots, m$).

† This also follows from equations (α′).

‡ When $k = m$, A will be unitary.

§ This also follows from equations (β′).

3. We can find a quasi-orthogonal substitution of order 2 changing x_1 into $a_1x_1+a_2x_2+\ldots+a_mx_m$, where $a_1, a_2, \ldots, a_m$ are assigned constants, such that

$$\sum_{t=1}^{m} \epsilon_{tk} a_t^2 \equiv a_1^2+\ldots+a_k^2-a_{k+1}^2-\ldots-a_m^2 = 1.$$

In fact, the substitution whose matrix* is

$$\left|\begin{array}{ccccccc}
a_1 & a_2 & \cdots & a_k & a_{k+1} & \cdots & a_m \\
a_2 & \dfrac{a_2^2}{a_1\pm1}\mp1 & \cdots & \dfrac{a_2a_k}{a_1\pm1} & \dfrac{a_2a_{k+1}}{a_1\pm1} & \cdots & \dfrac{a_2a_m}{a_1\pm1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_k & \dfrac{a_ka_2}{a_1\pm1} & \cdots & \dfrac{a_k^2}{a_1\pm1}\mp1 & \dfrac{a_ka_{k+1}}{a_1\pm1} & \cdots & \dfrac{a_ka_m}{a_1\pm1} \\
-a_{k+1} & -\dfrac{a_{k+1}a_2}{a_1\pm1} & \cdots & -\dfrac{a_{k+1}a_k}{a_1\pm1} & -\dfrac{a_{k+1}^2}{a_1\pm1}\mp1 & \cdots & -\dfrac{a_{k+1}a_m}{a_m\pm1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-a_m & -\dfrac{a_ma_2}{a_1\pm1} & \cdots & -\dfrac{a_ma_k}{a_1\pm1} & -\dfrac{a_ma_{k+1}}{a_1\pm1} & \cdots & -\dfrac{a_m^2}{a_m\pm1}\mp1
\end{array}\right|$$

has this property.

For it is easily verified that it satisfies equations (α), and therefore is quasi-orthogonal. But it is obviously quasi-symmetric, and therefore, by § 1, it is of order 2.

We also readily verify that (taking the upper signs) it has a pole $(a_1+1, a_2, a_3, \ldots, a_k, -a_{k+1}, \ldots, -a_m)$ corresponding to the root $\lambda = 1$ of its characteristic equation, and ∞^{m-2} poles

$$\left(-\frac{(a_2X_2+a_3X_3+\ldots+a_mX_m)}{a_1+1},\ X_2,\ X_3,\ \ldots,\ X_m\right)$$

corresponding to the $(m-1)$-ple root $\lambda = -1$.† Hence, when it is transformed into the canonical form, it becomes

$$x_1' = +x_1, \quad x_2' = -x_2, \quad \ldots, \quad x_m = -x_m;$$

which gives another proof that this substitution is of order 2.

Similarly, we can find a quasi-unitary substitution of order 2 changing x_1 into $a_1x_1+a_2x_2+\ldots+a_mx_m$, where $a_1, a_2, \ldots, a_m$ are assigned constants such that a_1 is real and $\sum_{t=1}^{m} \epsilon_{tk} a_t \bar{a}_t = 1$, namely, the substitution with matrix*

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* Either all the upper or all the lower signs being taken. This matrix was suggested to me by the formulæ which transform the equation of a conicoid when we rotate it through 180° about the line bisecting the angle between an axis of the conicoid and an axis of reference (cf. Hilton's *Mathematical Crystallography*, p. 81).

† For the notation throughout, see the third chapter of Hilton's *Theory of Groups of Finite Order*. This book will be referred to as "F.G."

$$\left|\begin{array}{ccccccc}
a_1 & a_2 & \cdot\,\cdot & a_k & a_{k+1} & \cdots & a_m \\
\bar{a}_2 & \frac{\bar{a}_2 a_2}{a_1 \pm 1} \mp 1 & \cdots & \frac{\bar{a}_2 a_k}{a_1 \pm 1} & \frac{\bar{a}_2 a_{k+1}}{a_1 \pm 1} & \cdots & \frac{\bar{a}_2 a_m}{a_1 \pm 1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bar{a}_k & \frac{\bar{a}_k a_2}{a_1 \pm 1} & \cdots & \frac{\bar{a}_k a_k}{a_1 \pm 1} \mp 1 & \frac{\bar{a}_k a_{k+1}}{a_1 \pm 1} & \cdots & \frac{\bar{a}_k a_m}{a_1 \pm 1} \\
-\bar{a}_{k+1} & -\frac{\bar{a}_{k+1} a_2}{a_1 \pm 1} & \cdots & -\frac{\bar{a}_{k+1} a_k}{a_1 \pm 1} & -\frac{\bar{a}_{k+1} a_{k+1}}{a_1 \pm 1} \mp 1 & \cdots & -\frac{\bar{a}_{k+1} a_m}{a_1 \pm 1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\bar{a}_m & -\frac{\bar{a}_m a_2}{a_1 \pm 1} & \cdots & -\frac{\bar{a}_m a_k}{a_1 \pm 1} & -\frac{\bar{a}_m a_{k+1}}{a_1 \pm 1} & \cdots & -\frac{\bar{a}_m a_m}{a_1 \pm 1} \mp 1
\end{array}\right|$$

Also, we can find a quasi-unitary substitution of order 2 changing x_m into $a_1 x_1 + a_2 x_2 + \ldots + a_m x_m$, where $a_1, a_2, \ldots, a_m$ are assigned constants such that a_m is real and $\sum_{t=1}^{m} \epsilon_{tk} a_t \bar{a}_t = -1$, namely, the substitution with matrix*

$$\left|\begin{array}{ccccccc}
-\frac{\bar{a}_1 a_1}{a_m \pm 1} \mp 1 & \cdots & -\frac{\bar{a}_1 a_k}{a_m \pm 1} & -\frac{\bar{a}_1 a_{k+1}}{a_m \pm 1} & \cdots & -\frac{\bar{a}_1 a_{m-1}}{a_m \pm 1} & -\bar{a}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\frac{\bar{a}_k a_1}{a_m \pm 1} & \cdots & -\frac{\bar{a}_k a_k}{a_m \pm 1} \mp 1 & -\frac{\bar{a}_k a_{k+1}}{a_m \pm 1} & \cdots & -\frac{\bar{a}_k a_{m-1}}{a_m \pm 1} & -\bar{a}_k \\
\frac{\bar{a}_{k+1} a_1}{a_m \pm 1} & \cdots & \frac{\bar{a}_{k+1} a_k}{a_m \pm 1} & \frac{\bar{a}_{k+1} a_{k+1}}{a_m \pm 1} \mp 1 & \cdots & \frac{\bar{a}_{k+1} a_{m-1}}{a_m \pm 1} & \bar{a}_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\bar{a}_{m-1} a_1}{a_m \pm 1} & \cdots & \frac{\bar{a}_{m-1} a_k}{a_m \pm 1} & \frac{\bar{a}_{m-1} a_{k+1}}{a_m \pm 1} & \cdots & \frac{\bar{a}_{m-1} a_{m-1}}{a_m \pm 1} \mp 1 & \bar{a}_{m-1} \\
a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{m-1} & a_m
\end{array}\right|$$

4. The main object of this paper is to call attention to a method of transforming a quasi-symmetric substitution A into a simpler form by means of a quasi-orthogonal substitution.

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* See note *, p. 275.

We shall see that it is possible to transform A by a quasi-orthogonal substitution into the form

$$x_1' = \lambda_1 x_1, \quad x_2' = \lambda_2 x_2, \quad \ldots, \quad x_r' = \lambda_r x_r,$$
$$x_i' = s_{i r+1} x_{r+1} + s_{i r+2} x_{r+2} + \ldots + s_{im} x_m \quad (i = r+1, r+2, \ldots, m),$$

where the characteristic determinant of

$$x_i' = s_{i\,r+1} x_{r+1} + s_{i\,r+2} x_{r+2} + \ldots + s_{im} x_m \quad (i = r+1, r+2, \ldots, m)$$

has no simple invariant-factor.

We shall frequently use the following theorem proved in *F.G.*, Ch. III, § 6 :—

"If $(X_1, X_2, \ldots, X_m)$ is a pole of a substitution Q, and $(Y_1, Y_2, \ldots, Y_m)$ is the corresponding pole of $P^{-1}QP$,

$$Y_t = p_{t1} X_1 + p_{t2} X_2 + \ldots + p_{tm} X_m \quad (t = 1, 2, \ldots, m). \qquad (\gamma)$$

5. First suppose that A is transformable into a multiplication. Then we can choose m poles of A,

$$(X_{11}, X_{21}, \ldots, X_{m1}), \quad (X_{12}, X_{22}, \ldots, X_{m2}), \quad \ldots, \quad (X_{1m}, X_{2m}, \ldots. X_{mm}),$$

such that, if T^{-1} is the substitution

$$x_i' = X_{i1} x_1 + X_{i2} x_2 + \ldots + X_{im} x_m \quad (i = 1, 2, \ldots, m),$$

T is a quasi-orthogonal substitution transforming A into a multiplication.

For, suppose $P^{-1}NP = A$, where N is the multiplication

$$x_i' = \lambda_i x_i \quad (i = 1, 2, \ldots, m),$$

and let $\qquad c_{ji} = c_{ij} = \epsilon_{1k} p_{1i} p_{1j} + \epsilon_{2k} p_{2i} p_{2j} + \ldots + \epsilon_{mk} p_{mi} p_{mj}.$

Denoting by $|c_{ij}|$ the determinant, with c_{ij} as the element in the i-th row and j-th column, we have $|c_{ij}| = |p_{ji}| \times |\epsilon_{jk} p_{ji}|$ by the ordinary rule for the multiplication of determinants. Therefore, since

$$\pm |p_{ji}| = |\epsilon_{jk} p_{ji}| \neq 0, \qquad |c_{ij}| \neq 0.$$

Now $NP = PA$ gives for each value of i

$$\left.\begin{array}{l} \lambda_i p_{1i} = a_{11} p_{1i} + a_{12} p_{2i} + \ldots + a_{1m} p_{mi} \\ \lambda_i p_{2i} = a_{21} p_{1i} + a_{22} p_{2i} + \ldots + a_{2m} p_{mi} \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ \lambda_i p_{mi} = a_{m1} p_{1i} + a_{m2} p_{2i} + \ldots + a_{mm} p_{mi} \end{array}\right\}. \qquad (\delta)$$

Multiply these equations by $\epsilon_{1k} p_{1j}, \epsilon_{2k} p_{2j}, \ldots, \epsilon_{mk} p_{mj}$ respectively and add.

Then
$$\lambda_i c_{ij} = \sum_{t=1}^{m} p_{ti}(\epsilon_{1k} a_{1t} p_{1j} + \epsilon_{2k} a_{2t} p_{2j} + \ldots + \epsilon_{mk} a_{mt} p_{mj})$$
$$= \sum_{t=1}^{m} \epsilon_{tk} p_{ti}(a_{t1} p_{1j} + a_{t2} p_{2j} + \ldots + a_{tm} p_{mj}) \quad (\text{since } a_{rt} = \epsilon_{rt} a_{tr})$$
$$= \lambda_j c_{ij}$$

[writing j for i in (δ)].

Hence $c_{ij} = 0$ when $\lambda_i \neq \lambda_j$.

By equations (δ), $(p_{1i}, p_{2i}, \ldots, p_{mi})$ is a pole of A corresponding to the root λ_i of its characteristic equation.

Hence, if the poles $(X_{1i}, X_{2i}, \ldots, X_{mi})$ and $(X_{1j}, X_{2j}, \ldots, X_{mj})$ correspond to unequal roots of the characteristic equation of A,

$$\sum_{t=1}^{m} \epsilon_{tk} X_{ti} X_{tj} = 0. \tag{ϵ}$$

Now suppose that $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_t \quad (t = 4, 5, \ldots, m)$:—

the reasoning is general. Then

$$c_{t1} = c_{t2} = c_{t3} = 0 \quad (t = 4, 5, \ldots, m).$$

Therefore, since $|c_{ij}| \neq 0$,
$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0.$$

The poles of N corresponding to the equal roots λ_1, λ_2, λ_3 of its characteristic equation are $(\xi, \eta, \zeta, 0, 0, \ldots, 0)$, where ξ, η, ζ may have any values not all zero. The corresponding poles of A are, by (γ), $(Z_1, Z_2, \ldots, Z_m)$, where
$$Z_t = p_{t1}\xi + p_{t2}\eta + p_{t3}\zeta \quad (t = 1, 2, \ldots, m).$$

Now $\sum_{t=1}^{m} \epsilon_{tk} Z_t^2 = c_{11}\xi^2 + c_{22}\eta^2 + c_{33}\zeta^2 + 2c_{23}\eta\zeta + 2c_{31}\zeta\xi + 2c_{12}\xi\eta.$

Since
$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0,$$

we can find independent linear homogeneous functions ξ', η', ζ' of ξ, η, ζ such that
$$\sum_{t=1}^{m} \epsilon_{tk} Z_t^2 = \epsilon_{1k}\xi'^2 + \epsilon_{2k}\eta'^2 + \epsilon_{3k}\zeta'^2\,;$$

and then Z_t takes the form $X_{t1}\xi' + X_{t2}\eta' + X_{t3}\zeta'$ where

$$\left.\begin{array}{lll} \sum_{t=1}^{m} \epsilon_{tk} X_{t1}^2 = \epsilon_{1k}, & \sum_{t=1}^{m} \epsilon_{tk} X_{t2}^2 = \epsilon_{2k}, & \sum_{t=1}^{m} \epsilon_{tk} X_{t3}^2 = \epsilon_{3k} \\ \sum_{t=1}^{m} \epsilon_{tk} X_{t2} X_{t3} = 0, & \sum_{t=1}^{m} \epsilon_{tk} X_{t3} X_{t1} = 0, & \sum_{t=1}^{m} \epsilon_{tk} X_{t1} X_{t2} = 0 \end{array}\right\}. \qquad (\zeta)$$

We see then that, corresponding to the three equal roots λ_1, λ_2, λ_3 of the characteristic equation of A, we can find three poles

$$(X_{11}, X_{21}, \ldots, X_{m1}), \quad (X_{12}, X_{22}, \ldots, X_{m2}), \quad (X_{13}, X_{23}, \ldots, X_{m3})$$

satisfying (ζ).

We deal similarly with $\lambda_4, \lambda_5, \ldots, \lambda_m$.

Combining (ϵ) and (ζ), we see that the substitution T^{-1} formed as described at the beginning of this section is quasi-orthogonal; and we readily verify $TN = AT$.

The determinant $|X_{ij}| \neq 0$; for the determinant Δ formed by the corresponding poles of $N \neq 0$ and by (γ),

$$|X_{ij}| = \Delta \times |p_{ij}|.$$

6. We now return to the more general theorem of § 4. It is sufficient to show that, if the characteristic determinant of A has a simple invariant-factor, A can be transformed by a quasi-orthogonal substitution into the form

$$x_1' = \lambda_1 x_1, \quad x_t' = b_{t2}x_2 + b_{t3}x_3 + \ldots + b_{tm}x_m \quad (t = 2, 3, \ldots, m), \qquad (\eta)$$

for the complete result is then at once established by induction.

We first assume that a pole $(X_1, X_2, \ldots, X_m)$ of A exists such that

$$\sum_{t=1}^{m} \epsilon_{tk} X_t^2 \equiv X_1^2 + \ldots + X_k^2 - X_{k+1}^2 - \ldots - X_m^2 \neq 0. \qquad (\theta)$$

Then, since we are only concerned with the ratios of $X_1, X_2, \ldots, X_m$, we may suppose

$$\sum_{t=1}^{m} \epsilon_{tk} X_t^2 = 1.$$

Let P be the first substitution of § 3 in which a_t is taken as $\epsilon_{tk} X_t$, and let $B = P^{-1}AP$ be the substitution

$$x_i' = b_{i1}x_1 + b_{i2}x_2 + \ldots + b_{im}x_m \quad (i = 1, 2, \ldots, m).$$

Since P is quasi-orthogonal, it follows from (a') and (γ) that $P^{-1}AP$ has $(1, 0, 0, \ldots, 0)$ as a pole, and hence

$$b_{21} = b_{31} = \ldots = b_{m1} = 0.$$

But, by § 2, B is quasi-symmetric. Hence

$$b_{12} = b_{13} = \ldots = b_{1m} = 0,$$

and B takes the form (η).

7. The quasi-symmetric substitution A cannot always be transformed into the form (η);* though this will always be possible by the method of § 6 if A has one pole satisfying (θ). We show now that, conversely, if A can be transformed into the form (η), A has one pole satisfying (θ).

Let A be transformed by P^{-1} into the canonical form N, which is

$$x_i' = \lambda_i x_i + \beta_i x_{i+1} \quad (i = 1, 2, \ldots, m);$$

where β_i is 1 or 0, and is certainly 0 when $i = m$ or $\lambda_i \neq \lambda_{i+1}$ (see, for instance, *Messenger of Mathematics*, 1909, p. 24).

As in § 5, we have (if $\beta_0 = 0$)

$$\lambda_i p_{ti} + \beta_{i-1} p_{t\,i-1} = a_{t1} p_{1i} + a_{t2} p_{2i} + \ldots + a_{tm} p_{mi} \quad (i, t = 1, 2, \ldots, m);$$

and thence

$$\lambda_i c_{ij} + \beta_{i-1} c_{i-1\,j} = \lambda_j c_{ij} + \beta_{j-1} c_{i\,j-1} \quad (i, j = 1, 2, \ldots, m),$$

and in particular if $\beta_{i-1} = 0$

$$(\lambda_i - \lambda_1) c_{i1} = 0, \quad (\lambda_i - \lambda_2) c_{i2} = \beta_1 c_{i1}, \quad (\lambda_i - \lambda_3) c_{i3} = \beta_2 c_{i2},$$
$$\ldots, \quad (\lambda_i - \lambda_m) c_{im} = \beta_{m-1} c_{i\,m-1}.$$

Hence, when $\beta_{i-1} = 0$,

$$\left.\begin{array}{ll} & \text{if } \lambda_i \neq \lambda_j, \quad c_{ij} = 0, \\ \text{while} & \text{if } \lambda_i = \lambda_j, \quad c_{i\,j-1} = 0, \quad \text{whenever } \beta_{j-1} \neq 0 \end{array}\right\}. \qquad (\iota)$$

From this we may readily show that, when $\beta_1 = 0$, so that A is transformable into the form (η), at least one pole of A satisfies (θ).

Suppose, for instance (the reasoning is general), that N is

$$(ax_1,\ ax_2 + x_3,\ ax_3 + x_4,\ ax_4,\ ax_5 + x_6,\ ax_6,\ ax_7 + x_8,\ ax_8,\ \beta x_9 + x_{10},\ \beta x_{10}).$$

$(\xi, \eta, 0, 0, \zeta, 0, \theta, 0, 0, 0)$ are poles of N for all values of the ratios $\xi : \eta : \zeta : \theta$; and by (γ) the corresponding poles of A are

$$(p_{11}\xi + p_{12}\eta + p_{15}\zeta + p_{17}\theta,\ p_{21}\xi + p_{22}\eta + p_{25}\zeta + p_{27}\theta,\ \ldots,\ \ldots).$$

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* For example, $x' = ax - \frac{1}{2}(a-b)\,y$, $y' = \frac{1}{2}(a-b)\,x + by$.

One of these satisfies (θ), provided c_{11}, c_{22}, c_{55}, c_{77}, c_{12}, c_{15}, c_{17}, c_{25}, c_{27}, c_{57} are not all zero. But, by (ι), $|c_{ij}|$ is

$$\begin{vmatrix}
c_{11} & 0 & 0 & c_{14} & 0 & c_{16} & 0 & c_{18} & 0 & 0 \\
0 & 0 & 0 & c_{24} & 0 & c_{26} & 0 & c_{28} & 0 & 0 \\
0 & 0 & c_{33} & c_{34} & 0 & c_{36} & 0 & c_{38} & c_{39} & c_{8\,10} \\
c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & c_{49} & c_{4\,10} \\
0 & 0 & 0 & c_{54} & 0 & c_{56} & 0 & c_{58} & 0 & 0 \\
c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} & c_{69} & c_{6\,10} \\
0 & 0 & 0 & c_{74} & 0 & c_{76} & 0 & c_{78} & 0 & 0 \\
c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} & c_{89} & c_{8\,10} \\
0 & 0 & c_{93} & c_{94} & 0 & c_{96} & 0 & c_{98} & c_{99} & c_{9\,10} \\
0 & 0 & c_{10\,3} & c_{10\,4} & 0 & c_{10\,6} & 0 & c_{10\,8} & c_{10\,9} & c_{10\,10}
\end{vmatrix}.$$

Form a determinant of the third order by taking the elements in any three columns and in the first, second, and fifth rows of $|c_{ij}|$, and multiply this determinant by the complementary determinant. Owing to the existence of the zeros in the first, second, and fifth, and seventh rows, this product will always be zero if $c_{11} = 0$. But this is impossible, for $|c_{ij}| \neq 0$. Hence $c_{11} \neq 0$, and one pole of A satisfies (θ).

8. We notice that, if *every* pole of any substitution Q satisfies (θ), the same is true of $P^{-1}QP$, where P is quasi-orthogonal.

For, by (γ), $$\sum_{t=1}^{m} \epsilon_{tk} Y_t^2 = \sum_{t=1}^{m} \epsilon_{tk} X_t^2.$$

Hence, if every pole of the quasi-symmetric substitution A satisfies (θ), the same is true of the substitution (η). But, if (Y_1, Y_2, ..., Y_m) is a pole of (η), so is (0, Y_2, ..., Y_m). Hence

$$\sum_{t=2}^{m} \epsilon_{tk} Y_t^2 \neq 0;$$

and it readily follows by induction that A is transformable by means of a quasi-orthogonal substitution into a multiplication. As an example of such a substitution we may take that in which $a_{ij} \times \sqrt{\epsilon_{ij}} = a_{ji} \div \sqrt{\epsilon_{ij}}$ is *real.**

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* In particular, a real symmetric substitution is transformable into a multiplication by a real orthogonal substitution. I take this opportunity of correcting an error in F.G., Ch. III, § 6, Ex. 24, and § 8, Ex. 11. For "symmetric substitution" read "real symmetric substitution."

In this case every root of the characteristic equation of A is real;* and therefore the ratios $X_1 : \dots : X_k : \sqrt{-1}\, X_{k+1} : \dots : \sqrt{-1}\, X_m$ are real if $(X_1, X_2, \dots, X_m)$ is any pole of A. Hence

$$\sum_{t=1}^{m} \epsilon_{tk} X_t^2 \neq 0.$$

9. We now point out the analogous results for a quasi-Hermitian substitution A.

If $$\sum_{t=1}^{m} \epsilon_{tk} X_t \bar{X}_t \equiv X_1 \bar{X}_1 + \dots + X_k \bar{X}_k - X_{k+1} \bar{X}_{k+1} - \dots - X_m \bar{X}_m \neq 0 \qquad (\kappa)$$

holds for the pole $(X_1, X_2, \dots, X_m)$ of A, the corresponding root λ of the characteristic equation of A is real.

For $\sum_{i,j} \epsilon_{ik} a_{ij} \bar{X}_i X_j = \sum_i \epsilon_{ik} \bar{X}_i (a_{i1} X_1 + a_{i2} X_2 + \dots + a_{im} X_m) = \lambda . \sum_i \epsilon_{ik} X_i \bar{X}_i.$

But $$\sum_i \epsilon_{ik} X_i \bar{X}_i,$$

and $$\sum_{i,j} (\epsilon_{ik} a_{ij} \bar{X}_i X_j + \epsilon_{jk} a_{ji} \bar{X}_j X_i) = \sum_{i,j} \epsilon_{ik} (a_{ij} \bar{X}_i X_j + \bar{a}_{ij} \bar{X}_j X_i)$$

are both real. Therefore λ is real when these quantities are not both zero.*

It may therefore be possible to transform a quasi-Hermitian substitution into a multiplication, even though (κ) is not satisfied by any pole.†

10. If $A = P^{-1}NP$, we have, as in § 7,

$$\lambda_i c_{ij} + \beta_{i-1} c_{i-1\,j} = \bar{\lambda}_j c_{ij} + \beta_{j-1} c_{i\,j-1},$$

where $$c_{ij} = \bar{c}_{ji} = \sum_{t=1}^{m} \epsilon_{tk} p_{ti} \bar{p}_{tj}.$$

Hence, as in § 7, we see that if λ_1 is a *real* root of the characteristic equation of A with a corresponding simple invariant-factor, there is a corresponding pole $(X_1, X_2, \dots, X_m)$ satisfying (κ); and thence that A can be transformed by means of a quasi-unitary substitution into the form

$$x_1' = \lambda_1 x_1, \; \dots, \; x_e' = \lambda_e x_e, \; x_i' = s_{i\,e+1} x_{e+1} + \dots + s_{ir} x_r \quad (i = e+1, \dots, r),$$
$$x_{r+1}' = \lambda_{r+1} x_{r+1}, \; \dots, \; x_m' = \lambda_m x_m,$$

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* In particular, if $k = m$, λ is certainly real: a well-known result.

† Consider, for example, $x' = ax + hy$, $y' = -\bar{h}x + by$, where $a - b$ is real and $4h\bar{h} > (a-b)^2$, in which $X\bar{X} - Y\bar{Y} = 0$ for both poles.

where $\lambda_1, \ldots, \lambda_e, \lambda_{r+1}, \ldots, \lambda_m$ are *real*,[*] and no simple invariant-factor corresponds to a *real* root of the characteristic equation of

$$x_i' = s_{i\,e+1}\,x_{e+1} + \ldots + s_{ir}\,x_r \quad (i = e+1, \ldots, r).$$

A result similar to that of § 5 is easily proved for a quasi-Hermitian substitution transformable into a multiplication with *real* coefficients.

Every Hermitian substitution (in which $k = m$) is included under this category, for every pole satisfies (κ) when $k = m$. Since the above was written Dr. Bromwich has kindly pointed out to me that this property of a Hermitian substitution is implied in a result established by him in *Proc. London Math. Soc.*, Vol. XXXII (1900), p. 349, and is also readily deducible from Muth's *Elementartheiler*, p. 180.

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[*] If $\lambda_1, \ldots, \lambda_e, \lambda_{r+1}, \ldots, \lambda_m$ are *not* all real, this substitution is not quasi-Hermitian; and therefore, by § 2, A cannot be transformed into it by means of a quasi-unitary substitution.

THE DETERMINATION OF ALL GROUPS OF RATIONAL LINEAR SUBSTITUTIONS OF FINITE ORDER WHICH CONTAIN THE SYMMETRIC GROUP IN THE VARIABLES

By W. BURNSIDE.

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THE symmetric group of n variables consists of the totality of the permutations of the variables. That, for all values of n, there are greater groups of linear substitutions on the variables, of finite order, which contain the symmetric group as a sub-group, is quite obvious both in the general case and when the coefficients are restricted to be rational numbers. With the latter restriction, the substitutions will be called rational substitutions.

I have shewn (*Proc. London Math. Soc.*, Series 2, Vol. 7, pp. 8–11) that a group of rational linear substitutions of finite order can always be transformed into a group of rational integral substitutions in which the coefficients are rational integers.

In a memoir with the title "Zur Theorie der positiven quadratischen Formen" (*Crelle's Journal*, Vol. CI, pp. 196–202), the late Prof. Minkowski shews* that the order of a group of rational integral substitutions on n variables is equal to or is a factor of

$$\prod_q q^{[n/(q-1)] + [n/q(q-1)] + [n/q^2(q-1)] + \dots},$$

where $[a/b]$ denotes the greatest positive integer contained in the fraction a/b, and is therefore zero if a is less than b. The product is extended to all primes, but clearly only primes which are equal to or less than $n+1$ give effective terms in the product. Taken in conjunction with the previous result, Minkowski's theorem holds for groups of rational linear substitutions of finite order. It is quoted here as use is made of it in dealing with the orders of certain special groups which come under consideration.

The investigations of this paper were suggested by the known representation (§ 13) of the group of the twenty-seven lines on a cubic surface as a group of rational linear substitutions on six variables. This group, suitably transformed, contains the symmetric group in the variables. It has a self-conjugate sub-group of index 2, which is a simple group, and its order is $2^7 . 3^4 . 5$. It was anticipated that groups of rational linear substitutions would exist for larger values of n analogous to this known group for the case $n = 6$. With two exceptions, this proves not to be the case; and it is here shewn that, in general, a group of rational linear substitutions of finite order, containing the symmetric group, is one of a limited number of comparatively familiar types. There are, however, two exceedingly interesting exceptions. One is a group of linear substitutions on seven symbols, of order $2^{10}.3^4.5.7$, which gives a representation of the group of the twenty-eight bitangents to a quartic. The other, of order $2^{14}.3^5.5^2.7$, is on eight symbols, and gives a representation of Jordan's hypo-Abelian group (case $2n = 8$). Neither of these two groups of rational linear substitutions of relatively high order have hitherto been recognized.

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* I have given an independent proof of this result in my *Theory of Groups*, 2nd edition, pp. 482–484.

The table of § 15, which gives in explicit form the substitutions of the last mentioned group of eight symbols, has been indirectly checked in a very curious way. For this I owe many thanks to Mr. Grace. In his memoir on "Systems of Circles, Spheres, and Linear Complexes" (*Camb. Phil. Soc. Trans.*, Vol. XVI, pp. 153–190), Mr. Grace considers systems of spheres which intersect six by six in common points. I had noticed that the tactical problem underlying this geometrical configuration was identical with that of the sets of products of six symbols formed from twenty-seven which occurs in the group on six variables. At my request, Mr. Grace undertook the formidable geometrical problem of dealing with systems of spheres which intersect eight by eight in common points. The scheme which he arrived at for the complete representation of such a system gives the check referred to.

1. A group of finite order of linear substitutions with real coefficients on $x_1, x_2, \ldots, x_n$, must necessarily leave invariant some definite quadratic form in the x's. In fact, if

$$s_2 = x_1^2+x_2^2+\ldots+x_n^2,$$

and, if the N substitutions of the group change s_2 into $s_2^{(i)}$ $(i = 1, 2, \ldots, N)$, then $\sum\limits_i s_2^{(i)}$ is such a definite form.

A limited converse* of this statement is that, if a group of linear substitutions on the x's with rational integral coefficients leaves a definite quadratic form invariant, the group must be one of finite order. In fact, if f is the form and m any assigned positive number, there are only a finite number of integral sets of values of the x's satisfying the condition

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* See *Theory of Groups*, 2nd edition, pp. 493–495, for a more general converse.

$$f < m,$$

and these sets of values are permuted by the substitutions of the group. Moreover, by taking m sufficiently large, it may clearly be ensured that the only substitution which leaves all the sets of values unchanged is the identical substitution. The group is therefore, as a group of permutations, necessarily of finite order.

If

$$s_1 = x_1 + x_2 + \dots + x_n,$$

the only quadratic forms which are invariant for all permutations of the x's are

$$s_2 + \lambda s_1^2,$$

where λ is a constant; and this is a definite form only when

$$\lambda > -\frac{1}{n}.$$

A group of linear substitutions of finite order on the x's with real coefficients, which contains the symmetric group of the x's as a sub-group, has therefore necessarily an invariant quadratic form

$$s_2 + \lambda s_1^2,$$

where λ is greater than $-1/n$.

2. Denote by G_n the symmetric group of permutations of the x's, and by H a group of linear substitutions on the x's which contains G_n and whose coefficients are rational numbers. G_n has substitutions whose multipliers are $n-1$ 1's and one -1; in fact, the transposition $(x_1 x_2)$ is such a substitution. If the substitutions of G_n with these multipliers constitute a conjugate set in H, then G_n is a self-conjugate sub-group of H. Now, except when $n = 6$, G_n is a complete group. When $n = 6$, G_n admits an outer isomorphism which permutes the two conjugate sets whose substitutions are of order 3. But when the symmetric group of degree 6 is represented by the permutations of six symbols, these two conjugate sets have different characteristics, viz., 0 and 3. Hence, even when $n = 6$, any substitution of H gives an inner isomorphism of G_n. It follows* that H must be the direct product of G_n and another group of finite order.

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* *Theory of Groups*, 2nd edition, p. 94.

Now the only linear substitutions which are permutable with every permutation of the x's are of the form

$$x_i' = ax_i + \beta s_1 \quad (i = 1, 2, \ldots, n).$$

This gives

$$s_1' = (a + n\beta) s_1 ;$$

and a, β being rational numbers, this can only be an operation of finite order if

$$a + n\beta = \pm 1$$

and

$$a \qquad = \pm 1.$$

Hence there are just four substitutions of finite order of the required form, viz.,

$$\left.\begin{array}{l} x' = x_i \\ x_i' = -x_i \\ x_i' = x_i - \dfrac{2}{n}\, s_1 \\ x_1' = -x_i + \dfrac{2}{n}\, s_1 \end{array}\right\} \quad (i = 1, 2, \ldots, n).$$

These constitute a group of order 4, with three sub-groups of order 2.

If, then, G_n is a self-conjugate sub-group of H, the order of H is either $n!$, $2.n!$, or $4.n!$. There is one type for orders $n!$ and $4.n!$, and there are three distinct types for order $2.n!$.

3. Suppose, next, that H contains substitutions whose multipliers are $n-1$ 1's and one -1, which do not belong to G_n, and are conjugate to a transposition. Let T be such a substitution of H, and let

$$\gamma = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

be the one linear function of the x's whose sign is reversed by T. Then the form of T must be

$$x_i' = x_i + k_i \gamma \quad (i = 1, 2, \ldots, n).$$

Since

$$\gamma' = \; -\gamma,$$

$$0 = 2 + \sum_i k_i c_i,$$

and, when this relation holds among the constants, T is actually a substitution of order 2 with the assigned multipliers.

Since H is, by supposition, a group of finite order containing G_n, it must have an invariant quadratic form $s_2 + \lambda s_1^2$.

Now, for T,

$$s_1' = s_1 + \; \gamma \sum_i k_i,$$

$$s_2' = s_2 + 2\gamma \sum_i k_i x_i + \gamma^2 \sum_i k_i^2.$$

Hence $$0 \equiv 2\lambda\gamma\, \Sigma\, k_i x_i + \gamma^2 \sum_i k_i^2 + 2\gamma s_1 \sum_i k_i + \lambda\gamma^2 (\sum_i k_i)^2,$$

or, since, by supposition, γ is not identically zero,

$$\gamma = Ps_1 + Q\, \Sigma\, k_i x_i,$$

where P and Q are constant coefficients. Hence, if

$$\sum_i k_i x_i = a,$$

T is $$x_i' = x_i + k_i (Ps_1 + Qa) \quad (i = 1, 2, \ldots, n),$$

and, since T reverses the sign of $Ps_1 + Qa$,

$$0 = 2 + Pl_1 + Ql_2,$$

where $$l_1 = \sum_i k_i, \qquad l_2 = \sum_i k_i^2.$$

Let S be any substitution of G_n, *i.e.*, any permutation of the x's, and write $$S^{-1}TS = T'.$$

Then T' will be derived from T by carrying out of the k's such a permutation K that when K and S are simultaneously effected on the k's and the x's, $$k_1 x_1 + k_2 x_2 + \ldots + k_n x_n$$

is unchanged. Moreover, by suitably choosing S, K may be *any* permutation $$\begin{pmatrix} k_1, & k_2, & \ldots, & k_n \\ k_1', & k_2', & \ldots, & k_n' \end{pmatrix}$$

of the k's.

Then, denoting $k_1' x_1 + k_2' x_2 + \ldots + k_n' x_n$ by β, the substitution T' is

$$x_i' = x_i + k_i' (Ps_1 + Q\beta).$$

If the only possible value of a were s_1, T would be permutable with every substitution of G_n, and we should fall back on the case of the previous paragraph. We may therefore assume that a is not s_1, and also that S is chosen so that β is not a.

Then the three linear functions s_1, a, β are transformed among themselves by T and T'. In fact,

$$\begin{array}{rl} T \text{ gives} & s_1' = (1+Pl_1)\, s_1 + Ql_1\, a, \\ & a' = Pl_2\, s_1 + (1+Ql_2)\, a, \\ & \beta' = P\mu\, s_1 + Q\mu a + \beta, \end{array}$$

where $$\mu = \sum_i k_i k_i';$$

$$\begin{array}{rl} \text{and } T' \text{ gives} & s_1' = (1+Pl_1)\, s_1 + Ql_1 \beta, \\ & a' = P\mu\, s_1 + a + Q\mu\beta, \\ & \beta' = Pl_2\, s_1 + (1+Ql_2)\beta, \end{array}$$

On calculation it is found that, so far as s_1, α, and β are concerned, TT' is

$$\begin{aligned} s_1' &= [(1+Pl_1)^2+\mu l_1 PQ]\,s_1 + Ql_1\,\alpha + Ql_1[1+Pl_1+Q\mu]\,\beta, \\ \alpha' &= [(1+Pl_1)Pl_2+P\mu(1+Ql_2)]\,s_1+(1+Ql_2)\,\alpha + [Q\mu(1+Ql_2)+PQl_1l_2]\,\beta, \\ \beta' &= P[Q\mu^2+\mu(1+Pl_1)+l_2]\,s_1 + Q\mu\,\alpha+[Q^2\mu^2+PQ\mu l_1+1+Ql_2]\,\beta. \end{aligned}$$

The characteristic χ of TT', then, is given by

$$(Q\mu+Pl_1)^2-1 = \chi,$$

so that

$$Q\mu+Pl_1 = \pm\sqrt{1+\chi}$$

Now, by supposition, TT' is a substitution of finite order, and all the symbols occurring in the coefficients are rational numbers. Hence χ can only be -1, 0, or 3. Moreover, if χ were 3, T and T', so far as they affect s_1, α, and β, would be the same substitution, and β would be the same as α, which is not the case. Hence χ is either -1 or 0, and

$$Q\mu+Pl_1 = -1,\ 0, \text{ or } +1.$$

This is true whatever permutations of the k's is denoted by K; and therefore whatever permutation

$$\begin{pmatrix} k_1, & k_2, & \ldots, & k_n \\ k_1', & k_2', & \ldots, & k_n' \end{pmatrix}$$

may be, $\sum\limits_i k_i k_i'$ is susceptible, at most, of four values, including $\sum\limits_i k_i^2$.

The cases $n=2$ and $n=3$ are easily dealt with. If $n>3$, there is no difficulty in verifying that, when three k's are distinct, $\sum\limits_i k_i k_i'$ always takes more than four values. If r k's are k_1 and $n-r$ are k_2 $(r \leqslant n-r)$, $\sum\limits_i k_i k_i'$ takes $r+1$ distinct values. Hence, when $n>3$, the only possible forms of α are

(i) $k_1x_1+k_2(x_2+x_3+x_4+\ldots+x_n)$,

(ii) $k_1(x_1+x_2)+k_2(x_3+x_4+\ldots+x_n)$,

(iii) $k_1(x_1+x_2+x_3)+k_2(x_4+\ldots+x_n)$.

Further, if $n<6$, the first two of these are the only possible distinct forms forms for α.

4. Since every substitution of G_n is permutable with the substitution

$$X_i = x_i+ms_1 \quad (i=1, 2, \ldots, n),$$

the form of the substitution T may be further modified without altering the form of G_n.

Writing $$S_1 = \sum_i X_i$$
and $$A = \sum_i k_i X_i,$$
then $$S_1 = (1+nm)s_1$$
and $$A = a+ml_1 s_1.$$

Hence, with the X's as variables, T becomes

$$\begin{aligned} X_i' &= x_i'+ms_1', \\ &= x_i+[m(1+Pl_1)+Pk_i]s_1+Q(ml_1+k_i)a, \\ &= X_i+(ml_1+k_i)\left(\frac{P-Qml_1}{1+nm}S_1+QA\right), \\ &= X_i+(ml_1+k_i)\left(-\frac{2+Q(ml_1^2+l_2)}{l_1(1+nm)}S_1+QA\right). \end{aligned}$$

Hence, if $$ml_1+k_2 = 0,$$ $$Q(k_1-k_2)^2 = q,$$

then, when a is of the form (i), T is

$$\begin{aligned} X_1' &= (1+q)X_1-(2+q)S_1, \\ X_i' &= X_i \qquad (i > 1); \end{aligned}$$

when a is of the form (ii), T is

$$\begin{aligned} X_1' &= X_1+q(X_1+X_2)-(1+q)S_1, \\ X_2' &= X_2+q(X_1+X_2)-(1+q)S_1, \\ X_i' &= X_i \qquad (i > 2); \end{aligned}$$

and, when a is of the form (iii), T is

$$\begin{aligned} X_1' &= X_1+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_2' &= X_2+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_3' &= X_3+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_i' &= X_i \qquad (i > 3). \end{aligned}$$

Now, if G_n and T belong to a group of finite order, the latter must leave invariant a definite quadratic form

$$S_2+\lambda S_1^2.$$

A simple calculation shews that, accordingly as T is of the first, second, or third of the above forms, λ is

$$\text{(i)}\quad -\frac{q+2}{q}, \qquad \text{(ii)}\quad -\frac{q+1}{2q}, \qquad \text{(iii)}\quad -\frac{3q+2}{9q}.$$

When T is

$$\begin{aligned} X_1' &= (1+q)X_1-(2+q)S_1, \\ X_i' &= X_i \qquad (i>1), \end{aligned}$$

T', the result of transforming T by (X_1X_2), is

$$\begin{aligned} X_2' &= (1+q)X_2-(2+q)S_1, \\ X_i' &= X_i \qquad (i\neq 2), \end{aligned}$$

and TT' is

$$\begin{aligned} X_1' &= (q+1)(q+3)X_1+(q+2)X_2+(q+1)(q+2)(X_3+\dots+X_n), \\ X_2' &= \quad -(q+2)X_1- \quad X_2- \quad (q+2)(X_3+\dots+X_n), \\ X_i' &= X_i \qquad (i>2). \end{aligned}$$

This is a substitution of finite order, with rational coefficients, when q is -1, -2, or -3, and in no other case.

If q is -3, then λ is $-\frac{1}{3}$, and the quadratic form is not definite (n being assumed greater than 3).

Hence, when a is of the first form, the variables may be chosen so that, G_n being the symmetric group in them, T is either

$$X_1' = -X_1, \qquad X_i' = X_i \quad (i>1),$$

or

$$X_1' = -S_1, \qquad X_i' = X_i \quad (i>1).$$

Moreover, the corresponding invariant forms are S_2 and $S_2+S_1^2$.

When T is

$$\begin{aligned} X_1' &= X_1+q(X_1+X_2)-(1+q)S_1, \\ X_2' &= X_2+q(X_1+X_2)-(1+q)S_1. \\ X_i &= X_i \qquad (i>2), \end{aligned}$$

T'', the result of transforming T by (X_2X_3), is

$$\begin{aligned} X_1' &= X_1+q(X_1+X_3)-(1+q)S_1, \\ X_2' &= X_2, \\ X_3' &= X_3+q(X_1+X_3)-(1+q)S_1, \\ X_i' &= X_i \qquad (i>3). \end{aligned}$$

The substitution TT'' is

$$\begin{aligned} X_1' &= (q+1)X_1+ \quad q(q+2)X_2+0\,.\,X_3+\dots, \\ X_2' &= (q+1)X_1+(q+1)(q+2)X_2+ \quad X_3+\dots, \\ X_3' &= \quad -X_1- \quad (q+1)X_2+0\,.\,X_3+\dots, \\ X_i' &= X_i \qquad (i>3). \end{aligned}$$

It follows from a consideration of the characteristic that this cannot be a substitution of finite order with rational coefficients unless q is either -1, -2, or -3. The values -2 and -3 of q make λ respectively $-\frac{1}{4}$ and $-\frac{1}{3}$, so that (n being assumed greater than 3) the quadratic form is not definite.

Hence, when a is of the second form, the variables may be so chosen that, G_n being the symmetric group in them, T becomes

$$X_1' = -X_2, \quad X_2' = -X_1, \quad X_i' = X_i \quad (i > 2),$$

and the corresponding definite quadratic form is S_2.

Lastly, when T is

$$\begin{aligned} X_1' &= X_1+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_2' &= X_2+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_3' &= X_3+q(X_1+X_2+X_3)-(\tfrac{2}{3}+q)S_1, \\ X_i' &= X_i \qquad\qquad\qquad\qquad\qquad (i > 3), \end{aligned}$$

it may be shewn exactly as in the preceding cases by a consideration of the substitution TT''', where T''' is the result of transforming T by (X_3X_4), that q again must be either -1, -2, or -3. These values of q give for λ the values $-\frac{1}{9}$, $-\frac{2}{9}$, $-\frac{7}{27}$. For the last value the form is indefinite when $n > 3$; the second gives a definite form when $n = 4$ and the first when $n < 9$. For values of n greater than 8, then T cannot be of the last form; and since, when $n < 6$, the form of a is either

$$k_1x_1+k_2(x_2+\ldots+x_n) \quad \text{or} \quad k_1(x_1+x_2)+k_2(x_3+\ldots+x_n),$$

T can be of this last form only when n is either 6, 7, or 8.

($n = 2$) *and* ($n = 3$).

5. For the sake of completeness the results in the cases $n = 2$ and $n = 3$ will be stated. The verification is quite simple and may be omitted.

A group of rational substitutions on x_1, x_2 of finite order, which contains the transposition (x_1x_2) may be brought to one of the following forms, of which the generating substitutions and orders are given:—

(i) (x_1x_2); $N = 2$.

(ii) (x_1x_2); $x_1' = -x_1, \quad x_2' = -x_2$; $N = 4$.

(iii) (x_1x_2); $x_1' = -x_1-x_2, \quad x_2' = x_2$; $N = 6$.

(iv) (x_1x_2); $x_1' = x_2, \quad x_2' = -x_1$; $N = 8$.

(v) (x_1x_2); $x_1 = -x_1-x_2, \quad x_2' = x_2$; $x_1' = -x_1, \quad x_2' = -x_2$; $N = 12$.

The greatest possible order, given by Minkowski's theorem, is 24; it is not attained.

A group of rational substitutions on x_1, x_2, x_3 of finite order, which contains G_3, the symmetric group in the variables, may be brought to one of the following forms, of which the generating substitutions and orders are given.

(i) G_3; $\qquad N = 6.$

(ii) G_3; $x_1' = -x_1,\ x_2' = -x_2,\ x_3' = -x_3$; $\qquad N = 12.$

(iii) G_3; $x_1' = x_1 - \frac{2}{3}s_1,\ x_2' = x_2 - \frac{2}{3}s_2,\ x_3' = x_3 - \frac{2}{3}s_3$; $\qquad N = 12.$

(iv) G_3; $x_1' = -x_1 + \frac{2}{3}s_1,\ x_2' = -x_2 + \frac{2}{3}s_1,\ x_3' = -x_3 + \frac{2}{3}s_1$; $\qquad N = 12.$

(v) G_3; $\left.\begin{array}{l} x_1' = -x_1,\ x_2' = -x_2,\ x_3 = -x_3; \\ x_1' = x_1 - \frac{2}{3}s_1,\ x_2' = x_2 - \frac{2}{3}s_1,\ x_3' = x_3 - \frac{2}{3}s_1; \end{array}\right\}$ $\qquad N = 24.$

(vi) G_3; $x_1' = -s_1,\ x_2' = x_2,\ x_3' = x_3$; $\qquad N = 24.$

(vii) G_3; $x_1' = -x_1,\ x_2' = x_2,\ x_3' = x_3$; $\qquad N = 48.$

In this case the group generated by

$$\begin{array}{l} G_3;\ x_1' = -s_1,\ x_2' = x_2,\quad x_3' = x_3; \\ x_1' = -x_1,\ x_2' = -x_2,\ x_3' = -x_3; \end{array}$$

is not, in fact, distinct from type (vii), though for larger values of n the two corresponding types are distinct.

The greatest possible order, given by Minkowski's theorem, is 48, and it is attained.

6. Leaving now for later consideration the exceptional cases in which the linear function, whose sign is reversed by T, is of the form

$$k_1(x_1+x_2+x_3)+k_2(x_4+\ldots+x_n),$$

which can only occur when n is 6, 7, or 8, the general result may be summarized as follows.

A group of linear substitutions H, of finite order, on $x_1, x_2, \ldots, x_n$ with rational coefficients, which contains the symmetric group G_n of the n variables as a sub-group which is not self-conjugate, can always be so transformed (the coefficients remaining rational) as to contain either

(i) a sub-group H' generated by

$$G_n \text{ and } x_1' = -x_1,\quad x_i' = x_i \qquad (i > 1);$$

or (ii) a sub-group H'' generated by

$$G_n \text{ and } x_1' = -x_2,\quad x_2' = -x_1,\quad x_i = x_i \quad (i > 2);$$

or (iii) a sub-group H''' generated by

$$G_n \text{ and } x_1' = -s_1,\quad x_i' = x_i \qquad (i > 1).$$

Suppose, first, that H contains the sub-group H', and that U is a substitution of H with multipliers $(1, 1, \ldots, 1, -1)$ which does not belong to H'.

H' contains the substitution T_2, or

$$x_1 = x_1, \quad x_2' = -x_2, \quad x_i' = x_i \quad (i > 2).$$

If $k_1x_1+k_2(x_2+\ldots+x_n)$ is the linear function whose sign is reversed by U, then

$$k_1x_1-k_2x_2+k_2(x_3+\ldots+x_n)$$

is the linear function whose sign is reversed by T_2UT_2.

Unless two of the three numbers k_1, k_2, $-k_2$ are equal, no group containing T_2UT_2 can be of finite order. Hence, either

$$k_2 = 0 \quad \text{or} \quad k_2 = -k_1.$$

If $k_2 = 0$, U is $\quad x_1' = -x_1, \quad x_i' = x_i+a_ix_1 \quad (i > 1).$

When this is combined with T or

$$x_1' = -x_1 \quad x_i' = x_i \qquad (i > 1),$$

the resulting substitution is

$$x_1' = x_1, \quad x_i' = x_i+a_ix_1 \qquad (i > 1);$$

and this is not a substitution of finite order unless

$$a_2 = a_3 = \ldots = a_n = 0,$$

in which case U is the same as T.

If $k_2 = -k_1$, and if T_3 is

$$x_1' = x_1, \quad x_2' = -x_2, \quad x_3' = -x_3, \quad x_i' = x_i \quad (i > 3),$$

the linear function whose sign is reversed by T_3UT_3 is

$$k_1(x_1+x_2+x_3)-k_1(x_4+\ldots+x_n).$$

When $n \geqslant 6$, this comes under the exceptional case which has been excluded. It is therefore only necessary to examine the cases $n = 4$ and $n = 5$.

Since $\quad -x_1+x_2+\ldots+x_n$

is the linear function whose sign is reversed by U, that whose sign is reversed by TUT is s_1, and therefore TUT is

$$x_i' = x_i+a_is_1 \quad (i = 1, 2, \ldots, n).$$

Now s_2 is invariant for H, and therefore

$$0 = 2s_1\Sigma a_ix_i+s_1^2\Sigma a_i^2,$$

giving

$$a_i = -\frac{2}{n}.$$

The substitution UT is

$$x_1' = -x_1 + \frac{2}{n}\, s_1, \quad x_i' = x_i - \frac{2}{n}\, s_1 \quad (i > 2),$$

and its characteristic is $\qquad (n-2)\left(1-\dfrac{2}{n}\right).$

If $n = 5$, the substitution UT is not of finite order, and the exception cannot occur.

If $n = 4$, UT is a substitution of finite order, and it will be seen later that H' may be contained in a greater group of finite order.

Suppose, next, that $k_1(x_1+x_2)+k_2(x_3+\ldots+x_n)$ is the linear function whose sign is reversed by U. Then it may be shewn, exactly as in the preceding case, that, unless k_1+k_2 is zero, inadmissible substitutions arise; while, if k_1+k_2 is zero, the group again contains a substitution with multipliers $(1, 1, \ldots, 1, -1)$ which reverses the sign of s_1. The only possible exception is again one for the case $n = 4$.

7. Suppose, next, that H contains the sub-group H''. It is evident that H'' is contained in H', and that H' has substitutions with multipliers $(1, 1, \ldots, 1, -1)$ not contained in H''. If H does not contain H', it may again be shewn, as above, that H'' contains all the substitutions of H with multipliers $(1, 1, \ldots, 1, -1)$, unless $n = 4$; and that the only then possible exception is when H has a substitution that reverses the sign of s_1.

8. For the third case, when H contains H''', there is a substitution S, or

$$x_1' = x_1, \quad x_2' = -s_1, \quad x_i' = x_i \quad (i > 2);$$

and, if $k_1x_1+k_2(x_2+\ldots+x_n)$ is the linear function whose sign is reversed by U, the corresponding one for SUS is

$$(k_1-k_2)x_1 - k_2x_2.$$

This again gives a group whose order is not finite unless either k_1 or k_2 is zero. If k_2 is zero, U is

$$x_1' = -x_1, \quad x_2' = x_2 + a_2x_1, \quad \ldots, \quad x_n' = x_n + a_nx_1;$$

and, if $s_2+s_1^2$ is invariant for this substitution,

$$a_2 = \ldots = a_n = \frac{2}{n},$$

But the substitution thus determined when combined with G_n immediately leads to substitutions which are not of finite order. If k_1 is zero, the condition that $s_2+s_1^2$ is invariant gives for U

$$x_1' = s_1, \qquad x_i' = x_i - \frac{2}{n-1}(s_1-x_1) \quad (i>1);$$

and the substitution that results on combining this with (x_1x_2) is not of finite order. There can therefore be no substitution such as U contained in H and not in H'''. Precisely similar reasoning applies if the linear function, whose sign is reversed by U, is of the form

$$k_1(x_1+x_2)+k_2(x_3+\ldots+x_n).$$

9. The result of this discussion, then, shews that, with possible exceptions when n is 4, 6, 7, or 8, every substitution of the group H of § 6 whose multipliers are $(1, 1, \ldots, 1, -1)$ is contained either in a sub-group H' or H'' or H'''. Moreover, each of these sub-groups is clearly generated by its substitutions having the specified multipliers; and therefore H has either H', H'', or H''' as a self-conjugate sub-group.

Now H' obviously contains a self-conjugate Abelian sub-group of order 2^n whose substitutions merely change the signs of the n symbols. Every group of linear substitutions which contains H' self-conjugately must clearly (since H' is irreducible) contain the Abelian sub-group self-conjugately. It must, therefore, be a group of monomial substitutions on the n symbols; and hence, if its coefficients are rational, it must coincide with H'. The second group H'' contains a self-conjugate Abelian sub-group of order 2^{n-1}, and is itself a self-conjugate sub-group of H' of index 2. Any group of linear substitutions on the n symbols which contains H'' self-conjugately must again be a group of monomial substitutions, and, if the coefficients are rational, it must coincide with or be contained in H'. The third group H''', generated by

$$G_u \quad \text{and} \quad x_1' = -s_1, \quad x_i' = x_i \quad (i>1)$$

is the ordinary representation of the symmetric group of $n+1$ symbols as a group of linear substitutions of n symbols (in that form in which the characteristic of a transposition is $n-2$ and *not* $-n+2$). As pointed out above, if H''' is contained self-conjugately in a greater group, this greater group must be a direct product.

Now, since H''' is irreducible, the only linear substitution with rational coefficients which is permutable with every substitution of H''' is

$$x_i' = -x_i \quad (i = 1, 2, \ldots, n);$$

and the only group of linear substitutions with rational coefficients which contains H''' is the direct product of H''' and the group of order 2 generated by the preceding substitution.

10. The results thus arrived at may be stated as follows.

A group of finite order of linear substitutions on $x_1, x_2, \ldots, x_n$ with rational coefficients, which contains G_n the symmetric group of permutations of the n variables, may in general be transformed into one of the following groups, of which the generating substitutions and orders are given:—

$$
\begin{array}{lllll}
\text{(i)} & G_n; & & & N = n!. \\
\text{(ii)} & G_n; & x_i' = -x_i & (i = 1, 2, \ldots, n); & N = 2.n! \\
\text{(iii)} & G_n; & x_i' = x_i - \dfrac{2}{n}s_1 & (i = 1, 2, \ldots, n); & N = 2.n! \\
\text{(iv)} & G_n; & x_i' = -x_i + \dfrac{2}{n}s_1 & (i = 1, 2, \ldots, n); & N = 2.n! \\
\text{(v)} & G_n; & \left.\begin{array}{l} x_i' = -x_i \\ x_i' = x_i - \dfrac{2}{n}s_1 \end{array}\right\} & (i = 1, 2, \ldots, n); & N = 4.n! \\
\text{(vi)} & G_n; & x_1' = -s_1, \quad x_i' = x_i & (i > 1); & N = (n+1)! \\
\text{(vii)} & G_n; & \begin{array}{l} x_1' = -s_1, \quad x_i' = x_i \\ x_i' = -x_i \end{array} & \left.\begin{array}{r} (i > 1); \\ (i = 1, 2, \ldots, n); \end{array}\right\} & N = 2(n+1)! \\
\text{(viii)} & G_n; & x_1' = -x_1, \quad x_2' = -x_2, \quad x_i' = x_i & (i > 2); & N = 2^{n-1}.n! \\
\text{(ix)} & G_n; & x_1' = -x_1, \quad x_i' = x_i & (i > 1); & N = 2^n.n!
\end{array}
$$

where $s_1 = x_1 + x_2 + \ldots + x_n$.

There are exceptions for the values 4, 6, 7, 8 of n and for no others.

$(n = 4)$.

11. I now proceed to consider the exceptional cases.

When $n = 4$, it has been shewn that a group of rational linear substitutions in four variables, which contains G_4 and does not occur in the preceding list, can be transformed so as to contain either

(i) the group H_1 generated by

$$
\begin{array}{ll}
G_4, \quad x_1' = -x_1, \quad x_i' = x_i & (i > 1), \\
 x_i' = x_i - \tfrac{1}{2}s_1 & (i = 1, 2, 3, 4);
\end{array}
$$

or (ii) the group H_2 generated by

$$
\begin{array}{ll}
G_4, \quad x_1' = -x_1, \quad x_2' = -x_2, \quad x_i' = x_i & (i > 2), \\
 x_i' = x_i - \tfrac{1}{2}s_1 & (i = 1, 2, 3, 4).
\end{array}
$$

The latter group is clearly contained in the former, which may be considered first.

That H_1 is, in fact, a group of finite order may be shewn as follows. Take new variables defined by

$$x_1 = 2y_1 - y_2, \qquad x_2 = y_2 - y_3,$$
$$x_3 = y_3 - y_4, \qquad x_4 = y_4.$$

The generating substitutions of G_4 are $(x_1 x_2)$ and $(x_1 x_2 x_3 x_4)$. With the y's as variables, the four generating substitutions of H_1 are found to be

$$(x_1 x_2) \sim \quad y_1' = y_1, \quad y_2' = 2y_1 - y_2 + y_2, \quad y_3' = y_3, \quad y_4' = y_4;$$

$$(x_1 x_2 x_3 x_4) \sim \begin{cases} y_1' = y_1, & y_2' = 2y_1 - y_2 + y_3, \\ y_3' = 2y_1 - y_2 + y_4, & y_4' = 2y_1 - y_2; \end{cases}$$

$$\left.\begin{aligned} x_1' &= -x_1 \\ x_i' &= x_i \end{aligned}\right\} \sim \quad y_1' = -y_1 + y_2, \quad y_2' = y_2, \quad y_3' = y_3, \quad y_4' = y_4;$$

$$x_i' = x_i - \tfrac{1}{2}s_1 \sim \begin{cases} y_1' = -y_1, & y_2' = -3y_1 + y_2, \\ y_3' = -2y_1 + y_3, & y_4' = -y_1 + y_4. \end{cases}$$

With the y's as variables, the coefficients of the substitutions of H_1 are therefore integers. Moreover, since s_2 is invariant for H_1 with the x's as variables, the definite form

$$2y_1^2 - 2y_1 y_2 + y_2^2 - y_2 y_3 + y_3^2 - y_3 y_4 + y_4^2$$

is invariant with the y's as variables. The group H_1 is therefore (§ 1) a group of finite order.

It remains to determine the order of H_1 and the totality of its substitutions. For this purpose, having verified that it is actually a group of finite order, it is most convenient to retain the x's as variables.

The result of transforming

$$x_i' = x_i - \tfrac{1}{2}s_1 \quad (i = 1, 2, 3, 4)$$

by $\quad x_1' = x_1, \quad x_2' = -x_2, \quad x_3' = -x_3, \quad x_4' = -x_4,$

is

$$x_1' = \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$
$$x_2' = \tfrac{1}{2}(x_1 + x_2 - x_3 - x_4),$$
$$x_3' = \tfrac{1}{2}(x_1 - x_2 + x_3 - x_4),$$
$$x_5' = \tfrac{1}{2}(x_1 - x_2 - x_3 + x_4).$$

Hence, if

$\tfrac{1}{2}(x_1 + x_2 + x_3 + x_4)$, $\tfrac{1}{2}(x_1 + x_2 - x_3 - x_4)$, $\tfrac{1}{2}(x_1 - x_2 + x_3 - x_4)$, $\tfrac{1}{2}(x_1 - x_2 - x_3 - x_4)$

be denoted by α, β, γ, δ; and

$$\tfrac{1}{2}(-x_1+x_2+x_3+x_4),\quad \tfrac{1}{2}(x_1-x_2+x_3+x_4),$$
$$\tfrac{1}{2}(x_1+x_2-x_3+x_4),\quad \tfrac{1}{2}(x_1+x_2+x_3-x_4)$$

by α_1, β_1, γ_1, δ_1; the group H_1 contains the $24\times 2^4\times 3$ substitutions which replace x_1, x_2, x_3, x_4 by $\pm x_1$, $\pm x_2$, $\pm x_3$, $\pm x_4$ in any sequence; $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm\delta$ in any sequence; and $\pm\alpha_1$, $\pm\beta_1$, $\pm\gamma_1$, $\pm\delta_1$ in any sequence.

Now the substitutions of G_4 change each of the products $x_1x_2x_3x_4$, $\alpha\beta\gamma\delta$, and $\alpha_1\beta_1\gamma_1\delta_1$ into themselves. The substitution

$$x_i' = x_i - \tfrac{1}{2}s_1 \quad (i = 1, 2, 3, 4)$$

permutes $x_1x_2x_3x_4$ and $\alpha_1\beta_1\gamma_1\delta_1$, while it leaves $\alpha\beta\gamma\delta$ unchanged. The substitution

$$x_1' = -x_1,\quad x_i' = x_i \quad (i > 1)$$

changes $x_1x_2x_3x_4$ into $-x_1x_2x_3x_4$; $\alpha\beta\gamma\delta$ into $-\alpha_1\beta_1\gamma_1\delta_1$; and $\alpha_1\beta_1\gamma_1\delta_1$ into $-\alpha\beta\gamma\delta$. Hence H_1 consists of the above specified $2^7.3^2$ substitutions.

Moreover,

$$x_1' = -x_1,\quad x_2' = -x_2,\quad x_3' = x_3,\quad x_4' = x_4$$

permutes $\alpha\beta\gamma\delta$ with $\alpha_1\beta_1\gamma_1\delta_1$ and leaves $x_1x_2x_3x_4$ unchanged. It follows follows that H_2 is an actual sub-group of H_1, and that the order of H_2 is $2^6.3^2$.

That there is no group of rational linear substitutions of four variables with the maximum order $2^7.3^2.5$ is not difficult to verify. In such a group, a substitution of order 5 is permutable with the self-conjugate substitution of order 2, and obviously cannot be permutable with any other substitution except its own powers. A sub-group of order 5 must therefore be contained self-conjugately in one of order 10, 20, or 40. The first of these is the only possibility, since $2^5.3^2$ and $2^4.3^2$ are not congruent to unity (mod 5). In this case, however, the group would contain* a self-conjugate sub-group of order $2^7.3^2$; and this involves an obvious contradiction. The group H_1 is therefore the greatest group of rational linear substitutions of four variables which contains G_4; and the two groups H_1 and H_2 are the only exceptions for the case $n = 4$.

12. When n is 6, 7, or 8, it has been seen that T may be of the form

$$\begin{aligned} x_1' &= -x_2-x_3+\tfrac{1}{3}s_1, \\ x_2' &= -x_3-x_1+\tfrac{1}{3}s_1, \\ x_3' &= -x_1-x_2+\tfrac{1}{3}s_1, \\ x_i' &= x_i, \qquad (i > 3), \end{aligned}$$

and that the quadratic form $9s_2 - s_1^2$

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* *Theory of Groups*, second edition, p. 327.

is invariant for the group generated by G_n and T. That this is a group of finite order may be proved as in the previous case. If new variables be defined by

$$x_1 = 3y_1 - y_2, \quad x_2 = y_2 - y_3, \quad \ldots, \quad x_{n-1} = y_{n-1} - y_n, \quad x_n = y_n,$$

the substitutions of G_n and the substitution T becomes substitutions on the y's with rational integral coefficients, while for the group which they generate the definite form

$$4y_1^2 - 3y_1y_2 + y_2^2 - y_2y_3 + \ldots + y_{n-1}^2 - y_{n-1}y_n + y_n^2 \quad (n = 6, 7, 8),$$

is invariant. The group is therefore one of finite order.

$(n = 6)$.

13. We shall first determine the linear functions into which the substitutions of the group change x_1. Every substitution of the group is expressible in the form

$$S_1TS_2TS_3T \ldots,$$

where S_1, S_2, S_3, ... are substitutions of G_6.

Under substitutions of G_6, x_1 is changed into any one of x_1, x_2, x_3, x_4, x_5, x_6. Under substitutions of the form S_1T, x_1 is changed into any one of the preceding six forms, together with

$$\tfrac{1}{3}s_1 - x_2 - x_3, \quad \tfrac{1}{3}s_1 - x_3 - x_1, \quad \tfrac{1}{3}s_1 - x_1 - x_2.$$

Hence under substitutions of the form S_1TS_2, x_1 is changed into any one of the twenty-one functions that arise from x_1 and $\frac{1}{3}s_1 - x_2 - x_3$ by permutations of the x's. The only new functions into which x_1 is changed by substitutions of the form $S_1TS_2TS_3$ are those that arise from

$$\tfrac{1}{3}s_1 - x_1 - x_4 \quad \text{and} \quad \tfrac{1}{3}s_1 - x_4 - x_5.$$

Since T changes s_1 into $2s_1 - 3(x_1 + x_2 + x_3)$, $\frac{1}{3}s_1 - x_1 - x_4$ is unchanged by T, while $\frac{1}{3}s_1 - x_4 - x_5$ becomes $\frac{1}{3}s_1 - (x_1 + x_2 + x_3 + x_4 + x_5)$ or $x_1 - \frac{1}{3}s_1$. Hence under substitutions of the forms S_1, S_1TS_2, $S_1TS_2TS_3$, x_1 is changed into one of the twenty-seven functions that arise from

$$x_1, \quad \tfrac{1}{3}s_1 - x_2 - x_3, \quad x_1 - \tfrac{1}{3}s_1$$

by permutations of the x's. These twenty-seven functions are not only permuted among themselves by G_6 but also by T. Hence by every substitution of the group, x_1 is changed into one of these twenty-seven functions, and the group is simply isomorphic with a transitive permutation group of degree 27.

To exhibit all the substitutions of the group explicitly an abbreviated notation is convenient. Denote

$$\left.\begin{array}{ll} \tfrac{1}{3}s_1 - x_i - x_j & \text{by } \; x_{ij} \\ x_i - \tfrac{1}{3}s_1 & \text{by } \; \bar{x}_i \end{array}\right\} \quad (i, j = 1, \ldots, 6).$$

The product $x_1x_2x_3x_4x_5x_6$ is unchanged by G_6. Under the substitution T it becomes

$$x_{23}x_{31}x_{12}x_4x_5x_6,$$

and this, under the substitutions of G_6, becomes any one of the twenty products that arise by permutations of the suffixes.

The product $x_1x_2x_3x_{56}x_{64}x_{45}$ becomes under the substitution T

$$x_{23}x_{31}x_{12}\bar{x}_4\bar{x}_5\bar{x}_6,$$

and this again, under the substitutions of G_6, becomes any one of the twenty products that arise by the permutation of the suffixes.

The product $x_{14}x_{45}x_{51}x_2x_3x_6$ becomes under the substitution T

$$\bar{x}_1x_{12}x_{13}x_{14}x_{15}x_6,$$

and this, under the substitutions of G_6, becomes one of thirty products arising from permutations of the suffixes. Lastly, the product

$$\bar{x}_1\bar{x}_2\bar{x}_3x_{56}x_{64}x_{45}$$

is changed by T into $\qquad \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6.$

There thus arises a set of seventy-two products, viz.,

$$\begin{array}{lr} x_1\;x_2\;x_3\;x_5\;x_5\;x_6 \;\ldots\ldots & 1, \\ \bar{x}_1\;\bar{x}_2\;\bar{x}_3\;\bar{x}_4\;\bar{x}_5\;\bar{x}_6 \;\ldots\ldots & 1, \\ x_{23}x_{31}x_{12}x_4\;x_5\;x_6 \;\ldots\ldots & 20, \\ x_{23}x_{31}x_{12}\bar{x}_4\;\bar{x}_5\;\bar{x}_6 \;\ldots\ldots & 20, \\ \bar{x}_1\;x_{12}x_{13}x_{14}x_{15}x_6 \;\ldots\ldots & 30, \end{array}$$

which are permuted both by the substitutions of G_6 and by T.

The substitutions of $\{G_6, T\}$ then are the substitutions which replace x_1, x_2, x_3, x_4, x_5, x_6 by the factors of any one of these products taken in any sequence. The order of the group is

$$6! \times 72 = 2^7.3^4.5.$$

It is clear that $\{G_6, T\}$ does not contain the self-conjugate substitution

$$x_i' = -x_i \quad (i = 1, 2, \ldots, 6);$$

and therefore $\{G_6, T\}$ is contained self-conjugately in a greater group $\{G_6, T\}'$ of rational linear substitutions of twice its order.

Now the greatest possible order of a group of rational linear substitutions on six variables is $2^{10}.3^4.5$. Hence, if $\{G_6, T\}'$ is contained in a greater group of linear substitutions the order of the latter must be $2^9.3^4.5$ or $2^{10}.3^4.5$. In $\{G_6, T\}'$, the cyclical sub-group generated by $(x_2x_3x_4x_5x_6)$ is contained self-conjugately in a sub-group of order 80. By forming the most general rational linear substitution, which is permutable with $(x_2x_3x_4x_5x_6)$, it is easily verified that this sub-group of order 80 contains all the rational substitutions, for which $9s_2-s_1^2$ is invariant, which transform the cyclical sub-group into itself. Hence, if $\{G_6, T\}'$ were contained in a group of order $2^9.3^4.5$ or $2^{10}.3^4.5$, a cyclical sub-group of order 5 would be one of $2^9.3^4.5/80$ or $2^{10}.3^4.5/80$ conjugate sub-groups. This is impossible, since neither of these numbers is congruent to unity (mod 5). Hence $\{G_6, T\}$ and $\{G_6, T\}'$ are the only exceptions for $n = 6$.

The group $\{G_6, T\}$ of rational linear substitutions on six symbols was first shown to exist by Herr Burkhardt (*Math. Ann.*, Vol. XLI, pp. 320–326). The totality of its substitutions were presented in the above form by the author (*Quar. Jour.*, 1909, pp. 246–250). It is simply isomorphic with the group of permutations of the twenty-seven lines on a cubic surface. In fact, if the twenty-seven symbols x_i, $\bar{x}_i$, x_{ij} be regarded as denoting the twenty-seven lines on the surface, then the seventy-two products correspond to the halves of the thirty-six double sixes that can be formed from the lines; and the group of linear substitutions on the six x's is simply isomorphic with the group of permutations of the twenty-seven symbols which leaves invariant the set of seventy-two products.

($n = 7$).

14. As in the preceding case substitutions of the form S_1TS_2 change x_1 into any one of the twenty-eight functions that arise from

$$x_1 \quad \text{and} \quad \tfrac{1}{3}s_1-x_2-x_3,$$

by permutations of the x's. The only new functions that arise from these under substitutions of the form $S_1TS_2TS_3$ are

$$\tfrac{2}{3}s_1-(x_1+x_2+x_3+x_4+x_5) \quad \text{or} \quad x_6+x_7-\tfrac{1}{3}s_1,$$

and those resulting from this by permutation. From $x_2+x_3-\frac{1}{3}s_1$ there arises, under T, $-x_1$. Hence in this case every substitution of the group replaces x_1 by one of the fifty-six functions that arise from

$$\pm x_1, \quad \pm(\tfrac{1}{3}s_1 - x_2 - x_3),$$

by permutation of the symbols.

Denote

$$\left.\begin{array}{rcl} \tfrac{1}{3}s_1 - x_i - x_j & \text{by} & x_{ij}, \\ -\tfrac{1}{3}s_1 + x_i + x_j & \text{by} & \bar{x}_{ij}, \\ -x_i & \text{by} & \bar{x}_i. \end{array}\right\} \quad (i, j = 1, 2, \ldots, 7).$$

From the product

$$x_1 x_2 x_3 x_4 x_5 x_6 x_7$$

there arise, under the substitution T and subsequent permutations, thirty-five products, of which

$$x_{12} x_{23} x_{31} x_4 x_5 x_6 x_7$$

is typical. From

$$x_{14} x_{45} x_{51} x_2 x_3 x_6 x_7,$$

under T and subsequent permutations, there arise one hundred and five products, of which

$$\bar{x}_{67} x_{12} x_{13} x_{14} x_{15} x_6 x_7$$

is typical. From

$$x_1 x_2 x_3 x_{56} x_{64} x_{45} x_7$$

there arise, under T and subsequent permutations, one hundred and forty products, of which

$$x_{23} x_{31} x_{12} \bar{x}_{47} \bar{x}_{57} \bar{x}_{67} x_7$$

is typical. From

$$\bar{x}_{17} \bar{x}_{27} \bar{x}_{37} x_{56} x_{64} x_{45} x_7$$

there arise, under T and subsequent permutations, seven products, of which

$$\bar{x}_{17} \bar{x}_{27} \bar{x}_{37} \bar{x}_{47} \bar{x}_{57} \bar{x}_{67} x_7$$

is typical.

One of the latter, viz., $\bar{x}_{12} \bar{x}_{13} \bar{x}_{14} \bar{x}_{15} \bar{x}_{16} \bar{x}_{17} x_1$

is changed by T into $\bar{x}_2 \bar{x}_3 x_{23} \bar{x}_{14} \bar{x}_{15} \bar{x}_{16} \bar{x}_{17}$;

and the product $\bar{x}_6 \bar{x}_7 x_{67} \bar{x}_{12} \bar{x}_{13} \bar{x}_{14} \bar{x}_{15}$,

which arises from the preceding by a permutation is changed by T into

$$\bar{x}_2 \bar{x}_3 \bar{x}_6 \bar{x}_7 \bar{x}_{14} \bar{x}_{15} \bar{x}_{45}.$$

Finally the product $\bar{x}_{12} \bar{x}_{13} \bar{x}_{23} \bar{x}_4 \bar{x}_5 \bar{x}_6 \bar{x}_7$,

which arises from the preceding by a permutation is changed by T into

$$\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6 \bar{x}_7,$$

so that the group contains a self-conjugate substitution of order 2.

The group therefore has substitutions which change the product of the symbols into each product contained in the following sets, viz.,

$$\begin{array}{lr}
x_1 x_2 x_3 x_4 x_5 x_6 x_7 \;\ldots\ldots & 1, \\
\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6 \bar{x}_7 \;\ldots\ldots & 1, \\
x_1 x_2 x_3 x_4 x_{67} x_{75} x_{56} \;\ldots\ldots & 35, \\
\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_{67} \bar{x}_{75} \bar{x}_{56} \;\ldots\ldots & 35, \\
x_1 x_2 \bar{x}_{12} x_{34} x_{35} x_{36} x_{37} \;\ldots\ldots & 105, \\
\bar{x}_1 \bar{x}_2 x_{12} \bar{x}_{34} \bar{x}_{35} \bar{x}_{36} \bar{x}_{37} \;\ldots\ldots & 105, \\
x_1 \bar{x}_{12} \bar{x}_{13} \bar{x}_{14} x_{67} x_{75} x_{56} \;\ldots\ldots & 140, \\
\bar{x}_1 x_{12} x_{13} x_{14} \bar{x}_{67} \bar{x}_{75} \bar{x}_{56} \;\ldots\ldots & 140, \\
x_1 \bar{x}_{12} \bar{x}_{13} \bar{x}_{14} \bar{x}_{15} \bar{x}_{16} \bar{x}_{17} \;\ldots\ldots & 7, \\
\bar{x}_1 x_{12} x_{13} x_{14} x_{15} x_{16} x_{17} \;\ldots\ldots & 7.
\end{array}$$

There are in all here five hundred and seventy-six products, and the group therefore contains all the substitutions which change x_1, x_2. x_3, x_4, x_5, x_6, x_7 into the factors, taken in any sequence, of any one of these products. These constitute $2^{10}.3^4.5.7$ substitutions, so that the order of the group is equal to or greater than this number. Now the greatest possible order of a group of rational linear substitutions on seven variables is $2^{11}.3^4.5.7$, so that the order of $\{G_7, T\}$ is either $2^{10}.3^4.5.7$ or $2^{11}.3^4.5.7$. In $\{G_7, T\}$ the only substitution with which a substitution of order 7 is permutable, except its own powers, is the self-conjugate substitution of order 2. Moreover a substitution of order 7 is transformed into its cube by a substitution of order 6; and therefore the greatest sub-group containing a cyclical subgroup of order 7 self-conjugately has order 84. Now $2^{10}.3^4.5.7/84$ is, and $2^{11}.3^4.5.7/84$ is not, congruent to unity (mod 7). Hence, finally, the order of $\{G_7, T\}$ is $2^{10}.3^4.5.7$; it is not contained in a greater group of rational linear substitutions, and the totality of its substitutions is exhibited in the above table.

The existence of this remarkable group of rational linear substitutions on seven variables has not previously, so far as I am aware, been recognised. It may be easily verified that it is simply isomorphic with the group of permutations on the fifty-six symbols

$$x_i,\ \bar{x}_i,\ x_{ij},\ \bar{x}_{ij} \quad (i, j = 1, 2, \ldots, 7)$$

for which the set of five hundred and seventy-six products is invariant. This group of permutations is clearly imprimitive, permuting the fifty-six symbols in the twenty-eight pairs

$$x_i,\ \bar{x}_i\ ;\ x_{ij},\ \bar{x}_{ij}\quad (i, j = 1, 2, \dots, 7).$$

It contains a self-conjugate permutation which permutes the two symbols of each imprimitive system; and in respect of this self-conjugate permutation it is isomorphic with the group of permutations of twenty-eight symbols for which the two hundred and eighty-eight products, that arise from the preceding table on neglecting the distinction between x_i and $\bar{x}_i$ or x_{ij} and $\bar{x}_{ij}$, is invariant.

An examination of the resulting table shews it to be identical with the table of the two hundred and eighty-eight Aronhold systems* that can be formed from the twenty-eight tangents to a plane quartic. It follows that if S is the self-conjugate substitution

$$x_i' = -x_i\quad (i = 1, 2, \dots, 7),$$

then $\{G_7, T\}/\{S\}$ is simply isomorphic with the group of the twenty-eight bitangents to a plane quartic.

$(n = 8)$.

15. As in the preceding cases, substitutions of the forms S_1TS_2 and $S_1TS_2TS_3$ change x_1 into

$$x_1,\quad \tfrac{1}{3}s_1-x_2-x_3,\quad \tfrac{2}{3}s_1-(x_1+x_2+x_3+x_4+x_5),$$

and functions which arise from them by permutations. The last form may be written

$$x_6+x_7+x_8-\tfrac{1}{3}s_1.$$

The only new functions that arise from this and its permutations, under the substitution T, are

$$x_1-x_2,\quad \tfrac{1}{3}s_1-x_1-x_2-x_3,$$

and their permutations. From these in a similar way there arise

$$x_1+x_2-\tfrac{1}{3}s_1,$$

and its permutations; and from this

$$-x_1$$

and its permutations. Hence the linear functions into which x_1 is changed by the substitutions of the group are

$$\pm x_i,\quad \pm(\tfrac{1}{3}s_1-x_i-x_j),\quad \pm(\tfrac{1}{3}s_1-x_i-x_j-x_k),\quad x_i-x_j,$$
$$(i, j, k = 1, 2, \dots, 8).$$

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* Cayley, *Collected Works*, Vol. VII, p. 125.

Of these there are two hundred and forty, so that $\{G_8, T\}$ is simply isomorphic with a transitive permutation group of degree 240.

$$
\text{Denote}\qquad
\begin{array}{rcl}
-x_i & \text{by} & \bar{x}_i, \\
\tfrac{1}{3}s_1 - x_i - x_j & \text{by} & x_{ij}, \\
-\tfrac{1}{3}s_1 + x_i + x_j & \text{by} & \bar{x}_{ij}, \\
-\tfrac{1}{3}s_1 + x_i + x_j + x_k & \text{by} & x_{ijk}, \\
\tfrac{1}{3}s_1 - x_i - x_j - x_k & \text{by} & \bar{x}_{ijk}, \\
x_i - x_j & \text{by} & x_{(ij)}.
\end{array}
$$

The product $\prod_1^8 x_i$ is changed by every substitution of the group into the product of eight of these symbols. In particular, T changes it into

$$x_{23}x_{31}x_{12}x_4x_5x_6x_7x_8.$$

A permutation changes this into

$$x_{67}x_{78}x_{86}x_1x_2x_3x_4x_5,$$

and T changes this into $x_4x_5x_{23}x_{31}x_{12}x_{456}x_{457}x_{458}$.

The number of products that arise in this case is so great that it would be tedious to give, as in the preceding simpler cases, their actual construction. It is easy to verify that, as in the case $n = 7$, the group contains a self-conjugate substitution of order 2, so that, for example, from the last written product

$$\bar{x}_4\,\bar{x}_5\,\bar{x}_{23}\,\bar{x}_{31}\,\bar{x}_{12}\,\bar{x}_{456}\,\bar{x}_{457}\,\bar{x}_{458}$$

arises. By continuing the process begun above, I have verified directly that all the products in the following table arise from $\prod_1^8 x_i$ under the substitutions of the group. Each entry in the table is to be regarded as a typical product, and the number written against it is the number of products that arise from it under all permutations of the x's; or, which is the same thing, all permutations of the suffixes.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	......	1,
$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{x}_5$	$\bar{x}_6$	$\bar{x}_7$	$\bar{x}_8$	......	1,
x_1	x_2	x_3	x_4	x_5	x_{78}	x_{86}	x_{67}	......	56,
$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{x}_5$	$\bar{x}_{78}$	$\bar{x}_{86}$	$\bar{x}_{67}$	......	56,
x_1	x_2	x_3	x_{48}	x_{58}	x_{68}	x_{78}	x_{123}	......	280,
$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_{28}$	$\bar{x}_{38}$	$\bar{x}_{48}$	$\bar{x}_{58}$	$\bar{x}_{123}$	......	280,

$$\begin{array}{cccccccccr}
x_1 & x_2 & x_{123} & x_{124} & x_{125} & x_{126} & x_{127} & x_{128} & \ldots\ldots & 28, \\
\bar{x}_1 & \bar{x}_2 & \bar{x}_{123} & \bar{x}_{124} & \bar{x}_{125} & \bar{x}_{126} & \bar{x}_{127} & \bar{x}_{128} & \ldots\ldots & 28, \\
x_1 & x_2 & x_{78} & x_{86} & x_{67} & x_{123} & x_{124} & x_{125} & \ldots\ldots & 560, \\
\bar{x}_1 & \bar{x}_2 & \bar{x}_{78} & \bar{x}_{86} & \bar{x}_{67} & \bar{x}_{123} & \bar{x}_{124} & \bar{x}_{125} & \ldots\ldots & 560, \\
x_1 & x_{28} & x_{38} & x_{48} & x_{167} & x_{175} & x_{156} & x_{(18)} & \ldots\ldots & 1120, \\
\bar{x}_1 & \bar{x}_{28} & \bar{x}_{38} & \bar{x}_{48} & \bar{x}_{167} & \bar{x}_{175} & \bar{x}_{156} & x_{(81)} & \ldots\ldots & 1120, \\
x_1 & x_{28} & x_{38} & x_{48} & x_{58} & x_{68} & x_{78} & x_{(18)} & \ldots\ldots & 56, \\
\bar{x}_1 & \bar{x}_{28} & \bar{x}_{38} & \bar{x}_{48} & \bar{x}_{58} & \bar{x}_{68} & \bar{x}_{78} & x_{(81)} & \ldots\ldots & 56, \\
x_1 & x_{78} & x_{123} & x_{124} & x_{125} & x_{126} & x_{(17)} & x_{(18)} & \ldots\ldots & 840, \\
\bar{x}_1 & \bar{x}_{78} & \bar{x}_{123} & \bar{x}_{124} & \bar{x}_{125} & \bar{x}_{126} & x_{(71)} & x_{(81)} & \ldots\ldots & 840, \\
x_1 & x_{134} & x_{142} & x_{123} & x_{(15)} & x_{(16)} & x_{(17)} & x_{(18)} & \ldots\ldots & 280, \\
\bar{x}_1 & \bar{x}_{134} & \bar{x}_{142} & \bar{x}_{123} & x_{(51)} & x_{(61)} & x_{(71)} & x_{(81)} & \ldots\ldots & 280, \\
x_1 & x_{(12)} & x_{(13)} & x_{(14)} & x_{(15)} & x_{(16)} & x_{(17)} & x_{(18)} & \ldots\ldots & 8, \\
\bar{x}_1 & x_{(21)} & x_{(31)} & x_{(41)} & x_{(51)} & x_{(61)} & x_{(71)} & x_{(81)} & \ldots\ldots & 8, \\
x_{23} & x_{31} & x_{12} & x_{567} & x_{467} & x_{457} & x_{456} & \bar{x}_{123} & \ldots\ldots & 280, \\
\bar{x}_{23} & \bar{x}_{31} & \bar{x}_{12} & \bar{x}_{567} & \bar{x}_{467} & \bar{x}_{457} & \bar{x}_{456} & x_{123} & \ldots\ldots & 280, \\
x_{18} & x_{28} & x_{345} & x_{346} & x_{347} & x_{(38)} & x_{(48)} & \bar{x}_{128} & \ldots\ldots & 1680, \\
\bar{x}_{18} & \bar{x}_{28} & \bar{x}_{345} & \bar{x}_{346} & \bar{x}_{347} & x_{(83)} & x_{(84)} & x_{128} & \ldots\ldots & 1680, \\
x_{123} & x_{124} & x_{125} & x_{126} & x_{127} & x_{(18)} & x_{(28)} & \bar{x}_{12} & \ldots\ldots & 168, \\
\bar{x}_{123} & \bar{x}_{124} & \bar{x}_{125} & \bar{x}_{126} & \bar{x}_{127} & x_{(81)} & x_{(82)} & x_{12} & \ldots\ldots & 168, \\
x_{126} & x_{127} & x_{128} & x_{(13)} & x_{(14)} & x_{(15)} & \bar{x}_{12} & \bar{x}_{345} & \ldots\ldots & 1120, \\
\bar{x}_{126} & \bar{x}_{127} & \bar{x}_{128} & x_{(31)} & x_{(41)} & x_{(51)} & x_{12} & x_{345} & \ldots\ldots & 1120, \\
x_{123} & x_{124} & x_{(18)} & x_{(28)} & \bar{x}_{12} & \bar{x}_{678} & \bar{x}_{758} & \bar{x}_{568} & \ldots\ldots & 1680, \\
\bar{x}_{123} & \bar{x}_{124} & x_{(81)} & x_{(82)} & x_{12} & x_{678} & x_{758} & x_{568} & \ldots\ldots & 1680, \\
x_{12} & x_{13} & x_{14} & x_{15} & x_{(61)} & x_{(71)} & x_{(81)} & x_{678} & \ldots\ldots & 280, \\
\bar{x}_{12} & \bar{x}_{13} & \bar{x}_{14} & \bar{x}_{15} & x_{(16)} & x_{(17)} & x_{(18)} & \bar{x}_{678} & \ldots\ldots & 280, \\
x_{18} & x_{28} & x_{(38)} & x_{(48)} & x_{(58)} & x_{(68)} & x_{(78)} & \bar{x}_{128} & \ldots\ldots & 168, \\
\bar{x}_{18} & \bar{x}_{28} & x_{(83)} & x_{(84)} & x_{(85)} & x_{(86)} & x_{(87)} & x_{128} & \ldots\ldots & 168, \\
x_{234} & x_{134} & x_{124} & x_{123} & \bar{x}_{678} & \bar{x}_{578} & \bar{x}_{568} & \bar{x}_{567} & \ldots\ldots & 70.
\end{array}$$

There are here in all 17,280 distinct products, and the group contains a substitution which changes the x's into the eight factors of any one of these products taken in any sequence. The table therefore gives in explicit form $17280 \times 8!$, *i.e.*, $2^{14}.3^5.5^2.7$, distinct substitutions all belonging to $\{G_8, T\}$.

Now the order of any group of rational linear substitutions on eight symbols is a factor of $2^{15}.3^5.5^2.7$; and therefore the order of $\{G_8, T\}$ must be either equal to or one half of this number. In G_8 the cyclical substitution $(x_1x_2x_3x_4x_5x_6x_7)$ is transformed into its cube by a substitution of order 6. It is easily verified that the only rational substitutions, for which $9s_2-s_1^2$ is invariant, with which $(x_1x_2x_3x_4x_5x_6x_7)$ is permutable (its own powers excepted) are

$$x_i' = -x_i \quad (i = 1, 2, \ldots, 8),$$

$$x_8' = x_8, \quad x_i' = x_8 - x_i \quad (i = 1, 2, \ldots, 7),$$

and their product. Hence, in $\{G_8, T\}$, the order of the greatest subgroup which contains a sub-group of order 7 self-conjugately is 168. Now $2^{14}.3^5.5^2.7/168$ is, and $2^{15}.3^5.5^2.7/168$ is not, congruent to unity (mod 7). Hence, finally, the order of $\{G_8, T\}$ is $2^{14}.3^5.5^2.7$; all its substitutions are given explicitly by the preceding table; and it is not contained in a group of rational linear substitutions of greater order.

For the case $n = 8$, there is therefore again just one exception. The existence of this group of rational linear substitutions on eight variables has not hitherto been recognised. It contains a self-conjugate substitution S of order 2; and it may be shewn that the factor-group $\{G_8, T\}/\{S\}$ is simply isomorphic with Jordan's first hypo-Abelian group* on eight symbols. The proof of this statement, which is not directly connected with the subject matter of the present paper, is left for another occasion.

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* Jordan, *Traité des Substitutions*, pp. 195-206.

DEFORMABLE OCTAHEDRA

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THE following synopsis of the contents of the paper may be of assistance to the reader, though necessarily too brief to be intelligible in itself.

§§ 1. Preliminary.
2. The infinitesimally deformable octahedron.
3. Kinematics of the same.
4. Polarizing screws: groups of normals.
5. Polarizing quadrics.
6. Mechanical completion of the figure: eight rigid tetrahedra.
7. Spherical indicatrix: the double-four.
8. Kinematics of double-four: centres of rotation and perspective quadrilaterals.
9. Other perspective quadrilaterals.
10. The axially symmetric octahedron.
11. The axis as related to four concurrent planes.
12. The axis as related to four coplanar points.
13. The axis as related to a tetrahedron.
14. Congruences of circles associated with a four-plate mechanism.
15. The spherical double-four of the axially symmetric octahedron.
16. Plane double-four symmetric about centre or axis.
17. The axially symmetric skew hexagon: the general axially symmetric chain.
18. The equilateral and rectangular skew hexagon.
19. The prismatic case of the axially symmetric octahedron.
20. Congruent and mutually inscribed tetrahedra.
21. Locus of the reflecting axis of the tetrahedra.
22. Tetrahedron with two pairs of equal and opposite edges: cylindroid movement.
23. Tetrahedron with three pairs of equal and opposite edges. Three cylindroids.
24. Deformable octahedron symmetric about a plane.
25. The skew deformable octahedron: preliminary.
26. Spherical double-four composed of isogram linkworks.
27. Plane double-four composed of crossed isogram linkworks.
28. Construction of the skew deformable octahedron.
29. Properties of the skew deformable octahedron.
30. The two flat forms of the skew deformable octahedron.
31. Compound octahedral mechanisms.
32. Concluding remarks.

1. The common regular octahedron has three pairs of opposite vertices, four pairs of opposite triangular faces, six pairs of opposite edges, and

three diagonals. The octahedron here to be considered has elements of the same number and kind as the regular octahedron, but is, otherwise, arbitrary in form. Such an irregular octahedron may be supposed to be constructed with eight rigid plates for faces hinged along the edges, or with twelve rigid wires for edges articulated at the vertices; and the form so constructed is in general a stiff framework. An analytical investigation of all the special types of octahedron which are deformable has been made by Bricard;* who studies for that purpose the conditions under which three doubly quadratic equations in three variables may be poristic. He finds that the possibilities are exhausted by three different types of octahedron. The first type is symmetric about a line: the second about a plane. The third is unsymmetric; but it is shown to have the alternate angles at each vertex equal or supplementary; and a construction is given for either of two flat forms which the octahedron assumes during its deformation. Figures are given in representation of a model of each type. To the paper is appended a note of Mannheim, pointing out that, as an immediate consequence of the kinematics, the eight faces of each octahedron are concurrent in two points.

It is part of the purpose of this paper to give, later, some discussion of the geometry of the three types of deformable octahedron, and to develop the kinematic properties of the movement. But it is convenient to consider first the more general case of an octahedron which shall be only infinitesimally deformable: for any results true of this last case will necessarily be true of the more special cases in which the free working of the mechanism is of finite extent.

2. Let the three pairs of vertices of the octahedron be A and A', B and B', C and C'. The four pairs of triangular faces may be named as follows:—

$$\begin{aligned} AB'C' &= \alpha, & A'BC &= \alpha', \\ A'BC' &= \beta, & AB'C &= \beta', \\ A'B'C &= \gamma, & ABC' &= \gamma', \\ ABC &= \delta, & A'B'C' &= \delta'. \end{aligned}$$

Assuming then a small movement to be possible, consider the kinematics of the movement of ABC relatively to $A'B'C'$. The point A, as a vertex of the plate $AB'C'$, moves normally to the plane $AB'C'$; and hence the point A has α as its polar plane with regard to the screw of the relative movement; and similarly B and C have as polar planes β and γ. In the same way it appears that the polar planes of A', B', and C' with regard to the same screw are α', β', and γ'. Now, since A, B, and C lie in the plane δ, their polar planes α, β, γ pass through the point D of δ which is the pole of δ: that is, the planes $\alpha, \beta, \gamma, \delta$ are concurrent. Similarly, the planes α', β', γ', δ' are concurrent in the point D', which is the pole of δ'. The above table of notation may therefore be extended thus:—

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* *Liouville*, 1897, pp. 113–148.

$$\begin{array}{llll}
AB'C'D = \alpha, & A'BCD' = \alpha', & \alpha\beta'\gamma'\delta = A, & \alpha'\beta\gamma\delta' = A', \\
A'BC'D = \beta, & AB'CD' = \beta', & \alpha'\beta\gamma'\delta = B, & \alpha\beta'\gamma\delta' = B', \\
A'B'CD = \gamma, & ABC'D' = \gamma', & \alpha'\beta'\gamma\delta = C, & \alpha\beta\gamma'\delta' = C', \\
ABCD = \delta, & A'B'C'D' = \delta', & \alpha\beta\gamma\delta = D, & \alpha'\beta'\gamma'\delta' = D',
\end{array}$$

and the complete configuration is that of Möbius, capable of being exhibited in four different ways as consisting of the eight vertices and the eight faces of two mutually inscribed tetrahedra. The four pairs of tetrahedra are these:—

$$\begin{array}{ll}
A'BCD = \delta\gamma\beta\alpha', & AB'C'D' = \delta'\gamma'\beta'\alpha, \\
AB'CD = \gamma\delta\alpha\beta', & A'BC'D' = \gamma'\delta'\alpha'\beta, \\
ABC'D = \beta\alpha\delta\gamma', & A'B'CD' = \beta'\alpha'\delta'\gamma, \\
ABCD' = \alpha'\beta'\gamma'\delta, & A'B'C'D = \alpha\beta\gamma\delta'.
\end{array}$$

In this last table the faces of each tetrahedron are named in the order corresponding to that of the opposite vertices: and from it may be read off the alternative names of any one of the 24 lines: *ex. gr.*,

$$AB' = \alpha\beta', \qquad AB = \gamma'\delta.$$

It is perhaps worth noticing, in passing, that the relations of the points and planes and lines of this well-known configuration may be very concisely presented by attaching the names to the corners of a cube (Fig. 1, p. 312). One corner is named D, the three adjacent corners are named A, B, C, and the corners diagonally opposite are named D', A', B', C', respectively. Any corner and the three adjacent corners then give the names of four coplanar points. The name of the plane corresponding in this way to any corner is attached to that same corner: so here, to the corners already named, are given (in the same order) the additional names δ, α, β, γ, δ', α', β', γ'. Then the following results hold:—(i) a

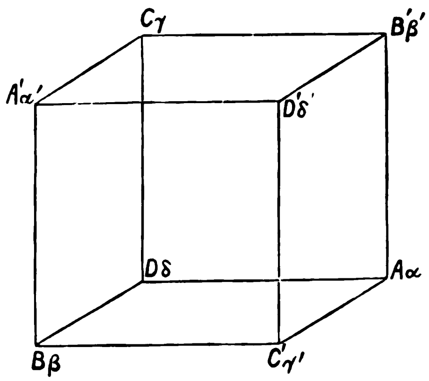

Fig. 1.

point is on a plane if the names are at the same corner or at adjacent corners; (ii) four points are coplanar, or four planes are concurrent, if their names are at a corner and the three adjacent corners; (iii) the line joining two points is the same as the line of intersection of two planes if their names are either at the ends of the same edge of the cube or else at the ends of the two diagonals of a face.

(It should be remarked that the cubical arrangement of the letters is one of four such arrangements essentially different, and is selected as the most convenient only on account of the alphabetical accident of notation. The customary pairing of A with α, etc., is, of course, merely literal and not logical.)

The 28 joins of the eight points taken in pairs are completed by the addition of the four lines AA', BB', CC', and DD'; and the four lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ complete the 28 intersections of the eight planes taken in pairs. It may be recalled here, as a known property of the configuration, that a pair of transversals meet the lines AA', BB', CC', DD', and separate each pair of points harmonically; and that the same transversals meet the lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$, and separate each pair of planes harmonically. They will be referred to as the harmonic transversals.* A consequence necessary to be quoted later (§ 9) is that the pair of planes projecting the pairs of points A and A', B and B', C and C', D and D' from any line which meets the harmonic transversals, form a pencil in involution, and that the double planes of the involution pass through the harmonic transversals.

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* Cf. *Quarterly Journal*, No. 168, 1911, pp. 379-386.

3. The kinematics of the relative movement of any two of the eight plates may now be considered. There are three types of pair. (i) There are twelve pairs with a hinge-line of connection, such as α' and δ with the hinge-line BC, the relative movement being rotation about the hinge-line. (ii) There are twelve pairs with a common point, such as α and δ with the common point A. At the point A are consecutively hinged the plates α, β', δ, γ' (in this order); and hence the relative motion of α and δ is rotation about the line $\beta'\gamma'$. (iii) There are four pairs, such as α and α', with no material point of connection, and these have each a screw for their relative motion. The relative motions of the 28 pairs may be arranged tabularly as follows:—

α								
β	$\gamma'\delta$							
γ	$\beta'\delta$	$\alpha'\delta'$						
δ	$\beta'\gamma'$	$\gamma'\alpha'$	$\alpha'\beta'$					
α'	S_1	$\alpha'\beta$	$\gamma\alpha'$	$\alpha'\delta$				
β'	$\alpha\beta'$	S_2	$\beta'\gamma$	$\beta'\delta$	$\gamma\delta$			
γ'	$\gamma'\alpha$	$\beta\gamma'$	S_3	$\gamma'\delta$	$\beta\delta$	$\alpha\delta$		
δ'	$\alpha\delta'$	$\beta\delta'$	$\gamma\delta'$	S_4	$\beta\gamma$	$\gamma\alpha$	$\alpha\beta$	
·	α	β	γ	δ	α'	β'	γ'	δ'

S_1, S_2, S_3, S_4 being screw motions and all the rest rotations. It may be noticed that the plates α, β, γ, δ, having the geometrical point D common, have as axes of relative rotation the lines of intersection of the plates α', β', γ', δ', having the point D' common; and also conversely.

4. It was seen (§ 2) that the mutually inscribed tetrahedra $\alpha\beta\gamma\delta'$ and $\alpha'\beta'\gamma'\delta$ are reciprocal with respect to the screw of the relative movement of plates δ and δ'. Similar results hold for the other three screws. The eight points, polarized with regard to any one of the four screws, give rise

to the eight planes according to the scheme:—

	A	B	C	D	A'	B'	C'	D'
S_1	δ	γ'	β'	α	δ'	γ	β	α'
S_2	γ'	δ	α'	β	γ	δ'	α	β'
S_3	β'	α'	δ	γ	β	α	δ'	γ'
S_4	α	β	γ	δ	α'	β'	γ'	δ'

And thus there are associated with each screw a pair of opposite faces of the octahedron and a pair of mutually inscribed and reciprocal tetrahedra; namely,

with the screw S_1, the faces α, α', and the tetrahedra $\alpha'\beta\gamma\delta$, $\alpha\beta'\gamma'\delta'$,
,, S_2, ,, β, β', ,, $\alpha\beta'\gamma\delta$, $\alpha'\beta\gamma'\delta'$,
,, S_3, ,, γ, γ', ,, $\alpha\beta\gamma'\delta$, $\alpha'\beta'\gamma\delta'$,
,, S_4, ,, δ, δ', ,, $\alpha\beta\gamma\delta'$, $\alpha'\beta'\gamma'\delta$.

For any one pair of reciprocal tetrahedra in relation to the polarizing screw the following results hold:—(i) the lines joining non-corresponding vertices are rays of the screw, (ii) the lines of intersection of non-corresponding faces are rays of the screw, (iii) the lines joining corresponding vertices are polars of the lines of intersection of the opposite faces, (iv) the polar lines of the edges of either tetrahedron are the edges of the other tetrahedron which are corresponding-opposite. From the last, since the axis of a screw and any two polar lines have a common normal, it follows that the six common normals of corresponding opposite edges of each pair of tetrahedra are normal to the axis of the associated screw. The same also appears immediately and kinematically on considering the relative movements of groups of three pieces. Consider a group consisting of α, α' and any one other. For α, α' and β', the movements are (§ 3) the screw S_1 and rotations on $\alpha\beta'$ and $\gamma\delta$. Hence the common normal of $\alpha\beta'$ and $\gamma\delta$ meets the axis of S_1. Replacing β' by each of the other five plates in turn, it appears that the common normals of corresponding-opposite edges of the tetrahedra $\alpha'\beta\gamma\delta$ and $\alpha\beta'\gamma'\delta'$ are all normal to the axis of S_1.

It may be observed that any pair of corresponding-opposite edges of two reciprocal tetrahedra occur also as opposite edges of another tetrahedron. For example, $\alpha\beta'$ and $\gamma\delta$, above, are opposite edges of the tetrahedron $\alpha\beta'\gamma\delta$. Each tetrahedron gives rise to three common normals of opposite edges, and the eight tetrahedra altogether to 24 such lines: and these may be grouped in four sets of six, each set being all of them normals to the axis of one of the four screws.

5. Less immediately kinematic, but convenient to include for the sake of reference, are the four polarizing quadrics of the configuration. The table of § 4 shows the four screws to be commutative and therefore reciprocal. Successive polarization with regard to any three of them has the same effect as polarization with regard to the quadric generated by their common rays. If Q_1 is the quadric derived from screws S_2, S_3, S_4, and similarly for Q_2, Q_3, Q_4, the following table gives the polar relations of the eight points and planes with regard to the four quadrics:—

	A	B	C	D	A'	B'	C'	D'
Q_1	δ'	γ	β	α'	δ	γ	β'	α
Q_2	γ	δ'	α	β'	γ'	δ	α'	β
Q_3	β	α	δ'	γ'	β'	α'	δ	γ
Q_4	α'	β'	γ'	δ'	α	β	γ	δ

The tetrahedra which are reciprocal with respect to any one of the screws are each self-polar with respect to the corresponding quadric.

The tables of §§ 4, 5 conjointly show each of the eight planes as the polar of any one of the eight points, provided that the polarizing screw or quadric is properly selected.

It may also be conveniently stated here that the two harmonic transversals (§ 2) are the common rays of the four screws; and hence that they are also common generators of the four quadrics.

6. The symmetry of the relationships which attach to the configuration of eight points and eight planes does not, so far, extend to the material of the mechanism itself: for the points D and D' play a part which is merely geometrical. This lack of agreement may, however, be removed. Regard the octahedron as constructed of twelve wires articulated at the points A, B, C, A', B', C'. Introduce three wires AD', BD', CD'; so that $ABCD'$ is now a rigid tetrahedron. The point D' has, therefore, relatively to the triangle $A'B'C'$, a movement normal to the plane $A'B'C'$; and three additional wires $A'D'$, $B'D'$, $C'D'$ will not prevent this infinitesimal movement. Similarly, six wires $A'D$, $B'D$, $C'D$, AD, BD, CD may further be added without constraining the infinitesimal freedom. The mechanism now consists, symmetrically, of 24 wires articu-

lated at the eight points A, B, C, D, A', B', C', D'. (The only non-material lines are the four diagonals AA', BB', CC', DD'.) This completed mechanism has the same system of kinematic relations as the previous and incomplete form. The complete mechanism may be regarded as a collocation of four different mechanisms of the original form; for, whereas omission of the wires having terminals at D or D' restores the original form, omission of the wires having terminals at A or A' gives a fresh form, and similarly for B or B' and C or C'. If constructed of plates instead of wires each of the partial mechanisms is an octahedron, and the complete mechanism consists of 32 plates. Each plate may be named by the three vertices, the vertices being named by three letters (out of four) all different, each either plain or accented. The four plates connecting coplanar points A, B, C, D lie all in one plane and permit of infinitesimal movement: there are eight such planes. The four plates connecting points A, B, C, D' form a rigid tetrahedron: there are eight such tetrahedra, consisting of the four pairs of § 2. Two tetrahedra which are a pair are not immediately connected and have one of the screws for their relative motion: but any two which are not a pair have a common edge, and the common edge is the axis of the rotation which is their relative movement. The table of § 3 gives, *ex. gr.*, $\alpha'\delta'$ as the axis of rotation for the plates β and γ: and in the completed figure the plates are represented by the rigid tetrahedra $A'BC'D'$ and $A'B'CD'$, with the common edge $A'D'$, which is $\alpha'\delta'$. Hence, for the completed mechanism, the axes of rotation are now, being material lines of connection, immediately apparent without the auxiliary table.

As an application of this consider the four tetrahedra $ABCD'$, $ABC'D$, $A'B'C'D$, $A'B'CD'$, forming a cycle of four rigid bodies hinged consecutively. The hinge-lines AB, $C'D$, $A'B'$, CD' must, in virtue of the infinitesimal freedom, be generators of a quadric. The same is true of the four remaining tetrahedra $AB'CD$, $A'BCD$, $A'BC'D'$, $AB'C'D'$, giving hinge-lines CD, $A'B$, $C'D'$, AB' as generators of a quadric. And the two quadrics are the same; for each set of four lines meet the other set of four. The eight lines are the sides of the skew quadrilaterals $ABA'B'$ and $CDC'D'$. It follows that AA' and BB' are polar lines, and that CC' and DD' are polar lines; and hence that the two harmonic transversals, meeting these four, are also polar lines. There are three such quadrics, each passing through all the eight points and touching all the eight planes, and together accounting (as generators) for all the 24 lines.

The fact which has here been sufficiently brought out in regard to the Möbius configuration is that, if the figure is materialized, it retains one degree of infinitesimal freedom, and that the kinematics of the movement accounts for much of the geometry associated with the configuration. And a simple analogue in uniplanar kinematics, which is perhaps not too familiar, may here be remarked. A four-bar linkwork gives six instantaneous centres of relative rotation for the six pairs of bars; namely, the pivot of connection for any two adjacent bars, and the point of intersection of either pair of opposite bars for the other two opposite bars. This may be symmetrized by replacing each bar by a triangular plate, the sides of the triangle being given by the line of the bar and of the two adjacent bars. The figure consists then of the four triangles obtained by omitting one line at a time from the figure of a complete quadrilateral. Each pair of triangles are pivoted at a common vertex, being a vertex of the quadrilateral, and the whole forms a mechanism with one degree of infinitesimal freedom. Three separate four-bar mechanisms are here collocated in the one complete mechanism, and are obtainable, each in turn, by the omission of the pivots at a pair of opposite vertices of the quadrilateral. (The plane analogue differs from the three-dimensional case in that the collocated mechanisms have finite freedom.)

7. Reverting to the original octahedral mechanism with its eight triangular plates, we may construct a spherical figure to represent, by great circles and their points of intersection, the angular positions of the plates and the directions of the edges. Such a figure may be called the spherical indicatrix of the octahedron. Transferring the notation of the octahedron to the spherical figure, the great circles may be denoted by the symbols α, β, γ, δ, α', β', γ', δ': and the directions of the edges will be represented by twelve of the sixteen points in which α, β, γ, δ are cut by α', β', γ', δ'; namely, when the four points $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ are excluded. Alternative names for the twelve points, such as AB' for $\alpha\beta'$ and $C'D'$ for $\alpha\beta$, follow the same rules as for the octahedron itself: and four points named AA', BB', CC', DD' represent the directions of the corresponding diagonals of the octahedron. Excluding these last four points, the figure consisting of the eight arcs and the twelve points of intersection may conveniently be termed a spherical double-four.

The rigidity of the plate $A'BC$ of the octahedron involves invariability of the angles of inclination of the edges $A'B$, $A'C$, BC, that is, of the lines $\alpha'\beta$, $\alpha'\gamma$, $\alpha'\delta$ of the octahedral figure: and hence the points $A'B$, $A'C$, BC of the spherical figure, that is, points $\alpha'\beta$, $\alpha'\gamma$, $\alpha'\delta$, may be regarded as points marked on a rigid arc α'. Similar results hold for all the eight arcs; and hence the whole spherical double-four may be regarded as a spherical mechanism of eight pieces articulated at the twelve crossing points. The infinitesimal deformability of the octahedron involves a like property in the spherical double-four. Such a spherical framework would in general, instead of having one degree of freedom, be quadruply stiff. The present double-four is of special type. For, in the octahedron, the directions of the three pairs of sides of the quadrangle $A'BCD$ are in involution, and hence in the spherical figure the three pairs of points in which a is cut by β and β', γ and γ', δ and δ', are in involution. Hence any three pairs of sides of the double-four cut each of the other two in involution. There are eight such involutions in all.*

8. The kinematics of the spherical mechanism may be considered. Each of the 28 different pairs of pieces gives rise to a centre of relative rotation. There are three types of pair. (i) There are 12 pairs such as a and β', hinged at the point $a\beta'$, so that the point $a\beta'$ is their centre of rotation. These are the 12 points of the double-four. (ii) There are 12 pairs such as a and β, or a' and β'. Since $a\gamma'\beta\delta'$ are the consecutive sides of a quadrilateral linkwork, the centre of rotation for a and β is the point $\gamma'\delta'$. Hence the centre of rotation for any two pieces which are sides of the quadrilateral $a\beta\gamma\delta$ is the corresponding-opposite vertex of the quadrilateral $a'\beta'\gamma'\delta'$, and *vice versa*. (iii) There are four pairs such as a and a': giving centres of rotation which may be named I_1, I_2, I_3, I_4.

The theorem of collinearity of the relative centres for any three pieces, if applied to a, a', and β, shows I_1 to be collinear with $a'\beta$ and $\gamma'\delta'$. If each of the other five is associated in turn with a and a', it then appears that the joins of corresponding-opposite vertices of the quadrilaterals $a'\beta\gamma\delta$ and $a\beta'\gamma'\delta'$ are all concurrent in the point I_1 and form a pencil in involution. A similar property holds for the other three pairs of quadrilaterals, namely, $a\beta'\gamma\delta$ and $a'\beta\gamma'\delta'$, $a\beta\gamma'\delta$ and $a'\beta'\gamma\delta'$, $a\beta\gamma\delta'$ and $a'\beta'\gamma'\delta$: the centres of the involution pencils being I_2, I_3, I_4.

9. The same property of perspective involution may be seen to be true of four other pairs of quadrilaterals. For the pairs of planes in involution (§ 2) through the line AA', in the octahedral figure, which project the pairs of points B and B', C and C', D and D', give rise to three pairs of lines in the spherical figure, concurrent in the point AA' and in involution. Now the six planes, in order, contain the lines AB, AB', AC, AC', AD, AD'; and these are lines of intersection of planes a, β', γ', δ. Hence in the spherical figure the six arcs pass through the points $\gamma'\delta$, $a\beta'$, $\beta'\delta$, $a\gamma'$, $a\delta$, $\beta'\gamma'$, that is, the pairs of vertices of the quadrilateral

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* Cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 284, § 14.

$\alpha\beta'\gamma'\delta$. Similarly the planes contain the lines $A'B$, $A'B'$, $A'C$, $A'C'$, $A'D$, $A'D'$, and hence the arcs pass through the points $\alpha'\beta$, $\gamma\delta'$, $\alpha'\gamma$, $\beta\delta'$, $\beta\gamma$, $\alpha'\delta'$, these being corresponding-opposite vertices of the quadrilateral $\alpha'\beta\gamma\delta'$ as compared with the former. Hence the joins of corresponding-opposite vertices of the quadrilaterals $\alpha'\beta\gamma\delta'$ and $\alpha\beta'\gamma'\delta$ are concurrent in the point AA' and form a pencil in involution. The same property holds for the other three pairs of quadrilaterals $\alpha\beta'\gamma\delta'$ and $\alpha'\beta\gamma'\delta$, $\alpha\beta\gamma'\delta'$ and $\alpha'\beta'\gamma\delta$, $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$; the centres of the involution pencils being the points BB', CC', DD'.

It may be remarked that the properties of §§ 8–9 are geometrical equivalents: for the involution property of § 7 which gives the figure its other properties allows of an exchange of any corresponding pair of lines, say α and α'. Moreover, from the one figure, by such exchanges of corresponding pairs of lines, eight different double-fours may be formed, each giving rise to a mechanism. It is thus the kinematic aspect only which is responsible for differentiating the 16 quadrilaterals as two distinct sets.

It may also be added that involution theorems corresponding to those of §§ 7–9 are true of the plane figure, consisting of four pairs of straight lines, which is obtained by taking any plane section of the Möbius configuration. If the plane is taken at infinity, the theorems become equivalent to the above spherical theorems which arise kinematically.

10. The first of the three types of continuously deformable octahedron found by Bricard is symmetrical about an axis. The three diagonals AA', BB', CC' are all perpendicularly bisected by a line z, the axis of symmetry. The eight triangular faces consist of four pairs of congruent triangles, each pair being images in z. The figure is geometrically completed by the points D and D' in which the planes of the faces are concurrent in two sets of four. The whole figure may be derived from the foregoing general case (§§ 2–6) by supposing one of the two harmonic transversals to remain a finite line z while the other becomes the line at infinity in any plane perpendicular to it.

That the figure gives a continuously deformable mechanism is apparent on dissecting it into two twin halves, taking the skew quadrilateral $BCB'C'$ as the line of separation. One half then consists of the four plates ABC, ACB', $AB'C'$, $AC'B$ with a common vertex A and consecutive hinge-lines AB, AC, AB', AC', and has one degree of freedom perpetually. The free edges BC, CB', $B'C'$, $C'B$ of the skew quadrilateral $BCB'C'$ are alternately equal, forming a skew isogram. (Here and elsewhere the name "isogram" is used for any four-sided figure, whether plane or skew or spherical, which has its alternate sides of equal length.) Now the diagonals BB', CC' of a skew isogram are bisected by their common normal z. Hence a half-turn about z converts the figure $ABCB'C'$, point for point, into a figure $A'B'C'BC$ which may be supplied by the second half of the dissected mechanism.

The specification of the figure of the octahedron depends on seven parameters: for the twelve sides of the four triangles ABC, ACB', $AB'C'$, $AC'B$ consist of six equal pairs of lengths, but are otherwise arbitrary, and the deformability adds one further parameter. It may be observed that the equalities among the lengths of the sides are such as to allow the four plates (one of each pair) to form a tetrahedron. The necessary inequalities, however, affecting the angles are not necessarily satisfied. Again, and differently, the figure may be obtained by taking an arbitrary triangle and reflecting it in an arbitrarily placed line. This, again, accounts for seven parameters.

11. It is profitable to take another view of the half mechanism $ABCB'C'$. Four plates δ, β', a, γ' may be supposed consecutively hinged along the four concurrent lines AB, AC, AB', AC', but the free edges may be left undetermined. It would then be an elementary problem to determine a set of points B, C, B', C', one on each hinge, such that the lengths of the edges BC, CB', $B'C'$, $C'B$ may be alternately equal. There should be a double infinity of solutions, each giving rise to a position for z, the common normal of BB' and CC', a half-turn about which completes the octahedron. The positions possible for z may be better found, however, from previous results, as follows.

The involution of planes through AA' projecting the three pairs of lines of the four-plane $\delta\beta'a\gamma'$ has as double planes (§ 9) the plane λ, say, through AA' perpendicular to z, and the plane λ', say, through AA' and z. The involution is a symmetric involution, with perpendicular double planes; and its pairs of planes exchange by a half-turn about z as well as by reflection in either λ or λ'. The involution is completed by the pairs of tangent planes through $\lambda\lambda'$ to the quadric cones inscribed in the four-plane. The following consequences arise:—(i) The line $\lambda\lambda'$ is a focal line of one of the cones. (ii) Two of the cones pass through $\lambda\lambda'$, one touching λ and the other touching λ'. (iii) One of the cones has λ as a principal plane, and another cone has λ' as a principal plane.

And so, to construct any one of the axes of symmetry z which may be associated with the four-plane, a plane of symmetry λ of any inscribed quadric cone is taken, and z is then any one of the normals of λ which lie in the companion plane λ'. It may be noted that a figure of four concurrent planes involves five parameters; and these, together with the two needed for fixing z, give seven, as before, for the determination of the octahedron.

12. The relation of the axis z to a set of four coplanar points, say $ABCD$, may also be noticed. A half-turn about z converts the quadrangle $ABCD$ and its plane δ into the quadrangle $A'B'C'D'$ and its plane δ'. The pairs of sides of the two quadrangles meet the line $\delta\delta'$ in the same three pairs of points in involution, namely, the points in which $\delta\delta'$ is cut by α and α', β and β', γ and γ'. The half turn about z, which is normal to $\delta\delta'$, exchanges the two points of each of these pairs: and hence the involution is symmetric, with one double point at infinity and the other the point $z\delta\delta'$. The line $\delta\delta'$ is therefore an asymptote of one of the conics through A, B, C, D; and z is a normal to $\delta\delta'$ at the point of contact of the other conic through A, B, C, D and touching $\delta\delta'$. It is easily seen that this construction, like the others, involves seven parameters.

13. One further construction, associating the axis z with any one of the tetrahedra, say $ABCD'$, may also be given for the sake of its application later (§ 21). It was seen (§ 6) that the skew quadrilaterals $BCB'C'$ and $ADA'D'$ have their sides all generators of a quadric of which the harmonic transversals are polar lines. Here the quadrilaterals are isograms and the axis z is a principal axis of the quadric. It follows that a screw of proper pitch with z as axis has as rays the four lines BC, $B'C'$, AD', $A'D$, these being generators all of the same system. Hence BC and AD', opposite edges of the tetrahedron $ABCD'$, belong as rays to the same screw with axis z. (The same result may also be seen independently by noticing that BC' and $B'C$, being images in z, are polar lines of a screw on z, and that BC and AD' meet both these polar lines and are therefore rays of the screw.) Thus, in relation to the tetrahedron $ABCD'$, the axis z satisfies three conditions; namely, it is the axis of three screws each having a pair of opposite edges as rays. A single infinity of positions of z are associated with the arbitrary tetrahedron $ABCD'$, giving seven parameters for the complete figure as before.

14. From the movement of the octahedron and the results of § 9 may be derived a property of any mechanism of four plates with concurrent hinges. For, consider the movement of the point A' relatively to the pieces of the four-plane with vertex A. Relatively to the plate ABC the point A', vertex of the triangle $A'BC$, describes a circle of which the line

BC is axis. Similarly the point A' describes, relatively to each of the other three plates, a circle of which a side of the isogram $BCB'C'$ is axis. Each circle may be regarded as a figure rigidly connected with its plate, and the four plates concurrent in A' may be removed. Then, during the motion of the four plates concurrent in A, the four circles are always concurrent. The two circles of two adjacent plates are sections of a sphere whose centre is a vertex of the isogram, and so have necessarily two common points. The solitary intersection of the circles of opposite plates is one of these two.

The double infinity of positions of z provided by § 11 give rise to a congruence of such circles associated with each of the plates and cutting the plate orthogonally. The circles of the four congruences are associated in sets of four, which remain concurrent in a point during the movement. The congruences have necessarily the property of being self-inverse in respect to the point A, each circle inverting into some other circle. The examination of the congruences in greater detail need not be developed here.

15. The spherical double-four, of §§ 7–9, may now be noticed, as derived specially from the axially symmetrical octahedron of §§ 10–13. Like the octahedron, the spherical indicatrix itself is also continuously deformable. The symmetry of the octahedron about the axis z implies central symmetry of the double-four about a point Z; but, regarding antipodal points as equivalent, it may be taken as symmetrical about a great circle λ, whose plane is parallel to the plane λ of § 11 and perpendicular to z. And further, it follows that the great circles α, β', γ', δ touch a sphero-conic having λ as one of its three axes; and hence the four pairs of lines of the double-four are tangents to a sphero-conic and symmetric about one of its axes. The figure with central symmetry equivalent to this consists of four pairs of tangents to a sphero-conic at the extremities of four diameters (through any one of the three centres).

That any three pairs of the lines cut each of the other two in involution (§ 7) is true here in virtue of the fact that symmetry about the line λ makes all the four pairs of tangents cut any arbitrary tangent whatsoever in involution.

The system of centres of rotation is given by § 8; the points I_1, I_2, I_3, I_4 being on the axis λ. The involution centres AA', BB', CC', DD' (§ 9) are also on λ. The involution perspectives of §§ 8–9 reduce all to the theorem that, if a quadrilateral circumscribes a sphero-conic, and if the vertices are joined each to the image of the companion vertex reflected in an axis of the conic, then the six lines so drawn intersect the axis in the same point.

The pair of quadrilateral linkworks $\alpha\beta'\delta\gamma'$ and $\alpha'\beta\delta'\gamma$ are images in the line λ. Their hinges account for eight of the twelve points of the double-four. The remaining four are the hinge-points $\alpha\delta'$, $\alpha'\delta$, $\beta\gamma'$, $\beta'\gamma$ connecting the two quadrilaterals. They are the intersections of sides which are corresponding-opposite; and they are the vertices of a crossed isogram having λ as axis of symmetry. The complete mechanism dissects into two parts, similarly, in two other ways; namely, by taking the pair of quadrilaterals $\alpha\gamma'\beta\delta'$ and $\alpha'\gamma\beta'\delta$ or the pair $\alpha\beta'\gamma\delta'$ and $\alpha'\beta\gamma'\delta$.

The quadrilateral mechanism $\alpha\beta'\delta\gamma'$, taken by itself, with the points $\alpha\delta'$, $\beta'\gamma$, $\delta\alpha'$, $\gamma'\beta$ marked on its sides, has the property that these four points remain the vertices of a crossed isogram during the deformation. There are a single infinity of such sets of points, each set being derivable by reflection of the quadrilateral in an axis λ of any conic inscribed in the quadrilateral, and that, too, for any position of the deforming quadrilateral. If the degenerate conic is taken which touches α and δ at their intersection and touches β' and γ' at their intersection, the set of four points obtained are $\alpha\delta$ on α, and δ and $\beta'\gamma'$ on β' and γ'. This process alone, continually repeated as the quadrilateral deforms, is sufficient to define the correspondence on the sides of the quadrilateral of such sets of points as remain corners of a crossed isogram. (It must not, however, be assumed that the whole extent of the sets of points are necessarily thus secured: a limited portion, only, may pass through the special form in which the points of the isogram lie on its axis. For the octahedron itself, the skew isogram $BCB'C'$ may or may not happen to pass through the plane parallelogram form during the motion.)

The property of § 14 has its spherical interpretation. With each side of the spherical quadrilateral $\alpha\beta'\delta\gamma'$ may be associated, as if rigidly connected, a sphero-conic with the side as one axis. The vertices of the isogram of points connecting $\alpha\beta'\delta\gamma'$ with its image $\alpha'\beta\delta'\gamma$ are poles of cyclic arcs of these conics. The four conics remain concurrent in the point AA' during the movement. So, for any arbitrary spherical quadrilateral mechanism there are a single infinity of such sets of four conics, carried by the four sides and perpetually concurrent.

16. Two plane mechanisms, each a double-four, one centrally symmetrical and the other axially symmetrical, are obtainable, as limiting cases, from the spherical figures, by keeping the dimensions finite while enlarging the sphere till it becomes a plane. They occur as varieties of a class of mechanism of eight pieces discussed by Kempe* and Darboux.†

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* *Proc. London Math. Soc.*, 1878, Vol. IX, pp. 133–147.

† *Bulletin des Sciences Mathématiques*, 1879, pp. 151–192.

Fig. 2 represents half the mechanism of the axially symmetrical type. It has been drawn without reference to the conic touching the eight lines

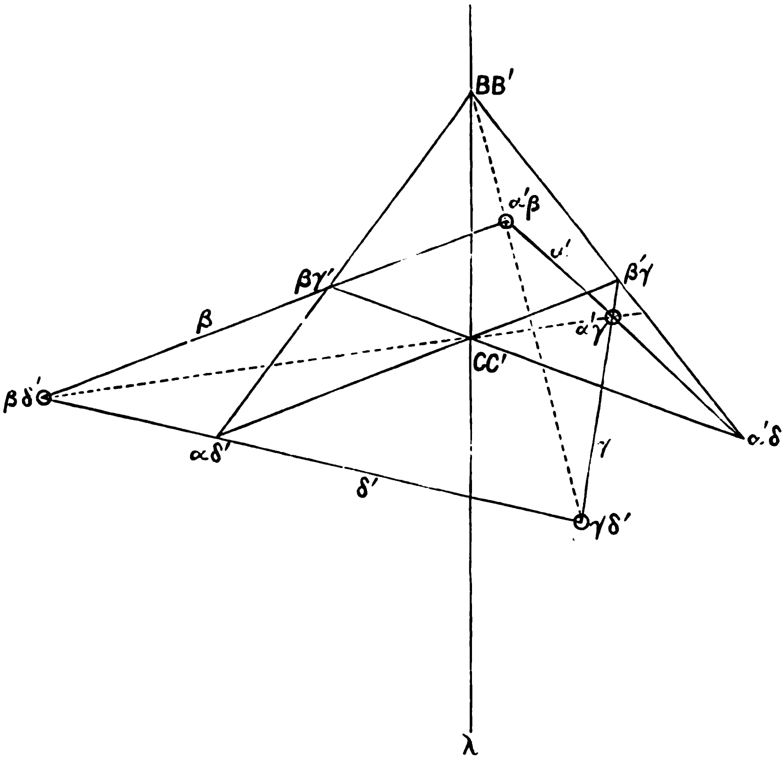

FIG. 2.

by use of the conditions of § 9. Starting with a crossed isogram, let its alternate vertices be $\alpha\delta'$ and $\alpha'\delta$, $\beta\gamma'$ and $\beta'\gamma$, images in a line λ, the intersections of alternate sides being points BB' and CC' on λ. (The precise notation of § 9 is preserved.) Arbitrary lines α', β, γ are drawn through the points $\alpha'\delta$, $\beta\gamma'$, $\beta'\gamma$. The line joining BB' to $\alpha'\beta$ cuts γ in $\gamma\delta'$: and the line joining CC' to $\alpha'\gamma$ cuts β in $\beta\delta'$. Then a line δ' passes through all three points $\gamma\delta'$, $\beta\delta'$, $\alpha\delta'$.

The jointed quadrilateral with consecutive sides α', β, δ', γ has thus its two chief diagonals each concurrent with a pair of opposite sides of the isogram. During the deformation of the quadrilateral these four points, one on each side, remain always vertices of an isogram. To complete the mechanism the lines α', β, γ, δ' are reflected in λ, giving α, β', γ', δ. The symmetric pair of jointed quadrilaterals, pivoted at the vertices of the isogram, give the plane deformable double-four with its twelve hinge-points. (The corresponding figure given by Kempe* seems to be wrongly drawn: for the necessary concurrences of the above lines are neglected.)

It may be easily seen from the results of § 11, and is suitably remarked here, that a section of the axially symmetric octahedron by a plane through the axis of symmetry and perpendicular to one of the diagonals AA', BB', CC' produces a plane figure of the type just described.

17. Some special forms and derivatives of the axially symmetric octahedron may now be briefly noticed. First, as remarked by Bricard, if a pair of opposite plates are removed from a deformable octahedron there remains a closed chain of six triangular plates consecutively hinged and freely deformable. If the plates omitted are ABC and $A'B'C'$, the hinge-lines of the six pieces are the sides of the skew hexagon $AB'CA'BC'$. Hence a skew hexagon symmetrical about an axis, with angles and sides all constant, is a deformable mechanism.

In general a closed chain of six rigid bodies, consecutively hinged, forms a stiff configuration; so that any deformable chain of six pieces is a singular mechanism. The above example, in which the chain consists of plates, may be rendered more general by supposing that the six hinge-lines have symmetry about an axis merely; the conditions of consecutive intersection being set aside. It may be seen that such a chain is deformable. For, suppose the hinge-lines, in order, to be the non-intersecting lines a, b, c, a', b', c', and let z be the axis of symmetry; so that a and a' are images in z, and likewise b and b', and c and c'. Points on the lines a and a' which are images in z may be called corresponding points. Among them occur the ends of the common normal of a, a', and z, other pairs being equidistant from these points. Suppose, now, that the chain is dissected into two twin halves by separation along the hinge-lines a and a'. Each half is then an unclosed chain of three bodies with two degrees of freedom of deformation; and whatever hinge movement, about b and c, is made by one half may be imitated by the same hinge movements of the other half about b' and c'. If these identical deformations are made arbitrarily, the correspondence of points initially noted on the lines a and a' will be upset. But at the expense of one degree of freedom of each triad of bodies (one degree of freedom surviving), the ends of the common normal of a and a' may be kept corresponding points. The two triads may thus, at any stage of their movement, be reunited along the lines a and a' without any alteration of the original fit. It follows that the axially symmetric chain is deformable.

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* *Loc. cit.*, p. 141, Fig. 8.

18. Among the skew hexagons of § 17 the equilateral rectangular hexagon deserves brief but special notice. It is a suitable form for a model, in that the complete cycle of its movement can be executed with no mutual interference of its parts. At six stages of the movement the form of the hexagon is such that the sides may be regarded as the edges of two adjacent faces of a cube, the common edge being omitted. It may be mentioned that with the same material may be formed a skew hexagon which does not appear during the movement; namely, a form given by the six consecutive edges of a cube which do not meet a selected diagonal. This form is stiff.

19. Degenerate forms of the axially symmetrical octahedron occur if a pair of vertices, say A and A', are at infinity. The four plates concurrent in either of these points form an ordinary prismatic mechanism with four parallel hinge-lines. In each of two cases two such mechanisms are combined with all the hinge-lines parallel: a normal section giving one or other of the plane double-fours of § 16. But in a third and more general case the axis of symmetry z to be associated with an arbitrary prismatic four-plate may be neither parallel nor perpendicular to the hinge-lines. Taking plane λ, of § 11, to be the plane at infinity, the companion plane λ' passes through the two lines, say x and y, midway between pairs of opposite hinge-lines, say b and b', c and c'. The plane λ' consequently cuts any normal section of the four-plate in the line of collinearity of the mid-points of the diagonals of the quadrilateral. And the line z may be taken to be any line in this plane, and so is any line meeting the parallels x and y. If the plane through xz normal to z meets b and b' in points B and B', and if the plane through yz normal to z meets c and c' in C and C', then $BCB'C'$ is a skew isogram with z for its axis. A half-turn about z completes the octahedron, which consists of two equal prismatic four-planes hinged along edges forming a skew isogram.

With a given prismatic four-plate it may be regarded as an elementary problem to determine points B, C, B', C' on the hinge-lines b, c, b', c' such that $BC = B'C'$ and $BC' = B'C$. It is easily found that the distances of the points (in order) from any normal plane section differ by an arbitrary constant from

$$\mu + U/\mu, \quad -\mu + V/\mu, \quad \mu - U/\mu, \quad -\mu - V/\mu,$$

where μ is arbitrary, and

$$16U = -p^2+q^2+r^2-s^2,$$

$$16V = \quad p^2+q^2-r^2-s^2;$$

p, q, r, s being the widths bc, cb', $b'c'$, $c'b$ of the plates. This result may be verified as in agreement with the above construction for the axis of symmetry.

20. An interesting problem of rigid geometry has its solution implicitly contained in the properties of the figure of the axially symmetric octahedron; and, without attempting here a complete and exhaustive study of the solution, we may gather such information as is readily forthcoming.

The problem in question is to place two given rigid tetrahedra in such a relative position as to secure that they shall be mutually inscribed. It is at once apparent that, if the tetrahedra are of arbitrary form, the problem is insoluble: for the eight conditions to be satisfied, though reducing to seven in virtue of Möbius' theorem, are in excess of the six degrees of freedom available. If, however, the tetrahedra are congruent, the problem is not only soluble, but has an infinite number of solutions. For it has been shown (§ 13) that with any tetrahedron $ABCD'$ may be associated a single infinity of lines z such that a half-turn about z brings the tetrahedron into the position $A'B'C'D$ with the two tetrahedra mutually inscribed. Hence a continuous movement of the two tetrahedra is possible with the property of mutual inscription maintained.

For present purposes the notation may be changed for convenience and the tetrahedra taken as $PQRS$ and $P'Q'R'S'$. Then, as regards the kinematics of the relative movement, since P' describes a curve in the plane QRS, the normal at P' to the plane QRS is a ray of the screw of the movement; and similarly for each of the other points. Hence the eight lines, drawn one through each vertex normal to the plane through it, are rays of the screw. The line z generates a ruled surface relatively to $PQRS$ such that half-turns of $PQRS$ about the generators give rise to all the positions of $P'Q'R'S'$. It follows that the axis of the screw movement is the common normal of consecutive generators. Hence the screw axis is normal to z, and touches the ruled surface at the point where the line of striction cuts z.

21. The form of the threefold restriction on z which associates it with the tetrahedron (§ 13) may now be modified. If a screw with axis z has

QR and PS as rays, it follows that screws of the same pitch on QR and PS have z as a common ray. Now screws of equal pitch on QR and PS determine (regardless of the magnitude of the pitch) a cylindroid of which QR and PS may be called a pair of symmetric generators, being images in either of the principal generators of the cylindroid; and z must be a normal to some one generator of this cylindroid. Similar results hold for the other pairs of edges, and hence the restriction on z may be put in this fresh form. Namely, each pair of opposite edges of the tetrahedron are taken as a pair of symmetric generators determining a cylindroid; and the line z is then a common normal of three generators, one of each cylindroid.

It may be noticed that the restriction on z which is immediately and obviously necessary consists in making the images of points P, Q, R, S in the line z lie each in the opposite face of the tetrahedron; and then also the image of each face passes through the opposite vertex. But this is a fourfold restriction. The foregoing geometry sufficiently indicates (the verifiable fact) that these four latter conditions are not independent, and that they are equivalent to the three more symmetrically used above.

Lacking any further investigation of the nature of the surface generated by z, the positions of a number of special generators, obtainable by elementary means, as legitimate reflectors, may at least be mentioned. There occur (i) the image, in the centre of the tetrahedron, of the common normal of any pair of opposite edges; (ii) the two perpendicular normals to an edge at its middle point, each equally inclined to the faces through the edge; (iii) any line which can be drawn meeting two opposite edges and bisecting, internally or externally, at each of its extremities, the angle subtended by the opposite edge. (For each pair of opposite edges there appear to be seven of these last lines, and apparently at least three are real.)

22. A simplification occurs in the surface locus of z if the tetrahedron has two pairs of opposite edges equal; say $PQ = RS$ and $QR = PS$. So $PQRS$ is a skew isogram with unequal diagonals PR and QS, having an axis of symmetry x perpendicularly bisecting PR and QS. The cylindroid of which PQ and RS are symmetric generators has x as a principal generator, and so has the cylindroid of which QR and PS are symmetric generators. Hence, if the first cylindroid is turned through one right angle about its axis x (in either sense) then any generator of the new cylindroid is normal to a generator of each of the three cylindroids. Hence, for a tetrahedron of this special type, part of the surface locus of z consists of a cylindroid. Half-turns of the tetrahedron $PQRS$

about the generators of the cylindroid give a continuous series of positions of the moving tetrahedron $P'Q'R'S'$.

To consider the nature of the movement, let the position of a generator z, normal to x, be denoted by the coordinates ζ, θ, with the relation $\zeta = \kappa \sin 2\theta$. A half-turn about this line brings the initial line into the position ζ', θ', where $\zeta' = 2\zeta$ and $\theta' = 2\theta$. Hence the movement in question is a screw movement on the permanent axis x with shift and turn related by the condition $\zeta' = 2\kappa \sin \theta'$. It is at once apparent that, for such a relative movement of two bodies, any point of either lies always on a circular cylinder with axis x, and also on a plane fixed relatively to the other body, and so describes an ellipse. A movement described by Mannheim and Darboux,* allied to the well known trammel motion, gives elliptic paths for all the points of one body relatively to the other. The present movement is a special case, with the trammel motion reduced to rotation about an axis: and, unlike the general case, the two bodies have each the same motion relatively to the other. Any point of either body describes an ellipse relatively to the other.

Incidentally it may be noticed that if half-turns of a fixed point K about the variable generator z give rise to the moving point K' with its elliptic locus, it follows that the mid-point of KK', namely, the foot of the perpendicular from K on z, has also an elliptic locus. This is the well known theorem that the generator-pedal of an arbitrary point with regard to a cylindroid is an ellipse. It attaches itself naturally to the finite screw motion here associated with the cylindroid, and not to the small screw movements more familiarly known in connection with the generators of the cylindroid.

23. If the tetrahedron has each edge equal to the opposite edge, then the results of § 22 occur three times over. The surface locus of the axis of reflection z gives rise to three cylindroids; and it may be shown, in fact, that these account for the whole of the real part of the locus.

The regular tetrahedron, in particular, is included under this last case.

24. A second type of deformable octahedron is symmetrical about a plane. It is of less interest geometrically than the other types, and may be described and dismissed briefly.

Consider four articulated wires forming a plane crossed isogram $BCB'C'$. If BC meets $B'C'$ in X, and BC' meets $B'C$ in Y, the line XY is the axis of symmetry of the isogram, and the plane ζ through

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* *Comptes Rendus*, 1881.

XY perpendicular to the plane of the isogram is the plane of symmetry. Take any point A in the plane ζ and connect it by wires, articulated at A, with B, C, B', C'. Then the isogram may deform and remain in a plane ξ, the point A remaining in the plane ζ. A second point A' may be taken in ζ, and four fresh wires $A'B$, $A'C$, $A'B'$, $A'C'$ added, without affecting the freedom. (Any number of such sets of four wires, indeed, may be added.) The twelve wires constitute the edges of a deformable octahedron, symmetrical about the plane ζ. The diagonal AA' lies in the plane ζ, and the diagonals BB', CC' are perpendicularly bisected by the plane ζ. Like the axially symmetric type, this octahedron has four pairs of equal triangles for its faces, and six pairs of equal edges; and the figure depends for its form on seven parameters. The points D and D', in this case, are the points of intersection of AX with $A'Y$ and AY with $A'X$. The harmonic transversals of the figure are the line XY, in the plane ζ, and the normal to ζ through the intersection of AA' and DD'.

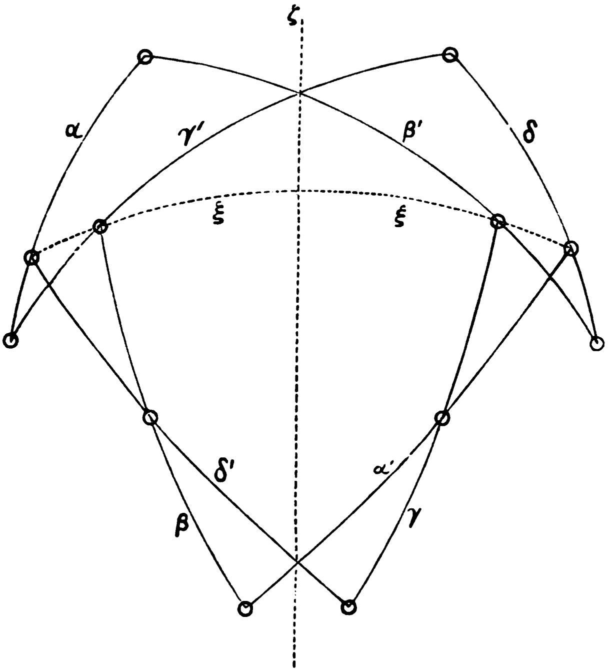

FIG. 3.

The corresponding spherical indicatrix consists of two crossed isograms, with a common line of symmetry ζ, connected at points which remain on a line ξ, normal to ζ, during the deformation. (Fig. 3.)

The corresponding plane mechanism is of the same character, and is one of the class already referred to (§ 16).

25. The third type of deformable octahedron has no centre or axis or plane of symmetry, and may be called the skew deformable octahedron. Its twelve edges are of unequal lengths. It is less easy to realize than the two symmetric types and is of greater geometrical interest. A progressive exploration of the figure, starting from the stage at which Bricard leaves it, and deriving later the spherical indicatrix, would not afford such a clear view as the reverse process. The spherical double-four associated with the octahedron shall be described first; and the octahedron, with this as preliminary basis, may then be built up. Several detailed properties will then be readily derivable.

26. Consider a spherical figure (Fig. 4) consisting of two concentric small circles, centre O, with radii r and r', cutting at an angle 2θ, and two intersecting tangents of lengths a and a' drawn at their extremities.

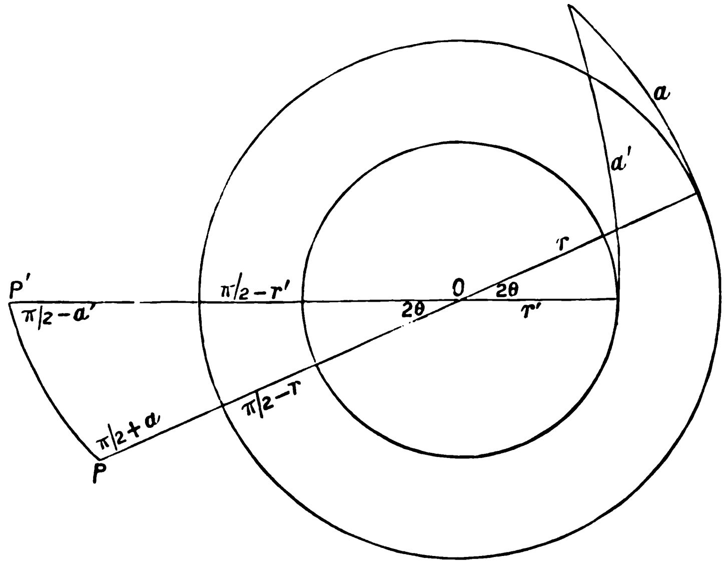

FIG. 4.

Taking the triangle OPP' formed by lines of which PP' is the polar of the point of intersection of the tangents, and the others are the radii produced, the analogies of Napier give

$$\tan\theta \tan\tfrac{1}{2}(a+a') = \sin\tfrac{1}{2}(r-r')/\cos\tfrac{1}{2}(r+r'),$$

$$\tan\theta \cot\tfrac{1}{2}(a-a') = -\cos\tfrac{1}{2}(r-r')/\sin\tfrac{1}{2}(r+r').$$

If r and r' and θ vary, but the ratios

$$\tan\theta : \sin\tfrac{1}{2}(r-r')/\cos\tfrac{1}{2}(r+r') : \cos\tfrac{1}{2}(r-r')/\sin\tfrac{1}{2}(r+r')$$

are kept constant, the sum and difference of a and a' remain constant; and therefore also a and a' themselves. The figure also shows that

$$\cos a \cos r = \cos a' \cos r',$$

and hence $\cos r/\cos r'$ is constant, which is in agreement with the constancy of the last of the above ratios.

Take now two other variable angles ϕ, ψ, and let the further ratios $\tan\theta : \tan\phi : \tan\psi$ remain constant. Then the lengths of the tangents b and b' associated with ϕ are constant, and also c and c' associated with ψ; and, further,

$$\cos a/\cos a' = \cos b/\cos b' = \cos c/\cos c'.$$

Draw (Figs. 5, 6) tangents α, β, γ, δ to the circle of radius r, at points of which the azimuths, measured from an initial radius, are respectively

$$\begin{gathered}\theta-\phi-\psi,\\ -\theta+\phi-\psi,\\ -\theta-\phi+\psi,\\ \theta+\phi+\psi;\end{gathered}$$

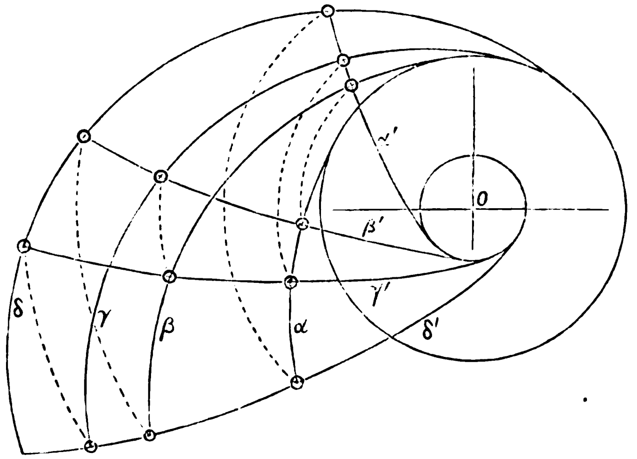

FIG. 5.—Deformable spherical double-four composed of isogram linkworks, shown in stereographic projection. The points lie, as three pairs of couples, in equal and opposite latitudes. (If the points on δ and δ' are replaced by the antipodal points the isograms become crossed isograms.)

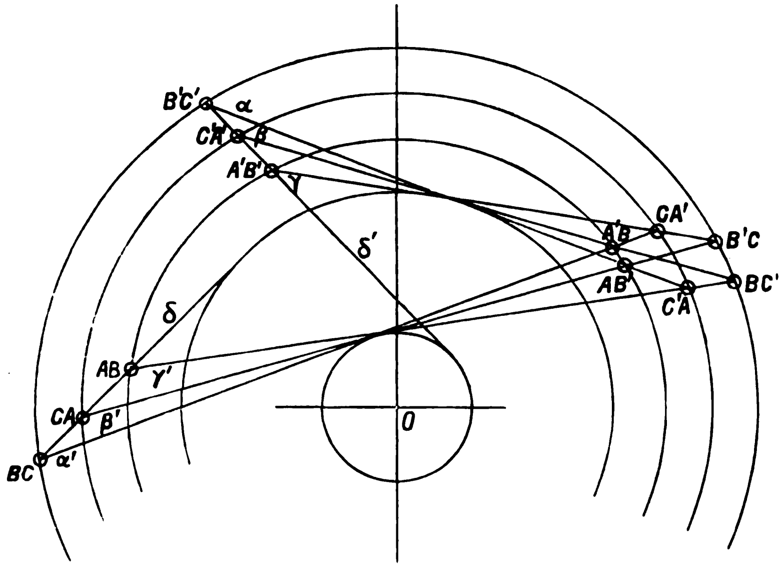

FIG. 6.—Deformable plane double-four composed of crossed isogram linkworks. The figure also sufficiently represents the corresponding spherical double-four (§ 26) of which Fig. 5 shows a modified equivalent.

and draw also tangents α', β', γ', δ' to the circle of radius r' at points with azimuths respectively equal to

$$
\begin{gathered}
-\theta+\phi+\psi, \\
\theta-\phi+\psi, \\
\theta+\phi-\psi, \\
-\theta-\phi-\psi.
\end{gathered}
$$

The tangential distances of the twelve points of the double-four from the circle of radius r are then as shown in the table :—

	α'	β'	γ'	δ'
α		$-c$	$-b$	$+a$
β	$-c$		$-a$	$+b$
γ	$-b$	$-a$		$+c$
δ	$+a$	$+b$	$+c$	

and a similar table gives the tangential distances $\pm a'$, $\pm b'$, $\pm c'$ from the circle of radius r'. The tangents being all constant, their algebraic differences are also constant, and hence the relative distances of the three points on any one of the eight lines are constant. It follows that the variable spherical double-four is a continuously deformable mechanism. During the deformation the quadrilaterals $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ circumscribe concentric circles, and the points of contact are permanently the same points of the bars. The twelve hinge-points consist of three sets of four, each set lying on a circle with the same centre O. The three pairs of quadrilateral mechanisms into which the complete mechanism dissects are all crossed isograms. The figure depends on five parameters, which may, in particular, be taken to be the radii of the five concentric circles.

Since the quadrilateral $\alpha\beta\gamma\delta$ circumscribes a circle, the projection of its pairs of vertices from the centre gives a symmetric involution. The azimuths of the pairs of vertices are $\pm\theta$, $\pm\phi$, $\pm\psi$; and hence the line which counts as initial line is one of the two perpendicular double lines of the involution. The vertices of the quadrilateral $\alpha'\beta'\gamma'\delta'$ give rise to the same involution; vertices which are corresponding-opposite (such as $\beta\gamma$ and $\alpha'\delta'$) lying on the same ray.

As the angular radii r and r' vary, subject to the condition that $\cos r/\cos r'$ is constant, one of the angles assumes all values, and the other oscillates about the value $\frac{1}{2}\pi$. For this mean value of the oscillating angle the other takes alternately the values $\pm\frac{1}{2}\pi$; and on each of these two different occasions the circles are coincident and the eight pieces of the mechanism all lie in line.

A special case of this mechanism occurs if the radii r and r' are equal. The two circles then perpetually coincide and the figure is symmetrical about the initial line.

27. The spherical mechanism of § 26 gives as an ultimate form, when the sphere is flattened, a plane mechanism (Fig. 6) precisely analogous. For the plane figure the ratios

$$\tan\theta : \tan\phi : \tan\psi : r-r' : 1/(r+r')$$

must remain constant; where r and r' are the lengths of the radii of the concentric circles. The lengths of the tangents from the hinge-points satisfy the equations

$$a^2-a'^2 = b^2-b'^2 = c^2-c'^2 = r^2-r'^2.$$

During the deformation one radius varies from $+\infty$ to $-\infty$, the other oscillating from $+\infty$ to a minimum and back again. For each of the two extreme cases the circles becomes two identical straight lines, and the mechanism takes a rectilinear form.

It is of interest to notice that each of these rectilinear forms, besides being a critical shape through which the mechanism of crossed isograms passes, is also a shape from which the mechanism may pass through a series of parallelogram forms (Fig. 7). This is at once apparent from the table of § 26 (used here for rectilinear measurements) and a comparison of the figures. In virtue of the relations

$$a^2-a'^2 = b^2-b'^2 = c^2-c'^2,$$

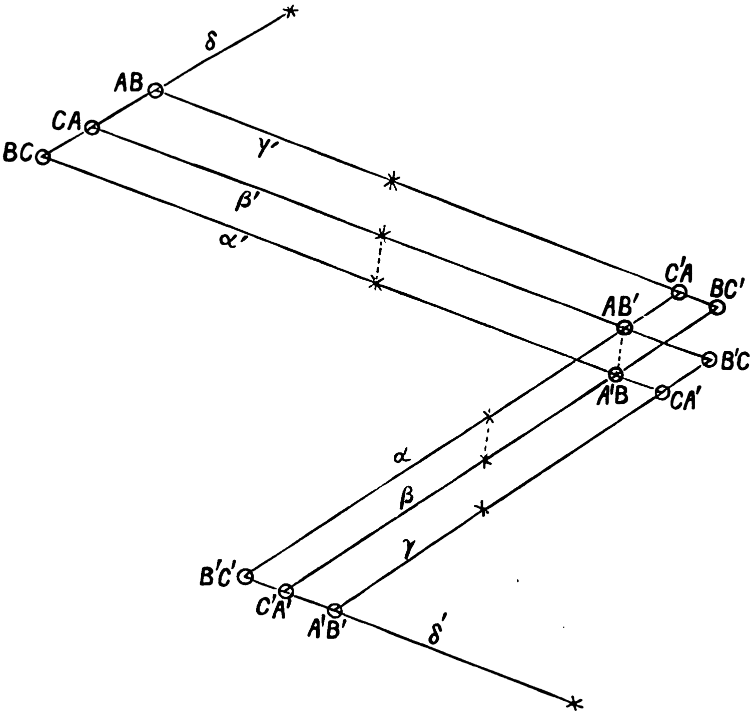

FIG. 7.—Parallelogram linkwork composed of the same material as Fig. 6.

it follows that the pairs of parallelograms $\alpha\beta'\gamma'\delta$ and $\alpha'\beta\gamma\delta'$, $\alpha\beta'\gamma\delta'$ and $\alpha'\beta\gamma'\delta$, $\alpha\beta\gamma'\delta'$ and $\alpha'\beta'\gamma\delta$ are reversely similar. On each rod may also be marked, in the parallelogram figure, the point which remains the point of contact with the circle in the crossed isogram figure. The line joining these points on α and β or on α' and β' is in each case parallel to the diagonal line joining $\alpha\beta'$ to $\alpha'\beta$. Hence the set of contact points on α, β, γ, δ form a quadrangle whose three pairs of sides are parallel to the six diagonal lines of the double four, and similarly for the contact points on α', β', γ', δ'.

A special case of the mechanism arises if the radii r and r' are equal and the two circles coincide. The figure is then symmetrical about an axis of symmetry. This last figure seems to be given by Darboux* where the more general case might have been expected.

Another special case arises if $b = c$ and $b' = -c'$. Two of the circles through the hinge-points then coincide. The sides of the isogram $\alpha\beta'\delta\gamma'$ are all zero, and the four rods are hinged all at a point. The complete mechanism consists of a crossed isogram $\alpha'\beta\delta'\gamma$ with four concurrent rods hinged each to a point on one side.

28. By aid of the spherical double-four of § 26 the skew deformable octahedron may now be readily constructed.

Take two points of space, D and D', on a line whose direction is represented by the point O of the spherical figure. Through D draw planes α, β, γ, δ parallel to those of the corresponding great circles of the spherical figure: and through D' draw planes α', β', γ', δ' parallel to the planes of the second set of great circles. Then the lines $\alpha\delta$ and $\beta'\gamma'$ have (§ 26) the same azimuth relatively to a plane λ through the line DD', and therefore intersect in a point A; and, similarly, $\beta\gamma$ and $\alpha'\delta'$ meet in a point A'. Similarly also for four other sets of concurrent planes, giving points B, B', C, C'. The figure formed is thus a case of the Möbius configuration of § 2.

Consider the octahedron with vertices A, A', B, B', C, C'. The triangle ABC has its angles represented in the figure of the spherical mechanism by the constant arcs separating the points BC, CA, AB on the great circle δ: and similarly for all the other faces of the octahedron. Hence, if the double-four goes through its mechanical movement of deformation, the octahedron derived from it has constant angles for all its triangular faces. The twelve edges have therefore constant ratios, and it is only necessary that the length DD', on which the scale of the figure depends, should suitably vary, in order that the lengths of the edges should remain constant. The deformability of the skew octahedron above constructed is therefore established.

The figure of the spherical mechanism is specified by means of five parameters (§ 26), and hence that of the octahedron by means of six. This is one less than it was in the case of either of the symmetric types of octahedron (§§ 10, 24).

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* *Loc. cit.*, p. 174, Fig. 4.

29. Some simple and essential properties of the skew octahedron may now be described.

The planes α, β, γ, δ through D are tangent planes to a circular cone, say Γ, with vertex D and axis DD': and the planes α', β', γ', δ' are tangent planes to a circular cone Γ' with vertex D' and the same axis. These cones Γ, Γ' cut each other in two circles K_1, K_2 in planes κ_1, κ_2, perpendicular to DD'. Each of the circles K_1, K_2 is touched by all the eight planes α, β, γ, δ, α', β', γ', δ'. Let the plane through DD' corresponding to the line of symmetry of the spherical figure be λ_1, and let λ_2 be the plane through DD' perpendicular to λ_1. Then the section of the figure made by the plane κ_1 is a plane symmetrical double-four (§ 27) with the axis of symmetry $\kappa_1\lambda_1$; and the section by the plane κ_2 is a plane symmetrical double-four with axis of symmetry $\kappa_2\lambda_2$. These lines are perpendicular diameters of the circles K_1, K_2. They are, moreover, the harmonic transversals (§ 2) of the figure. For consider the line $\alpha\alpha'$. The lines $\kappa_1\alpha$ and $\kappa_1\alpha'$ are symmetric about the line $\kappa_1\lambda_1$ and intersect on $\kappa_1\lambda_1$; and hence the line $\alpha\alpha'$ meets $\kappa_1\lambda_1$: similarly $\alpha\alpha'$ meets $\kappa_2\lambda_2$. So the lines $\kappa_1\lambda_1$ and $\kappa_2\lambda_2$ meet all the four lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ and are therefore the harmonic transversals.

Consider the skew quadrilateral $BCB'C'$ with its consecutive sides $\alpha'\delta$, $\beta'\gamma$, $\alpha\delta'$, $\beta\gamma'$. Taking the opposite sides $\beta\gamma'$ and $\beta'\gamma$, a rotation $2(\phi-\psi)$ about DD' brings planes β' and γ to coincide with γ' and β respectively. Hence the line $\beta\gamma'$ is obtainable from the line $\beta'\gamma$ by a rotation $2(\phi-\psi)$ about DD': and similarly $\alpha'\delta$ is got from $\alpha\delta'$ by the rotation $2(\phi+\psi)$ about DD'. Hence the four sides of the skew quadrilateral $BCB'C'$ are generators of a hyperboloid of revolution H_1 with DD' for its axis. Hyperboloids H_2 and H_3, similarly, have as generators the sides of the quadrilaterals $CAC'A'$ and $ABA'B'$. It follows that the pairs of edges AB, AB' and AC, AC' are each of them a pair of image lines in the plane ADD' through the axis DD'. A similar property holds good at each of the eight points. The skew octahedron has, in detail, these well-marked symmetries.

The plane α is the plane of the triangular plate $AB'C'$, whose edges $B'C'$, $C'A$, AB' are respectively generators of H_1, H_2, H_3. Hence the plane α is a tangent plane of all three hyperboloids, and so also for the other planes. Hence each of the three hyperboloids H_1, H_2, H_3 has all the eight planes as tangent planes, and has Γ and Γ' as tangent cones.

The sum of the projections of the sides of the quadrilateral $BCB'C'$ upon DD', the axis of the hyperboloid of revolution containing them, is algebraically zero. Hence if, conveniently, the lengths of the edges are denoted algebraically by BC, CB', $B'C'$, $C'B$, the contained signs of each depending on the sense of the projection on DD', then

$$BC+CB'+B'C'+C'B = 0.$$

And hence the edges of the octahedron satisfy the three relations

$$\left.\begin{aligned} BC+B'C' &= BC'+B'C \\ CA+C'A' &= CA'+C'A \\ AB+A'B' &= AB'+A'B \end{aligned}\right\}. \tag{i}$$

And further relations connecting the lengths come as follows. From the triangles $AB'C'$, $BC'A'$, $CA'B'$ we obtain

$$\frac{AC'}{AB'} = \frac{\sin(a+c)}{\sin(a+b)}, \quad \frac{BA'}{BC'} = \frac{\sin(b+a)}{\sin(b+c)}, \quad \frac{CB'}{CA'} = \frac{\sin(c+b)}{\sin(c+a)},$$

and hence $$BC'.CA'.AB'+B'C.C'A.A'B = 0.$$

And in this way are obtained the four equations

$$\left.\begin{aligned} BC'.CA'.AB'+B'C.C'A.A'B &= 0 \\ BC'.CA.A'B'+B'C.C'A'.AB &= 0 \\ B'C'.CA'.AB+BC.C'A.A'B' &= 0 \\ BC.C'A'.AB'+B'C'.CA.A'B &= 0 \end{aligned}\right\}. \tag{ii}$$

The seven equations (i) and (ii) make the lengths of the twelve edges depend on five parameters, which is (§ 28) the number necessary for specifying the material of the octahedron.

The permanence of the points of contact of the tangents to the circles of the spherical double-four (§ 26) gives a further property of the octahedron. Namely (speaking conveniently of DD' as if kept vertical) the slope lines which may be supposed traced on any plate are permanent, and hence also the level lines. Any horizontal section of the octahedron therefore cuts the eight planes in lines which are permanently level; and it gives therefore a deformable plane double-four of the type described in § 27, the rods being supposed articulated at the junctions. But a distinction is to be observed if hinge-lines are used in place of mere articulations. In the case of the articulated double-four of § 27 (as indeed for any plane mechanism) hinge-lines all normal to the plane may replace the articulations. If, however, the double-four got from the octahedron is supposed cut out, as a thin slice, by a pair of parallel planes close together, then the lines of the edges of the octahedron may be taken as hinge-lines of the new mechanism. The fact that this latter remains flat, although consistent with the property of the articulated double-four, is more extensive in meaning, and is indeed tantamount to the whole theory of the skew octahedron itself.

Among the horizontal sections of the octahedron may be noticed the eight sections which pass each through one of the eight vertices. The plane double-four mechanism so derived is of the type consisting of a crossed isogram and four concurrent bars each linked to a side.

Two other horizontal sections, already mentioned, are made by the planes κ_1, κ_2. They give double-fours of the symmetric type. Any one of the planes, say a, touches the circles K_1, K_2 in these two planes, at points which are permanently the same points of the plate. The projection of the constant slope line, joining the points, on to the line DD' gives the join of the centres of the circles K_1, K_2; and hence the width between the planes κ_1, κ_2 varies as $\cos r$, and therefore also as $\cos r'$. The extremities of parallel diameters of K_1, K_2 are, moreover, vertices of a plane crossed isogram with constant sides, and hence the product of the radii of K_1 and K_2 remains constant. Any one of these equivalent results supplies the information needed (§ 28) in regard to the scale of the figure during deformation.

30. When the spherical indicatrix passes through either of its critical and linear forms (§ 26) the skew octahedron becomes flat, all its eight plates being coplanar; and the chief features of these two plane figures (Figs. 8, 9) may be derived as corollaries of those of the octahedron.

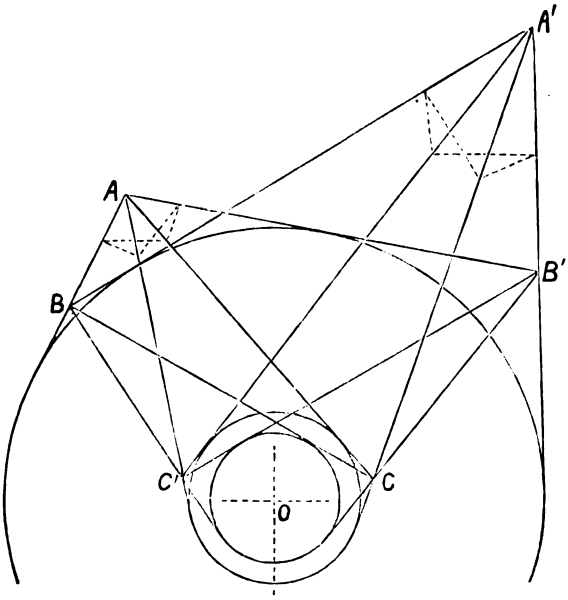

FIG. 8.—One of the two flat forms of a skew deformable octahedron. The dotted isograms show lines, one on each plate, which may remain level lines during the movement.

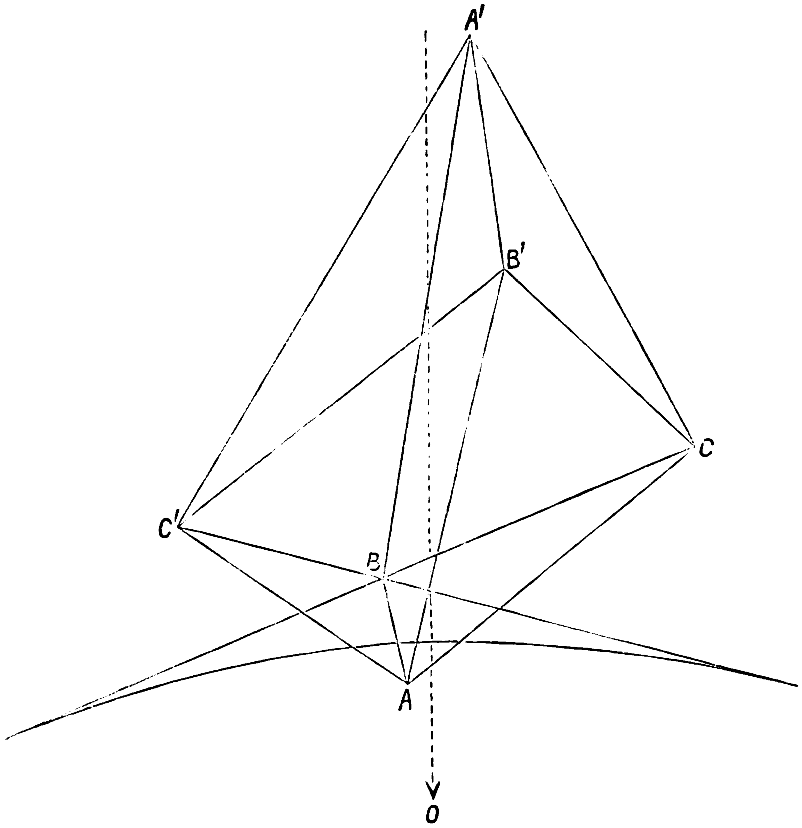

FIG. 9.—The companion form of Fig. 8. The plates α, β, γ, δ have turned over. (A portion of one of the three inscribed circles with centre O is shown; the rest falls outside the limits of the figure.)

At the point A the angle-bisectors of the angle BAB' are the same as those of the angle CAC'. Similarly for each of the other points B, C, A', B', C'. The two companion figures are of the same type as those given by the six centres of rotation connecting four arbitrary positions of a two-dimensional body.* The present figures are special, however, in depending on five parameters instead of six, and have the following additional properties.

Each of the three quadrilaterals $BCB'C'$, $CAC'A'$, $ABA'B'$ has the sum of two sides equal to the sum of the other two sides. Also a circle touches the four sides BC, CB', $B'C'$, $C'B$ of the first quadrilateral, and similarly for the other two. The three circles have a common centre O. Further, the angles AOA', BOB', COC' have the same angle-bisectors.

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* *Proc. London Math. Soc.*, 1910, p. 284, § 13.

Taking one point of each pair a triangle is formed, and the remaining three points form a companion triangle. If non-corresponding vertices are joined a hexagon is constructed. The products of the alternate sides of the hexagon are equal. This occurs in four different ways.

From these properties, and in various ways, the special type of figure, depending in any case on five parameters for its form and size, may be drawn. One method is this:—Take the points O, A, B, C arbitrarily. Then to find A', B', C', reflect CA in CO, AB in AO, BC in BO, and take their respective intersections with the reflections of BA in BO, CB in CO, and AC in AO. This construction is virtually equivalent to that devised by Bricard.*

A more symmetrical construction is derivable from the fact that if the plane double-four of § 27 (Fig. 6) is projected from points D and D' on the normal through the centre O, the four lines α, β, γ, δ being projected from D and the other four from D', a skew deformable octahedron is obtained. And if D and D' are taken very near O, then in the limit the octahedron is constructed in a flat form. The point A of the octahedron is got from the double-four as the intersection of the lines joining D and D' to the points AD and AD' respectively. These last are the points of intersection of opposite sides of the isogram linkwork $\alpha\beta'\delta\gamma'$ with vertices AB, AC, AB', AC', and lie on its axis of symmetry through O. The orthogonal projection of A on the plane lies on the line at a point such that, associated with AD, AD' and O, one of the cross-ratios is equal to OD/OD'. This persists in the limit when D and D' coincide with O and A is in the plane. Hence, on the axis of the crossed isogram linkwork AB, AC, AB', AC', a point A is taken giving an assigned value to the cross-ratio (O, A, AD, AD'), and similarly in turn for each of the six isograms. The set of points A, B, C, A', B', C' thus symmetrically constructed give the figure of a deformable skew octahedron in one of its flat forms. (The number of parameters involved in this construction is, including the value of the cross-ratio, six instead of five: but the extra parameter is accounted for by the fact that the flat octahedron is here derived from a section of it by its own plane, and this section is one of a single infinity of different figures all of which would give the same octahedron.)

Another symmetrical method, avoiding the use of a cross-ratio, depends on a representation of the octahedron in one plane diagram which may be got by taking the sections through D and D' and projecting either on to the other. The plane α gives lines a_0 and a, say, and the plane α' gives lines α' and α_0'. The diagram consists of lines α, β, γ, δ, α', β', γ', δ' forming a plane double-four, together with lines α_0, β_0, γ_0, δ_0, α_0', β_0', γ_0', δ_0', concurrent in the centre and respectively parallel to the lines of the double-four. The point A of the octahedron, being the point common to planes α, β', γ', δ, is represented in the plane diagram by the point of concurrence of the joins of corresponding vertices of the quadrilaterals $\alpha\beta_0'\gamma_0'\delta$ and $\alpha_0\beta'\gamma'\delta_0$. These are similar isograms with a common axis, and the point A is their centre of similitude. The set of points A, B, C, A', B', C' are thus immediately derivable from the figure of the plane double-four: and if D and D' coincide (which does not affect the construction) the six points are the vertices of a flat octahedron.

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* *Loc. cit.*, p. 144.

31. Two instances may be given in which skew deformable octahedra may be united to form a compound mechanism.

First, taking a plane double-four (Fig. 6), projection from points D_1 and D_1' on its axis gives the octahedron $\alpha_1\beta_1\gamma_1\delta_1\alpha_1'\beta_1'\gamma_1'\delta_1'$, say the octahedron Ω_1: and projection from other points D_2D_2' on the axis gives an octahedron Ω_2. These two octahedra may then be united by hinges along the lines of the plane double-four. The compound mechanism consists of 16 plates with 32 hinges, four in each plate: the hinge lines being the 12 hinges of each octahedron, and the eight hinges of connection $\alpha_1\alpha_2$, etc. The set of eight plates $\alpha_1\beta_1\gamma_2'\delta_2'$ and $\beta_2\alpha_2\delta_1'\gamma_1'$ constitute a deformable octahedron, each, in the order named, being hinged to the non-corresponding three of the other set of four. It is an octahedron of the type of § 24. There are six such octahedra included in the compound mechanism.

Another compound mechanism may be built up of four skew octahedra. Let one octahedron be Ω_1: let its images in the perpendicular principal planes λ_1 and λ_2 through the axis DD' be Ω_2 and Ω_4: and let the common image of either of these last be Ω_3. So that a half-turn about DD' brings Ω_1 to coincide with Ω_3 and Ω_2 with Ω_4. The four planes $\alpha_1\alpha_2'\alpha_3\alpha_4'$ touch the circles K_1, K_2 at the ends of parallel diameters, at points forming the plane crossed isogram of § 29: and the four plates may be supposed consecutively hinged along the tangents at those points. Similarly for $\alpha_1'\alpha_2\alpha_3'\alpha_4$, and so also for the other sets of four. The compound mechanism consists of 32 plates with 80 hinges, five in each plate: namely, 12 for each of the four octahedra, and 16 hinges of connection, tangents to each of the circles. The set of eight plates $\alpha_1\alpha_3\beta_2\beta_4$ and $\beta_3'\beta_1'\alpha_4'\alpha_2'$ constitute a deformable octahedron: each plate being hinged to the non-corresponding three of the other set of four. It is a special case of the skew type. There are 12 such octahedra included in the compound mechanism.

32. The questions of obvious interest which suggest themselves in connection with the deformable octahedral mechanisms are far from exhausted by what has been exhibited in this paper. To mention only a few: in regard to the relative motion of opposite faces, for any one of the three types, it is natural to enquire concerning the locus of any point, the envelope of each plate relatively to the other, and the nature of the axodes of the relative screw motions. The possibility (mentioned in § 18) of forming a stiff octahedron with the same material and the same connectivity as a deformable octahedron has not been treated. The spherical mechanisms of §§ 15, 24, 26 belong presumably to a class of such mechanisms, as yet unexplored: and similarly for closed chains of six pieces. The movement of §§ 20–23 merits, too, a more complete study.

The recommendation by Mannheim* of the movements associated with the octahedra as " dignes des efforts des géomètres "—an encouragement which reached the author only when the greater part of the above work had been completed—is still not seriously impaired.

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* *Liouville*, 1897, p. 150.

ON THE NATURE OF THE SUCCESSIONS FORMED BY THE COEFFICIENTS OF A FOURIER SERIES

By W. H. YOUNG, Sc.D., F.R.S.

[Received and Read June 8th, 1911.]

1. Given two ordered successions of constants

$$a_1,\ a_2,\ \ldots,$$
$$b_1,\ b_2,\ \ldots,$$

what conditions must hold, in order that a summable function may exist of which they are the Fourier constants?

It is known that, if Σa_n^2 and Σb_n^2 both converge, such a function exists, namely, one whose square is summable.* Thus a sufficient condition is known. On the other hand, the consideration of the integrated series of a Fourier series shews that, if, as usual, we regard the b's as the coefficients of the sine terms, it is a necessary condition that $\Sigma b_n/n$ should converge. A less easily obtained, but even better known, necessary condition is that a_n and b_n should each have the unique limit zero, as n increases indefinitely.

If the function in question is to have bounded variation, it is a necessary condition that Σa_n should converge, and that $\Sigma b_n/n^q$ should converge for all positive values of q. On the other hand, the convergence of Σa_n, as well as Σb_n, is neither a necessary, nor a sufficient condition, that some function should exist of which a_n and b_n are the Fourier constants. This condition is certainly not necessary, since $\Sigma n^{-q} \sin nx$ and $\Sigma n^{-q} \cos nx$ are both ordinary Fourier series. That it is not sufficient follows from the fact that the series $\Sigma (-)^n \sin nx/\log n$ is not a Fourier series, since its integrated series diverges at the point $x = \pi$.

Returning to the sufficient condition first stated, it will be remarked that in this condition there is no distinction between the a's and the b's. Moreover an infinite number of interchanges may be made among the a's by themselves, and among the b's by themselves, without affecting the truth of the condition. Further, any finite or infinite number of the constants may be omitted, and the gaps filled up by means of suitable movements in the remaining constants. No such change will affect the fact that the trigonometrical series considered is the Fourier series of a function whose square is summable.

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* F. Riesz, "Sur les systèmes orthogonaux de fonctions," 1907, *Comptes Rendus*. "Ueber orthogonale Funktionensysteme," *Göttinger Nachrichten*, 1907, pp. 116–122. E. Fischer, "Sur la convergence en moyenne," 1907, *Comptes Rendus*.

The main object of the present paper is to shew that this simplicity at once disappears, equally whether we occupy ourselves with the question whether a given series is the Fourier series of a function whose properties are more restricted than those of a function whose square is summable, or whether we examine the conditions that must hold in order that two successions of constants may be Fourier constants at all. For this purpose I first point out the essential nature of the distinction between the a's and the b's. Mere inspection of two of the series of Bernoulli serves to shew that the Fourier series of a function of bounded variation may cease to be the Fourier series of such a function, when the a's and b's are interchanged. I shew that even if the function of bounded variation is an integral, the effect of such a change may be to render the new series divergent. I further shew that the omission of a single constant, accompanied by the moving up of the subsequent constants, may equally well produce a similar effect.

On the other hand, it appears that, if we consider Fourier series in general, an interchange of the a's and b's may deprive the series of its Fourier character altogether, and that the same result may equally well follow from the omission of a single constant.

The only obvious properties then shared by the successions of the a's and the b's with the constants of a convergent series are (1) that of converging to zero, (2) that of allowing any finite number of the terms to be removed and replaced by any other constants. In particular it is plain that no condition depending on the convergence alone of series involving the a's and the b's can possibly constitute a necessary and sufficient condition that a given trigonometrical series should be a Fourier series, or that it should be the Fourier series of, for example, a function of bounded variation.

I terminate the paper by proving the following theorem, which serves to illustrate the distinction between the a's and the b's :—

If $\Sigma(a_n \cos nt + b_n \sin nt)$ *and* $\Sigma(b_n \cos nt - a_n \sin nt)$ *are respectively the Fourier series of any function* $f(t)$ *and its allied series, and* A_n, B_n *are the Fourier constants of another function, then*

$$\Sigma A_n (a_n \cos nt + b_n \sin nt)$$

and

$$\Sigma B_n (b_n \cos nt - a_n \sin nt)$$

are both Fourier series, and this equally whether the allied series of the Fourier series of $f(x)$ *is itself a Fourier series, or not.*

2. To convince ourselves that the allied series of the Fourier series of a function of bounded variation, though of course necessarily a Fourier series, need not be the Fourier series of a function of bounded variation, we have only to consider the two series investigated by D. Bernoulli,

$$\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}\tfrac{1}{2}x \sim \sum_{n=1} n^{-1}\cos nx$$

and

$$f(x) \sim \sum_{n=1} n^{-1}\sin nx,$$

where, if $0 < x$, $f(x) = \frac{1}{2}(\pi - x)$, and, if $x > 0$, $f(x) = -\frac{1}{2}(\pi + x)$.

Here the second series has for Fourier function a function of bounded variation, whereas its allied series, viz., the former of the two series, corresponds to a function whose variation is not bounded in any interval containing the origin.

3. The point raised in the preceding article is most simply elucidated by mere inspection of the series of Bernoulli there cited. The example given in the present article, which will be utilised in the sequel, might have been used for the same purpose. It requires, however, more than mere inspection to perceive this. The principle used depends on the necessary and sufficient condition, which I have lately obtained,* in order that the allied series of the Fourier series of a function $F(x)$ of bounded variation should converge at the point x, viz., that

$$\int_{\pi/n}^{1} \frac{1}{t}\,[F(x+t) - F(x-t)]\,dt$$

should have a unique finite limit as the inferior limit of integration approaches zero. If the function $F(x)$ of bounded variation be such that this limit is not unique, or is infinite in value, it follows that the allied series of the Fourier series of $F(x)$ has as Fourier function one whose variation cannot be bounded. It will be found that this is the case with the function which is equal to

$$\frac{\log\log\dfrac{1}{|t|}}{\log\dfrac{1}{|t|}}\operatorname{sgn} t,$$

---

* "Zur Theorie der verwandten Reihe einer Fourier'schen Reihe," 1911, *Münchener Bericht.*

in the neighbourhood of the origin. Writing, for shortness, y for $\log 1/t$, we have

$$F(t) = -F(-t) = \log y/y \quad (0 < t),$$

and we may put $F(0) = \underset{t \to 0}{\mathrm{Lt}}\, F(t) = \underset{y \to +\infty}{\mathrm{Lt}} \log y/y = 0.$

Then, if
$$f(t) = f(-t) = \frac{dF}{dt} = (-1+\log y)/ty^2 \quad (0 < t),$$
$$f(0) = 0,$$

$f(t)$ is bounded in the interval (ϵ, t), where $0 < \epsilon < t$, and therefore by Lebesgue's theorem,

$$F(t) - F(\epsilon) = \int_\epsilon^t f(t)\,dt.$$

Since $F(t)$ is continuous at $t = 0$, we get, letting ϵ approach zero as limit,

$$F(t) = \int_0^t f(t)\,dt,$$

the latter integral existing in the Lebesgue sense, since f is positive, provided t is sufficiently small, *i.e.*, provided $0 < t < e^{-e}$, e being the base of the system of logarithms we are using. Also

$$F(-t) = -F(t) = \int_0^{-t} f(t)\,dt.$$

Again, writing ϵ' for $\log 1/\epsilon$,

$$\int_\epsilon^t \frac{F(t)}{t}\,dt = -\int_{\epsilon'}^{y} \frac{\log y}{y}\,dy = \tfrac{1}{2}(\log \epsilon')^2 - \tfrac{1}{2}(\log y)^2.$$

Now as ϵ approaches zero, ϵ' approaches $+\infty$, and therefore

$$\underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t \frac{F(t)}{t}\,dt = +\infty.$$

Thus
$$\underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t \frac{F(t) - F(-t)}{t}\,dt = +\infty.$$

Thus this limit is infinite, and therefore the allied series of the Fourier series of $F(x)$ diverges at the point $x = 0$, and hence the function of which it is the Fourier series cannot have bounded variation.

4. A slight modification of the function $F(t)$ just given enables us to shew that the allied series of a Fourier series need not be a Fourier series at all.

Let us write $$t = \frac{1}{e}\sin \tfrac{1}{2}u,$$

$$G(u) = F(t).$$

Then $G(u)$ is an integral which has the property that

$$G(\pi) = G(-\pi) = 0.$$

Hence, $G(u)$ being a periodic integral, the derived series of its Fourier series is a Fourier series, namely, it is the Fourier series of the function

$$g(u) = \frac{dG}{du} = \frac{d}{du}\,\frac{\log\log(e \operatorname{cosec} \frac{1}{2}u)}{\log(e \operatorname{cosec} \frac{1}{2}u)} = g(-u) \quad (0 < u).$$

But the allied series of the Fourier series of $g(u)$ has for integrated series the allied series of the Fourier series of $G(u)$, and this certainly does not converge to an integral, for, like the allied series of $F(t)$, it diverges at the origin, since

$$\int_\eta^u \frac{G(u)}{u}\,du = \int_\epsilon^t \frac{F'(t)}{t}\,\frac{2et}{\sqrt{1-e^2t^2}\,\sin^{-1} et}\,dt = 2\int_\epsilon^t \frac{F'(t)}{t}\,\{1+tP(t)\}\,dt,$$

where $P(t)$ is a bounded function of t, and therefore, by § 2,

$$\operatorname*{Lt}_{\eta\to 0}\int_\eta^u \frac{G(u)}{u}\,du = +\infty,$$

and also $$\operatorname*{Lt}_{\eta\to 0}\int_\eta^u \frac{G(u)-G(-u)}{u}\,du = +\infty.$$

Hence the allied series of the Fourier series of $G(u)$ does not converge to an integral, and therefore the allied series of the Fourier series of $g(u)$ is not a Fourier series.

5. We now consider the question of the possibility of omitting one or more of the constants of a Fourier series, and moving up the others so as to fill the gap. Since any finite number of the constants may be made zero without disturbing the Fourier character of the series, it is clearly sufficient to examine the character of the series

$$\sum_{n=1}^{\infty}\{a_{n+1}\cos nt + b_{n+1}\sin nt\}, \tag{1}$$

and, we may add, of the series

$$\sum_{n=2}^{\infty}\{a_{n-1}\cos nt + b_{n-1}\sin nt\}. \tag{2}$$

Let $$f(t) \sim \sum_{n=1}^{\infty}\{a_n\cos nt + b_n\sin nt\},$$

then $$2f(t)\cos t \sim \sum_{n=0}^{\infty} \{(a_{n+1}+a_{n-1})\cos nt+(b_{n+1}+b_{n-1})\sin nt\}, \tag{3}$$

$$2f(t)\sin t \sim \sum_{n=0}^{\infty} \{(b_{n+1}-b_{n-1})\cos nt-(a_{n+1}-a_{n-1})\sin nt\}. \tag{4}$$

By (3) we see that the necessary and sufficient condition that (1) and therefore also (2) should be Fourier series, is that

$$\sum_{n=1}^{\infty} \{(a_{n+1}-a_{n-1})\cos nt+(b_{n+1}-b_{n-1})\sin nt\} \tag{5}$$

should be a Fourier series. But (5) is the allied series of (4).

Hence, if the allied series of the Fourier series of $f(t)\sin t$ is not a Fourier series, *i.e.*, if the allied series of the Fourier series of $\int f(t)\sin t\,dt$ does not converge to an integral, (1) and (2) are not Fourier series.

The following example (§ 6) shews that (1) and (2) are not in general Fourier series. It appears also, as is shewn by another example (§ 7), that the process here investigated does not in general lead from the Fourier series of a function of bounded variation to the Fourier series of another function of bounded variation. On this question no light is thrown by the series of Bernoulli considered in § 2. In fact the Fourier series $\Sigma n^{-1}\sin t$, whose Fourier function $f(t)$ has bounded variation, leads to the Fourier series of other functions of bounded variation, namely,

$$\pm \sin t \log \tfrac{1}{4}\operatorname{cosec}^2 \tfrac{1}{2}t+2f(t)\cos t \sim \Sigma n^{-1}\sin(n\pm 1)\,t.$$

6. Let us then write $u=v-c$, and, using the function $g(u)$ constructed in § 4, let

$$\left.\begin{aligned} h(v) = g(u) &= \frac{d}{du}\,\frac{\log\log e\operatorname{cosec}\frac{1}{2}u}{\log e\operatorname{cosec}\frac{1}{2}u} \\ h(c+u) &= h(c-u) \end{aligned}\right\} \quad (0<u\leqslant \pi),$$

$$h(c)=0 \quad\text{and}\quad h(v\pm 2\pi)=h(v).$$

Writing, for brevity, $$H(v)=\int_0^v h(v)\sin v\,dv,$$

we have

$$H(c+t)-H(c-t)=\int_{c-t}^{c+t} h(v)\sin v\,dv=\int_{-t}^{t} h(u+c)\sin(u+c)\,du,$$

and therefore, taking account of the nature of h, is equal to

$$2\sin c\int_0^t h(c+u)\cos u\,du.$$

Hence

$$\{H(c+t)-H(c-t)\}/t$$

$$= 2\sin c \int_0^t h(c+u)\,du/t - 2\sin c \int_0^t h(c+u)(1-\cos u)\,du/t$$

$$= 2\sin c\, P(t) - 2\sin c\, Q(t),$$

where $P(t) = \int_0^t h(c+u)\,du/t = \int_0^t g(u)\,du/t = \dfrac{\log\log e \operatorname{cosec} \frac{1}{2}t}{t \log e \operatorname{cosec} \frac{1}{2}t}$,

so that, by §3, $\underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t P(t)\,dt = +\infty$,

and $Q(t) = \int_0^t g(u)\, 2\sin^2 \frac{1}{2}u\,du/t = \int_0^t 2\sin^2 \frac{1}{2}u \dfrac{d}{du} \dfrac{\log\log e \operatorname{cosec} \frac{1}{2}u}{t \log e \operatorname{cosec} \frac{1}{2}u}\,du.$

This last integrand is bounded in the closed interval $(0, t)$; let us denote it by $q(u)/t$. Then

$$Q(t) = \frac{1}{t}\int_0^t q(u)\,du, \qquad \underset{u=0}{\mathrm{Lt}}\, q(u) = 0,$$

so that $Q(t)$ is, like $q(t)$, bounded for all values of t including zero. Hence

$$\underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t Q(t)\,dt = \int_0^t Q(t)\,dt,$$

so that this limit is unique and finite for each value of t, and lies, if t varies, between finite upper and lower bounds.

We have therefore proved that

$$\int_\epsilon^t [\{H(c+t)-H(c-t)\}/t]\,dt$$

has, as ϵ approaches zero, the unique limit $+\infty$. Hence the allied series of the function $H(t)$ diverges properly at the point c, and cannot therefore converge to an integral. Therefore the series got by differentiating it term-by-term is not a Fourier series; that is, the allied series of the Fourier series of $h(t)\sin t$ is not a Fourier series; and therefore, by what was shewn above, the Fourier series of $h(t)$ ceases to be a Fourier series, if we omit a_1 and b_1, and move down the remaining constants each one place, or if we move each of the constants on one place, and omit the terms $\cos t$ and $\sin t$.

This proves that the process considered in the present article, and at the beginning of § 5, does not in general preserve the Fourier character of the successions of Fourier constants.

7. Next, retaining the notation $v = u+c$, let us write

$$K(v) = G(u) = \frac{\log\log(e \operatorname{cosec} \tfrac{1}{2}u)}{\log(e \operatorname{cosec} \tfrac{1}{2}u)} \quad (0 < u),$$

$$K(c+u) = -K(c-u), \quad K(c) = 0.$$

Then $K(c+u)\sin(c+u) - K(c-u)\sin(c-u) = 2K(v)\sin c\cos u$;

and therefore

$$\underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t \frac{1}{u}\left[K(c+u)\sin(c+u) - K(c-u)\sin(c-u)\right] du$$
$$= \underset{\epsilon=0}{\mathrm{Lt}} \int_\epsilon^t \frac{\cos u}{u} K(v)\, du = \underset{\epsilon-0}{\mathrm{Lt}} \int_\epsilon^t \frac{\cos u}{u} G(u) du = +\infty,$$

by § 4, since $\dfrac{1-\cos u}{u} G(u)$ is a bounded function of u in the closed interval $(0, t)$, and has therefore a finite integral over this interval.

Hence the allied series of the periodic integral $K(v)$, although of course a Fourier series, diverges at the point c, and cannot therefore be the Fourier series even of a function of bounded variation.

8. It remains to prove the following theorem given in the introduction.

THEOREM.—*If* $\Sigma(a_n\cos nt + b_n\sin nt)$ *and* $\Sigma(b_n\cos nt - a_n\sin nt)$ *are respectively the Fourier series of any summable function* $f(t)$ *and its allied series, and* A_n, B_n *are the Fourier constants of another summable function* $g(x)$, *then*

$$\Sigma(a_n A_n \cos nt + b_n A_n \sin nt)$$

and

$$\Sigma(b_n B_n \cos nt - a_n B_n \sin nt)$$

are both Fourier series, and this equally whether the allied series of the Fourier series of $f(t)$ *is itself a Fourier series, or not.*

Writing, as usual, $F(x)$ for $\int_0^x f(x)\,dx$,

$$F(x \pm t) = C + \Sigma\left[\cos nt(a_n\sin nx - b_n\cos nx) \mp \sin nt(a_n\cos nx + b_n\sin nx)\right]/n.$$

Now F is a continuous function of bounded variation, and $g(x)$ is summable, therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(x\pm t)\, g(t) dt$$
$$= \Sigma\left[A_n(a_n\sin nx - b_n\cos nx) \mp B_n(a_n\cos nx + b_n\sin nx)\right]/n.$$

Thus it is sufficient to shew that the functions of x represented by the left-hand side of this equation are both integrals.

Now, since $F(x)$ is an integral, we can, after choosing e, find E such that the total increment of $F(x+t)$ over every set of non-overlapping intervals on the x-axis of content less than E is less than e. This is not only true for each fixed value of t, but it remains true when t varies, since the increment of $F(x+t)$ over an interval (x_i, x_i+h_i) is the same as the increment of $F(x)$ over the interval (x_i+t, x_i+t+h_i), and therefore the total increment of $F(x+t)$ over a certain set of intervals of content less than E is equal to that of $F(x)$ over another set of intervals of content less than E, and is therefore less than e, whatever value t may have.

Hence it follows that the total increment of $\int_{-\pi}^{\pi} F(x+t)\, g(t)\, dt$ over a set of intervals of content less than E is less than $e\int_{-\pi}^{\pi} |g(t)|\, dt$, and is therefore as small as we please. Thus $\int_{-\pi}^{\pi} F(x+t)\, g(t)\, dt$ is an integral when regarded as a function of x, and, in like manner, the same is true of $\int_{-\pi}^{\pi} F(x-t) g(t)\, dt$. The same is therefore true of the sum and difference of these two functions, that is, the series

$$\Sigma A_n (a_n \sin nx - b_n \cos nx)/n$$

and

$$\Sigma B_n (a_n \cos nx + b_n \sin nx)/n$$

converge throughout the closed interval $(-\pi, \pi)$ to integrals. But this is the necessary and sufficient condition that the derived series of these two series should be Fourier series, which proves the theorem.

ON THE ZEROS OF JACOBIAN FUNCTIONS

By H. F. BAKER.

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M. POINCARÉ obtained, in 1883 (*Bull. de Sc. Math. de France*, t. XI, p. 132), a formula for the number and sum of the common solutions of two theta functions of two variables. Herr Wirtinger, in 1896 (*Monatsh. f. Math. u. Physik*, VII, Jahrg., pp. 1–25), obtained the corresponding results for n Jacobian functions of n variables, but under the condition that only one bilinear relation connects the periods of any two arguments. M. Humbert, in 1899 (*Liouville's Jour. de Math.*, 1899, pp. 254, 270), obtained, for two Jacobian functions of two variables based on the periods arising on a Riemann surface when subject to a further bilinear relation —so called intermediary functions—a formula for the number of solutions, deduced from Poincaré's formula for theta functions.

It remained, unless unknown to me this has already been done, to ascertain what bilinear relations among the periods, in addition to the universal bilinear relation, are necessary for the existence of Jacobian functions of n variables based on the most general possible periods; and to obtain a formula for the number, and for the sum, of the common solutions of n such functions. The results are contained in the following pages (see p. 373).

In order to make clear the fundamental assumptions it is necessary to give references, and I have ventured to refer to my own volume, *Multiply-Periodic Functions*, Cambridge, 1907, using the abbreviation *M.P.F.* This volume contains an exposition of Wirtinger's paper quoted above, which is also the basis of the process here adopted, but it will be seen that in what follows, though the case dealt with is much more general, a considerable simplification is introduced into the inductive argument. Further, as the notation of matrices, which is used throughout, offers difficulties for some readers, I have given details of the work and examples, in some cases at greater length than is strictly necessary; and for similar reasons have included two preliminary sections dealing with formulæ which essentially are well known.

The sections are as follows :—

§ 1. Known Results for Symmetric Functions, pp. 354–356.
2. A Relation connecting a System of Matrices, pp. 357–360.
3. The Conditions for the Existence of a Jacobian Function, pp. 360–372.
4. The Number and Sum of the Common Zeros of n Jacobian Functions of n Variables, pp. 372–388.
5. Some Particular Cases, pp. 388–395.

1. *Known Results for Symmetric Functions.*

Considering n fundamental quantities (roots) $\alpha, \beta, \gamma, \ldots, \lambda$, and a set of positive integral exponents $p, q, r, \ldots$, or $p_1, p_2, p_3, \ldots$, we denote $\Sigma\alpha^p$, extended to all the roots, by (p); we denote $\Sigma\alpha^p\beta^q$, extended to every two different roots, by (p, q); similarly, $\Sigma\alpha^p\beta^q\gamma^r$ by (p, q, r); and so on. Then, by $(p, q)_\alpha$ is meant (p, q), extended to all the roots except α; by $(p, q, r)_\alpha$ is meant (p, q, r), extended to all the roots except α; and so on.

We manifestly have

$$(p, q) = (p, q)_\alpha + \alpha^p(q)_\alpha + \alpha^q(p)_\alpha, \quad (q) = (q)_\alpha + \alpha^q, \quad (p) = (p)_\alpha + \alpha^p,$$

and hence $$(p, q)_\alpha = (p, q) - \alpha^p(q) - \alpha^q(p) + 2\alpha^{p+q}.$$

Again, using this last formula to express $(q, r)_\alpha$, $(r, p)_\alpha$, and $(p, q)_\alpha$, and, noticing that

$$(p, q, r) = (p, q, r)_\alpha + \alpha^p(q, r)_\alpha + \alpha^q(r, p)_\alpha + \alpha^r(p, q)_\alpha,$$

we deduce that

$$(p, q, r)_\alpha = (p, q, r) - [\alpha^p(q, r) + \alpha^q(r, p) + \alpha^r(p, q)] \\ + (2!)\,[\alpha^{q+r}(p) + \alpha^{r+p}(q) + \alpha^{p+q}(r)] - (3!)\,\alpha^{p+q+r}.$$

In general, by the obvious formula,

$$(p_1, p_2, \ldots, p_k) \\ = (p_1, p_2, \ldots, p_k)_\alpha + \alpha^{p_1}(p_2, p_3, \ldots, p_k)_\alpha + \ldots + \alpha^{p_k}(p_1, p_2, \ldots, p_{k-1})_\alpha,$$

we find

$$(p_1, p_2, \ldots, p_k)_\alpha \\ = (p_1, p_2, \ldots, p_k) - \sum_1 \alpha^{p_1}(p_2, \ldots, p_k) + (2!) \sum_2 \alpha^{p_1+p_2}(p_3, \ldots, p_k) \\ - (3!) \sum_3 \alpha^{p_1+p_2+p_3}(p_4, \ldots, p_k) + \ldots,$$

where $\sum_i$ denotes a summation extending to every $k-i$ of the exponents $p_1, p_2, \ldots, p_k$; and hence, as we obviously have

$$(p_1, p_2, \ldots, p_k, p) = \alpha^p(p_1, \ldots, p_k)_\alpha + \beta^p(p_1, \ldots, p_k)_\beta + \ldots + \lambda^p(p_1, \ldots, p_k)_\lambda,$$

we find

$$
\begin{aligned}
(p_1, p, \dots, p_k, p) \\
&= (p)(p_1, p_2, \dots, p_k) - \sum_1 (p+p_1)(p_2, \dots, p_k) \\
&\qquad + (2!) \sum_2 (p+p_1+p_2)(p_3, \dots, p_k) - \dots, \qquad \text{(A)}
\end{aligned}
$$

from which any form $(p, q\ r, \dots)$ can be calculated. For example, we find

$$
\begin{aligned}
(p, q) &= (p)(q) - (p+q) \\
(p, q, r) &= (p)(q)(r) - (p)(q+r) - (q)(r+p) - (r)(p+q) + 2(p+q+r) \\
(p, q, r, s) &= (p)(q)(r)(s) \\
&\quad - (p)(s)(q+r) - (q)(r)(p+s) - \dots - \dots - \dots - \dots \\
&\quad + (p+s)(q+r) + \dots + \dots \\
&\quad + 2(p)(q+r+s) + \dots + \dots \\
&\quad - 6(p+q+r+s).
\end{aligned}
$$

In, particular, in formula (A), if $p_1 = p_2 = \dots = p_k = p = 1$, and $(p_1, \dots, p_k)$ be replaced by $(k!)\,\Sigma a_1 a_2 \dots a_k,\ = (k!)P_k$, say, we obtain

$$
\begin{aligned}
&[(k+1)!]\,P_{k+1} \\
&\quad = s_1 P_k(k!) - k s_2[(k-1)!]\,P_{k-1} + (2!)\,\frac{k!}{2!\,(k-2)!}\,s_3[(k-2)!]\,P_{k-2} - \dots
\end{aligned}
$$

or
$$(k+1)P_{k+1} = s_1 P_k - s_2 P_{k-1} + s_3 P_{k-2} - \dots,$$

the well known formula, usually called after Newton, where

$$s_k = a^k + \beta^k + \dots,$$

from which is found

$$
P_k = \frac{1}{k!}\begin{vmatrix}
s_1 & s_2 & s_3 & \dots & s_k \\
k-1 & s_1 & s_2 & \dots & s_{k-1} \\
0 & k-2 & s_1 & \dots & s_{k-2} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots\ 1 & s_1
\end{vmatrix}.
$$

Thus

$$
\begin{aligned}
&(x-a)(x-\beta)\dots(x-\lambda) \\
&\quad = x^n - x^{n-1}s_1 + x^{n-2}\,\frac{s_1^2 - s_2}{2!} - x^{n-3}\,\frac{s_1^3 - 3s_1 s_2 + 2s_3}{3!} \\
&\qquad + x^{n-4}\,\frac{s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4}{4!} - \dots.
\end{aligned}
$$

Conversely, if we replace α by

$$h_1\alpha^p+h_2\alpha^q+h_3\alpha^r+\ldots,$$

replace β by

$$h_1\beta^p+h_2\beta^q+h_3\beta^r+\ldots,$$

and so on, the coefficient of $h_1h_2\ldots h_k$ in P_k is $(p, q, r, \ldots)$, where there are k exponents. For, by this substitution, s_1 becomes

$$\sigma_1 = h_1(p)+h_2(q)+\ldots,$$

s_2 becomes

$$\sigma_2 = (2!)\, h_1h_2(p+q)+\ldots,$$

where it is immaterial to write the terms involving squares of $h_1, h_2, \ldots$ and s_3 becomes

$$\sigma_3 = (3!)\, h_1h_2h_3(p+q+r)+\ldots,$$

and so on; and hence the coefficient of h_1h_2 in $\dfrac{\sigma_1^2-\sigma_2}{2!}$ is $(p)(q)-(p+q)$, the coefficient of $h_1h_2h_3$ in

$$\frac{1}{3!}\,(\sigma_1^3-3\sigma_1\sigma_2+2\sigma_3)$$

is

$$\frac{1}{3!}\,\{(3!)(p)(q)(r)-3\,\Sigma(p)\,2(q+r)+2(3!)(p+q+r)\}$$

or

$$(p)(q)(r)-\Sigma(p)(q+r)+2(p+q+r),$$

and the coefficient of $h_1h_2h_3h_4$ in

$$\frac{1}{4!}\,(\sigma_1^4-6\sigma_1^2\sigma_2+8\sigma_1\sigma_3+3\sigma_2^2-6\sigma_4)$$

is

$$\frac{1}{4!}\,\{(p)(q)(r)(s)-6\,\Sigma\,2(p)(q)\,2(r+s)+8\,\Sigma(p)(3!)(q+r+s)$$
$$+3\,\Sigma\,8(p+q)(r+s)-6(4!)(p+q+r+s)\}$$

or

$$(p)(q)(r)(s)-\Sigma(p)(q)(r+s)+2\,\Sigma(p)(q+r+s)$$
$$+\Sigma(p+q)(r+s)-(3!)(p+q+r+s).$$

And evidently the same is true in general.

The formula (A) is unsymmetrical with regard to the k suffixes. If we take the $(k+1)$ possible forms of the right side and add, and afterwards change $k+1$ into k, we have the symmetrical formula

$$k(p_1, p_2, \ldots, p_k)$$
$$= \sum_1 (p_1)(p_2, p_3, \ldots, p_k)-(2!)\sum_2 (p_1+p_2)(p_3, p_4, \ldots, p_k)$$
$$+(3!)\sum_3 (p_1+p_2+p_3)(p_4, \ldots, p_k)-\ldots, \qquad \text{(B)}$$

where, as before, $\sum_i$ denotes a summation containing a term corresponding to every selection of i of the exponents $p_1, p_2, \ldots, p_k$.

2. *A Relation connecting a System of Matrices.*

Let D be a matrix of $2n$ rows and columns which is the product of two skew symmetrical matrices also of $2n$ rows and columns, of non-vanishing determinant, say equal to ΔK. The matrix $\Delta-\lambda K^{-1}$ is skew symmetrical, and its determinant is a perfect square; the determinant $|D-\lambda|$ is therefore also a perfect square, and its roots will consist, say, of $\xi_1, \ldots, \xi_n$, each repeated. Similarly the roots of $|D^r-\lambda|$ will consist of $\xi_1^r, \ldots, \xi_n^r$, each repeated. Thus, if σ_r be the sum of $\xi_1^r, \ldots, \xi_n^r$,

$$\sigma_r = \xi_1^r+\ldots+\xi_n^r = \tfrac{1}{2}\,[\text{sum of } r\text{-th powers of the roots of } D]$$

$$= \tfrac{1}{2}\sum_{l=1}^{2n}(D^r)_{l,l},$$

where $(D^r)_{l,l}$ is the (l, l)-th element of the matrix D^r, that is, the l-th diagonal element of this matrix. If, then, we take (cf. § 1 above)

$$\varpi_1 = \sigma_1, \qquad \varpi_2 = \frac{1}{2!}(\sigma_1^2-\sigma_2), \qquad \varpi_3 = \frac{1}{3!}(\sigma_1^3-3\sigma_1\sigma_2+2\sigma_3),$$

and so on, the matrix D satisfies the equation

$$D^n-\varpi_1 D^{n-1}+\varpi_2 D^{n-2}-\varpi_3 D^{n-3}+\ldots = 0.$$

Herein suppose that D is $h_1D_1+\ldots+h_nD_n$, where $h_1, h_2, \ldots, h_n$ are undetermined parameters, and each of $D_1, \ldots, D_n$ is a matrix of $2n$ rows and columns which is the product of two skew symmetrical matrices of the same type; in the application to be made below $D_1, D_2, \ldots$ are of the forms $\Delta_1 K, \Delta_2 K, \ldots$, where K is the same for each. Then every term in the equation becomes a polynomial in $h_1, h_2, \ldots, h_n$ of dimension n; we desire to pick out the coefficient of the product $h_1h_2\ldots h_n$ in the result.

For this let $(D')_{12}$ denote $D_1D_2+D_2D_1$, let $(D')_{123}$ denote the sum of the six products of D_1, D_2, D_3 in all possible orders, and so on. Then let δ_1 denote half the sum of the diagonal elements of D_1, let δ_{12} denote half the sum, divided by $(2!)$, of the diagonal elements of $(D')_{12}$, let δ_{123} denote half the sum, divided by $(3!)$, of the diagonal elements of $(D')_{123}$, and so on, so that

$$\delta_{12} = \frac{1}{2(2!)}\sum_{l=1}^{2n}(D_1D_2+D_2D_1)_{l,l},$$

$$\delta_{123} = \frac{1}{2(3!)}\sum_{l=1}^{2n}(D_1D_2D_3+D_1D_3D_2+D_2D_3D_1+D_2D_1D_3+D_3D_1D_2 \\ +D_3D_2D_1)_{l,l},$$

and so on. Then the term σ_r becomes

$$\tfrac{1}{2}\sum_{l=1}^{2n}\{(h_1D_1+\ldots+h_nD_n)^r\}_{l,l}$$

or
$$(r!)\sum_r h_1 h_2 \dots h_r \,\delta_{12\dots r},$$

where $\sum_r$ denotes a summation extending to every r of the letters $h_1, h_2, \dots, h_n$, and terms involving powers of $h_1, \dots, h_n$ above the first are not written down.

Also in D^n the coefficient of $h_1 h_2 \dots h_n$ is $(D')_{12\dots n}$, defined above. In D^{n-1}, the terms not containing squares or higher powers of $h_1, \dots, h_n$ are
$$\sum_1 h_1 h_2 \dots h_{n-1} (D')_{12\dots(n-1)};$$

multiplying this by ϖ_1, which is equal to
$$\sum_1 h_1 \delta_1,$$

the coefficient of $h_1 h_2 \dots h_n$ in $-\varpi_1 D^{n-1}$ is
$$-\sum_1 \delta_1 (D')_{23\dots n},$$

where $\sum_1$ relates to a summation of n constituents. Again, in D^{n-2} the terms not containing higher powers of $h_1, \dots, h_n$ are
$$\sum_2 h_3 h_4 \dots h_n (D')_{34\dots n};$$

while taking
$$\varpi_2 = \frac{1}{2!}(\sigma_1^2 - \sigma_2),$$

and retaining only terms which are of first order in each of $h_1, \dots, h_n$, we have to consider
$$\frac{1}{2!}\{2\Sigma h_1 h_2 \delta_1 \delta_2 - (2!)\Sigma h_1 h_2 \delta_{12}\}$$

or
$$\Sigma h_1 h_2 (\delta_1 \delta_2 - \delta_{12}).$$

Thus the coefficient of $h_1 h_2 \dots h_n$ in $\varpi_2 D^{n-2}$ is
$$\sum_2 (\delta_1 \delta_2 - \delta_{12})(D')_{34\dots n}.$$

Similarly the coefficient of $h_1 h_2 h_3$ in
$$\varpi_3 = \frac{1}{3!}(\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3)$$

is
$$\frac{1}{3!}\{3!\,\delta_1\delta_2\delta_3 - 3(2!)(\delta_1\delta_{23} + \delta_2\delta_{31} + \delta_3\delta_{12}) + 2(3!)\delta_{123}\},$$

and the coefficient of $h_1 h_2 \dots h_n$ in $-\varpi_3 D^{n-3}$ is
$$-\sum_3 [\delta_1\delta_2\delta_3 - (\delta_1\delta_{23} + \delta_2\delta_{31} + \delta_3\delta_{12}) + 2\delta_{123}](D')_{45\dots n}.$$

In general, let $\phi_{12\ldots k}$ denote the coefficient of $h_1 h_2 \ldots h_k$ in

$$\varpi_k = \frac{1}{k!}\begin{vmatrix} \sigma_1 & \sigma_2 & \cdots & & \sigma_k \\ k-1 & \sigma_1 & \cdots & & \sigma_{k-1} \\ \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & 1 & \sigma_1 \end{vmatrix};$$

then, as we have $k\varpi_k = \sigma_1\varpi_{k-1} - \sigma_2\varpi_{k-2} + \sigma_3\varpi_{k-3} - \ldots,$

and in σ_r the coefficient of $h_1 h_2 \ldots h_r$ is $(r!)\,\delta_{123\ldots r}$, while in ϖ_{k-r} the coefficient of $h_{r+1} h_{r+2} \ldots h_k$ is $\phi_{r+1,\ldots,k}$, we see that the function $\phi_{123\ldots k}$ can be calculated from the formula

$$k\phi_{12\ldots k} = \sum_1 \delta_1 \phi_{23\ldots k} - (2!) \sum_2 \delta_{12} \phi_{34\ldots k} + (3!) \sum_3 \delta_{123} \phi_{45\ldots k} - \ldots,$$

and has the same form in terms of $\delta_1, \delta_2, \ldots, \delta_{12}, \delta_{13}, \ldots, \delta_{123}, \ldots$ as has the function $(p_1, p_2, \ldots, p_k)$, considered in § 1, in terms of

$$(p_1), (p_2), \ldots, (p_1+p_2), (p_1+p_3), \ldots, (p_1+p_2+p_3), \ldots.$$

Thus, finally, we have the identity, connecting the matrices $D_1, \ldots, D_n$,

$$(D')_{12\ldots n} - \sum_1 \delta_1 (D')_{23\ldots n} + \sum_2 \phi_{12} (D')_{34\ldots n} - \sum_3 \phi_{123} (D')_{45\ldots n}$$
$$+ \ldots + (-1)^n \phi_{12\ldots n} = 0,$$

where $\sum_i$ denotes a summation extending to every i of the numbers $1, 2, \ldots, n$.

In particular, for $n = 2$ we have

$$D_1 D_2 + D_2 D_1 - \delta_1 D_2 - \delta_2 D_1 + \delta_1 \delta_2 - \delta_{12} = 0,$$

and for $n = 3$ we have

$$(D_1 D_2 D_3 + \ldots) - \Sigma\, \delta_1 (D_2 D_3 + D_3 D_2)$$
$$+ \Sigma (\delta_1 \delta_2 - \delta_{12}) D_3 - (\delta_1 \delta_2 \delta_3 - \delta_1 \delta_{23} \ldots + 2\delta_{123}) = 0.$$

If we further put, symbolically,

$$(D')_{12\ldots k} = D'_1 D'_2 \ldots D'_k,$$
$$\phi_{12\ldots k} = \phi'_1 \phi'_2 \ldots \phi'_k,$$

the identity can be written

$$(D'_1 - \phi'_1)(D'_2 - \phi'_2) \ldots (D_n - \phi_n) = 0.$$

Note I.—It is an evident consequence of what precedes that $\phi_{12\ldots n}$ is the coefficient of $h_1 h_2 \ldots h_n$ in the square root of the determinant of the matrix $h_1 D_1 + \ldots + h_n D_n$; and, in accordance with this, the function which $\phi_{12\ldots n}$ reduces to when the matrices $D_1, D_2, \ldots, D_n$ are all the same is equal to $(n!)$ times the square root of the determinant of $|D|$.

Note II.—If the expression on the left side of the equation above, after multiplication by $(-1)^n$, be denoted by $\psi_{12\dots n}$, so that, for instance,

$$\psi_1 = \phi_1 - D_1,$$

$$\psi_{12} = \phi_{12} - D_1\phi_2 - D_2\phi_1 + D_1D_2 + D_2D_1,$$

we have $\phi_{12} - D_1\psi_2 - D_2\psi_1 = \phi_{12} - D_1(\phi_2 - D_2) - D_2(\phi_1 - D_1) = \psi_{12}$,

$$\begin{aligned}\phi_{123} - D_1\psi_{23} - D_2\psi_{31} - D_3\psi_{12} &\\ = \phi_{123} - D_1 \{\phi_{23} - \phi_2D_3 - \phi_3D_2 + D_2D_3 + D_3D_2\} &\\ - D_2\{\phi_{31} - \phi_3D_1 - \phi_1D_3 + D_3D_1 + D_1D_3\} &\\ - D_3\{\phi_{12} - \phi_1D_2 - \phi_2D_1 + D_1D_2 + D_2D_1\} &\\ = \psi_{123}, &\end{aligned}$$

and so, in general,

$$\phi_{12\dots k} - D_1\psi_{23\dots k} - D_2\psi_{13\dots k} - \dots - D_k\psi_{12\dots(k-1)} = \psi_{12\dots k},$$

this depending merely on the fact that

$$D_1(D')_{23\dots k} + D_2(D')_{13\dots k} + \dots + D_k(D')_{12\dots(k-1)} = (D')_{12\dots k},$$

where, as before, $(D')_{12\dots m}$ is the sum of the $(m!)$ products in all possible orders of $D_1, D_2, \dots, D_m$.

3. *The Conditions for the Existence of a Jacobian Function.*

We know (M.P.F., p. 286) that for an integral function f, of n variables $u_1, \dots, u_n$, to satisfy $2n$ equations of the form

$$f(u_1 + a_{1,r}, u_2 + a_{2,r}, \dots, u_n + a_{n,r}) = e^{2\pi i b^{(r)}(u + \frac{1}{2}a^{(r)}) + c_r} f(u_1, \dots, u_n),$$

where $b^{(r)}(u + \frac{1}{2}a^{(r)}) = b_{1,r}u_1 + \dots + b_{n,r}u_r + \frac{1}{2}(b_{1,r}a_{1,r} + \dots + b_{n,r}a_{n,r})$,

it is necessary that the $2n$ sets of *periods*, a, should satisfy conditions expressible in the forms

$$aK\bar{a} = 0, \quad -iaK\bar{a}_0x_0x > 0.$$

$$\bar{b}a - \bar{a}b = \Delta,$$

where a, b denote the matrices of type $(n, 2n)$ formed with the elements $a_{\alpha,l}$, $b_{\beta,l}$, $\bar{a}$ and $\bar{b}$ denote the matrices of type $(2n, n)$ formed by transposing these, K is a skew symmetrical matrix of type $(2n, 2n)$ whose elements are integers, x denotes any row of n quantities and x_0 the row of n conjugate complex quantities, and Δ denotes some matrix of type $(2n, 2n)$ of which every constituent is an integer. It is a consequence of the theory that both K and Δ have a non-vanishing determinant.

With a view to investigating these relations more in detail we first repeat the well known argument[*] by which it is shown that without loss of generality we may suppose the matrix of periods a to have the form (e^{-1}, σ), wherein e^{-1}, of type (n, n), has zeros except in the diagonal, this diagonal consisting of the inverse positive integers $e_1^{-1}, e_2^{-1}, \ldots, e_n^{-1}$, of which e_{r+1}/e_r is an integer, and wherein σ denotes a symmetrical matrix of type (n, n) such that, when $\xi_1, \ldots, \xi_n$ are any real quantities, the real part of the quadratic form $i\sigma\xi^2$ is negative. To establish this result, take a unitary matrix g, of integers, of type $(2n, 2n)$, such that

$$g K \bar{g} = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix},$$

where e denotes the diagonal matrix whose elements are $e_1, \ldots, e_n$, and e_{r+1}/e_r is an integer, as we know to be possible (M.P.F., p. 311); then put, as the definition of ω and ω',

$$a = (\omega, \omega')g\,;$$

the condition

$$-ia K \bar{a}_0 x_0 x > 0$$

gives then

$$-i(\omega, \omega')\begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}\begin{pmatrix} \bar{\omega}_0 \\ \bar{\omega}'_0 \end{pmatrix} x_0 x > 0,$$

or

$$-i(\omega' e, -\omega e)\begin{pmatrix} \bar{\omega}_0 \\ \bar{\omega}'_0 \end{pmatrix} x_0 x > 0,$$

or

$$-i(\omega' e \bar{\omega}_0 x_0 x - \omega e \bar{\omega}'_0 x_0 x) > 0,$$

that is, if $\xi = \bar{\omega} x$, $\xi_0 = \bar{\omega}_0 x_0$,

$$-i(\omega' e \xi_0 x - \omega'_0 e \xi x_0) > 0\,;$$

this shows that ω is of non-vanishing determinant, since else we could choose $x_1, \ldots, x_n$ to make ξ or $\bar{\omega} x$ consist of n zeros, in which case the left side would not be > 0.

We can hence write, in place of $a = (\omega, \omega')g$,

$$a = \omega e (e^{-1}, \sigma) g,$$

where σ is defined by

$$\omega e \sigma = \omega'\,;$$

then the condition

$$a K \bar{a} = 0$$

---

[*] The quantities u, σ of the text are the quantities $\frac{1}{r}\mu^{-1}u$ and $\frac{\sigma}{r}$ of M.P.F., p. 227, and are those occurring in the theta functions of M.P.F., p. 283.

becomes
$$\omega e(e^{-1}, \sigma)\begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}\begin{pmatrix} e^{-1} \\ \bar{\sigma} \end{pmatrix}\overline{\omega e} = 0,$$

or
$$\omega e(\sigma e, -1)\begin{pmatrix} e^{-1} \\ \bar{\sigma} \end{pmatrix}\overline{\omega e} = 0,$$

or
$$\omega e(\sigma - \bar{\sigma})\overline{\omega e} = 0,$$

shewing that
$$\sigma = \bar{\sigma}.$$

The previous inequality then gives
$$-i(\omega e\sigma e\xi_0 x - \omega_0 e\sigma_0 e\xi x_0) > 0$$

or
$$-i(e\sigma e\xi\xi_0 - e\sigma_0 e\xi_0\xi) > 0,$$

where we have utilized the fact that, if μ be a matrix of type (n, n) and p, q two rows each of n quantities, $\mu pq = \bar{\mu}qp$; if $\eta = e\xi$, this is the same as
$$-i(\sigma - \sigma_0)\eta\eta_0 > 0,$$

or, if $\sigma = \rho + i\tau$, $\eta = \eta_1 + i\eta_2$, is the same as
$$-i\,2i\tau(\eta_1^2 + \eta_2^2) > 0,$$

or
$$\tau(\eta_1^2 + \eta_2^2) > 0,$$

which is the condition that the imaginary part of $i\sigma\eta_1^2$ should be negative. The transformation is thus established.

If now,* beside
$$a = \omega e(e^{-1}, \sigma)g,$$

in order to define β, of type $(n, 2n)$, and to define the new arguments $v_1, \ldots, v_n$, we put
$$\overline{\omega e}.b = \beta g \qquad u = \omega e.v,$$

---

* It is shewn, M.P.F., p. 298, that, if R be the least positive integer such that $R\Delta^{-1}$ consists of integers, and γ a unitary matrix such that $\gamma R\Delta^{-1}\bar{\gamma} = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$, where t is a diagonal matrix of positive integers $t_1, \ldots, t_n$ for which t_{s+1}/t_s is integral, and $t_1 = 1$, and $(\omega, \omega') = a\gamma^{-1}$, $\tau = (\omega t)^{-1}\omega'$, and if $w = (\omega t)^{-1}u + \text{constant}$, a single Jacobian function, after multiplication by the exponential of a (non-homogeneous) quadratic function of u, satisfies a defining equation
$$F(w + t^{-1}m + \tau m') = e^{-2\pi i Rm'(w + \frac{1}{2}\tau m')}F(w),$$
in which m, m' are rows of n arbitrary integers, and is the sum of $\sqrt{|\Delta|}$ theta functions of arguments Rw, with periods $R\tau$, that is, each of the form
$$\sum_n \exp\left\{2\pi i Rw\left(n + \frac{th}{R}\right) + i\pi R\tau\left(n + \frac{th}{R}\right)^2\right\},$$
wherein h is a row of positive integers in which $h_s < Rt_s^{-1}$.

But this reduction depends on Δ and, therefore on the matrix b proper to the function, while the reduction of the periods given in the text depends only on the original periods a, which are the same for all the Jacobian functions of the set to be considered subsequently.

and, when m is any row of $2n$ integers, we define a row of $2n$ integers, μ, by means of

$$gm = \mu,$$

we have
$$\begin{aligned} bm(u+\tfrac{1}{2}am) &= bm.\omega ev+\tfrac{1}{2}bm.\omega e(e^{-1}, \sigma)\mu \\ &= \overline{\omega e}.bmv+\tfrac{1}{2}\overline{\omega e}.bm(e^{-1}, \sigma)\mu \\ &= \beta\mu v+\tfrac{1}{2}\beta\mu(e^{-1}, \sigma)\mu \\ &= \beta\mu\left[v+\tfrac{1}{2}(e^{-1}, \sigma)\mu\right], \end{aligned}$$

which is of the same form in v, (e^{-1}, σ), β, μ as was the original in u, a, b, m.

While also

$$\begin{aligned} \bar{b}a-\bar{a}b &= \bar{b}\omega e(e^{-1}, \sigma)g-\bar{g}\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\overline{\omega e}\,b \\ &= \bar{g}\left[\bar{\beta}(e^{-1}, \sigma)-\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\beta\right]g, \end{aligned}$$

which, since g is a unitary matrix, shews that

$$\bar{\beta}(e^{-1}, \sigma)-\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\beta$$

is a matrix of integers. All the conditions previously to be satisfied by a, b, and u are thus satisfied by (e^{-1}, σ), β, and v.

The condition $aK\bar{a} = 0$ becomes, as we have seen, the identical equation

$$(e^{-1}, \sigma)\begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0;$$

we shall put
$$E = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix},$$
so that the condition becomes

$$(e^{-1}, \sigma)\,E\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0.$$

It may also be noticed, for the sake of a future application, that

$$(\bar{b}a-\bar{a}b)K$$

is
$$\bar{g}\left\{\left[\bar{\beta}(e^{-1}, \sigma)-\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\beta\right]E\right\}\bar{g}^{-1},$$

or, say,
$$\Delta K = \bar{g}\nabla E\,\bar{g}^{-1};$$

and hence, if
$$\Delta_1 = \bar{b}_1a-\bar{a}b_1, \qquad \Delta_2 = \bar{b}_2a-a\bar{b}_2,$$

$$\Delta_1K.\Delta_2K = \bar{g}\nabla_1E\nabla_2E\bar{g}^{-1}.$$

Also that
$$aK\Delta K\bar{a}$$
is equal to
$$aK(\bar{b}a-\bar{a}b)K\bar{a}$$
$$= \omega e(e^{-1}, \sigma)gK\bar{g}\left[\bar{\beta}(e^{-1}, \sigma)-\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\beta\right]gK\bar{g}\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\overline{\omega e}$$
$$= \omega e\left\{(e^{-1}, \sigma)E\nabla E\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\right\}\overline{\omega e},$$
and similarly that
$$aK\Delta K\Delta K\bar{a}$$
is equal to
$$\omega e\left\{(e^{-1}, \sigma)E\nabla E\nabla E\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}\right\}\overline{\omega e}.$$

In examining the conditions
$$\bar{b}a-\bar{a}b = \Delta,$$
we can thus without loss of generality suppose $a = (e^{-1}, \sigma)$. We shall continue to denote the arguments by u.

Then first, by multiplying $f(u)$ by an expression of the form
$$e^{\pi i A u^2},$$
where A is a symmetrical matrix of type (n, n), so that Au^2 is a quadratic form in $u_1, u_2, \ldots, u_n$, we can suppose that in the matrix b all the elements
$$b_{a,j} \quad (a = 1, 2, \ldots, n\,;\ j = 1, 2, \ldots, a)$$
are zero, so that b has the form
$$\begin{Bmatrix} 0, & b_{12}, & \ldots, & b_{1n}, & b_{1,n+1}, & \ldots, & b_{1,2n} \\ 0, & 0, & \ldots, & b_{2n}, & b_{2,n+1}, & \ldots. & b_{2,2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & \ldots, & 0, & b_{n,n+1}, & \ldots, & b_{n,2n} \end{Bmatrix}.$$
For, after such multiplication, when in $f(u)$ the arguments are simultaneously increased by the rows am, where m denotes a row of $2n$ integers, the quotient $f(u+am)/f(u)$ is $e^{2\pi i H}$, where, save for parts independent of u,
$$H = \tfrac{1}{2}A\,(u+am)^2-\tfrac{1}{2}Au^2+bm(u+\tfrac{1}{2}am),$$
and this is, effectively, $(Aa+b)m(u+\tfrac{1}{2}am)$,
namely, the matrix b is replaced by $b+Aa$. To obtain the specified reduction in the matrix b, it is thus sufficient to take the matrix A, so that in
$$A(e^{-1}, \sigma)+b$$
the elements (a, j) above described may all be zero; that is, if we write
$$b = (b_1, b_2),$$

in which each of b_1, b_2 is of type (n, n), it is sufficient to take A of such form that in

$$Ae^{-1}+b_1$$

the elements (a, j) above spoken of shall all be zero; this requires only

$$A_{a,j} = -b_{a,j}e_a$$

$$(a = 1, 2, \ldots, n;\ j = 1, 2, \ldots, a),$$

whereby the matrix A is determined. It may be noticed that, whatever form is given to the symmetrical matrix A, the replacement of b in $\bar{b}a-\bar{a}b$ by $b+Aa$ does not alter the value of $\bar{b}a-\bar{a}b$.

Assume this reduction in the matrix b, $=(b_1, b_2)$, and now write $\bar{b}a-\bar{a}b = \Delta$, that is, suppose

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}(e^{-1}, \sigma)-\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}(b_1, b_2) = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix}$$

or

$$\begin{pmatrix} \bar{b}_1e^{-1} & \bar{b}_1\sigma \\ \bar{b}_2e^{-1} & \bar{b}_2\sigma \end{pmatrix}-\begin{pmatrix} e^{-1}b_1 & e^{-1}b_2 \\ \sigma b_1 & \sigma b_2 \end{pmatrix} = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix}$$

or

$$\begin{pmatrix} \bar{b}_1e^{-1}-e^{-1}b_1 & \bar{b}_1\sigma-e^{-1}b_2 \\ \bar{b}_2e^{-1}-\sigma b_1 & \bar{b}_2\sigma-\sigma b_2 \end{pmatrix} = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix},$$

where Q is a matrix of type (n, n) consisting of integers, and P, R are skew symmetrical matrices of integers, also of type (n, n); further denote by P_1 the matrix of type (n, n) whose elements to the right of the principal diagonal are the negatives of those in P, the other elements in P_1 being zeros, so that

$$P = \bar{P}_1-P_1;$$

we may express this, perhaps, by saying that $-P_1$ is formed by the north east half of the elements of P, the remaining elements being zeros. Then the conditions to be satisfied are

$$\bar{b}_1e^{-1}-e^{-1}b_1 = \bar{P}_1-P_1,$$

$$e^{-1}b_2-\bar{b}_1\sigma = Q,$$

$$\bar{b}_2\sigma-\sigma b_2 = R.$$

The first of these is

$$e^{-1}b_1-P_1 = \bar{b}_1e^{-1}-\bar{P}_1,$$

and shews that $e^{-1}b_1-P_1$ is a symmetrical matrix; but in P_1 and in b_1, in consequence of the reduction supposed made in b, the elements below the principal diagonal are all zeros. Hence

$$b_1 = eP_1,$$

and then from the second equation above

$$b_2 = e(Q+\bar{P}_1 e\sigma),$$

so that the matrix b is entirely determined when the elements of the matrix Δ are given, in terms indeed of the elements of the first n rows of Δ. The remaining conditions

$$\bar{b}_2\sigma - \sigma b_2 = R$$

require, however,

$$(\sigma e P_1 + \bar{Q})\, e\sigma - \sigma e\,(Q+\bar{P}_1 e\sigma) = R,$$

or, as $P = \bar{P}_1 - P_1$, they require

$$\sigma e P e\sigma + \sigma e Q - \bar{Q} e\sigma + R = 0. \tag{C}$$

These conditions can be expressed in the form

$$(e^{-1}, \sigma) E\Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0;$$

for $$E\Delta E = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} = \begin{pmatrix} -e\bar{Q} & -eR \\ eP & -eQ \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -eRe & e\bar{Q}e \\ -eQe & -ePe \end{pmatrix},$$

and hence $$-(e^{-1}, \sigma) E\Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = (Re + \sigma e Q e,\ -\bar{Q}e + \sigma e P e) \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix}$$

$$= R + \sigma e Q - \bar{Q} e\sigma + \sigma e P e\sigma.$$

The matrix on the left side of (C), or the matrix $E\Delta E$, being skew symmetrical, the conditions (C) are equivalent to $\tfrac{1}{2}n(n-1)$ conditions; in fact, a relation

$$aM\bar{a} = 0,$$

wherein M is skew symmetrical, of type $(2n, 2n)$, is the same as

$$\sum_{i,\,j}^{1\ldots 2n} M_{ij}(a_{pi}a_{qj} - a_{pj}a_{qi}) = 0,$$

and expresses the same bilinear relation connecting the periods of every pair, such as u_p, u_q, of the n arguments. But while the numerical coefficients in the bilinear relations (C) are determined when the form of the matrix Δ is given, the converse is not true; for, evidently, in virtue of

$$(e^{-1}, \sigma)\, E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0,$$

the relation $$(e^{-1}, \sigma) E\Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0$$

would remain unchanged if Δ were replaced by $\Delta + rE^{-1}$, which is equally a skew symmetric matrix of integers, provided r is an integer divisible by e_n. This change is equivalent to replacing Q by $Q - re^{-1}$, which leaves unaffected the terms

$$\sigma e Q - \bar{Q} e \sigma$$

of the equation (C). In virtue of

$$b = \{eP_1,\ e(Q + \bar{P}_1 e\sigma)\},$$

it is also equivalent to replacing b by $b - r(0,\ 1)$, where 0 denotes the zero matrix of type $(n,\ n)$, and 1 denotes the unit matrix of type $(n,\ n)$.

Returning for a moment to the unreduced form for the periods, and the matrix b, the conditions

$$(e^{-1},\ \sigma)\, E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0, \qquad (e^{-1},\ \sigma) E\Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0,$$

are equivalent, as we have seen, respectively to

$$aK\bar{a} = 0, \qquad aK\nabla K\bar{a} = 0,$$

where ∇ is for the present used to denote $ba - \bar{a}b$. It is at once evident that the latter relation, being

$$aK\bar{b}\,.\,aK\bar{a} - aK\bar{a}\,.\,bK\bar{a} = 0,$$

follows from the former $aK\bar{a} = 0$. It is equally evident that

$$aK\nabla K\nabla K\bar{a} = 0\,;$$

for this is $$aK\bar{b}\,.\,aK\nabla K\bar{a} - aK\bar{a}\,.\,bK\nabla K\bar{a} = 0.$$

Similarly it can be proved that

$$a(K\nabla)^m \nabla^{-1}\bar{a} = 0,$$

whatever integer m may be. We have previously shewn (p. 357) the existence of an equation

$$\sum_{m=1}^{n} p_m (K\nabla)^m = 1,$$

where p_m is a number; from this we have

$$\nabla^{-1} = \sum_{m=1}^{n} p_m (K\nabla)^m \nabla^{-1},$$

so that also $$a\nabla^{-1}\bar{a} = 0.$$

And this again follows at once from

$$\nabla = \bar{b}a - \bar{a}b = \bar{A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A,$$

where $A = \begin{pmatrix} a \\ b \end{pmatrix}$ is of type $(2n, 2n)$, and 1 here denotes the unit matrix of type (n, n) (cf. M.P.F., p. 288). For this gives

$$\nabla^{-1} = A^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{A}^{-1},$$

and hence
$$\begin{pmatrix} a \\ b \end{pmatrix}\nabla^{-1}(\bar{a}, \bar{b}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

from which we deduce

$$a\nabla^{-1}\bar{a} = 0, \quad a\nabla^{-1}\bar{b} = 1, \quad b\nabla^{-1}\bar{b} = 0.$$

Resuming the results of this section so far obtained we thus have: The existence of a single-valued integral function $f(u)$, of the n variables $u_1, \ldots, u_n$, which with the periods $a = (e^{-1}, \sigma)$, subject to the relations

$$aE\bar{a} = 0,$$

satisfies, for any $2n$ integers m, equations of the form

$$f(u+am)/f(u) = \exp 2\pi i\left[bm\left(u+\tfrac{1}{2}am\right)+H_m\right],$$

where H_m is independent of u, requires the existence of further $\frac{1}{2}n(n-1)$ bilinear relations
$$aE\Delta E\bar{a} = 0,$$

where Δ, which we write in the form

$$\Delta = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix},$$

is a skew symmetrical matrix of integers of type $(2n, 2n)$, to which the matrix $\bar{b}a-\bar{a}b$ is required to be equal. Conversely, writing $P = \bar{P}_1 - P_1$, where $-P_1$ has zeros below the principal diagonal, but above this diagonal agrees with P, and assuming $aE\bar{a} = 0$, $aE\Delta E\bar{a} = 0$, the relation

$$\bar{b}a - \bar{a}b = \Delta$$

requires, and is ensured by taking

$$b = e(P_1, Q + \bar{P}_1 e\sigma).$$

The explicit forms of the necessary bilinear relations connecting the periods, other than the universal relations expressed by $aE\bar{a} = 0$, do not, however, determine the matrix Δ or the matrix b; when these bilinear relations are given we may equally take, for corresponding forms of b and

Δ, respectively

$$b = k\beta - r(0,\ 1),$$

$$\Delta = k\nabla + rE^{-1},$$

where k is any integer and r is any integer divisible by e_n, and if

$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & p \end{pmatrix}, \qquad p = \bar{p}_1 - p_1,$$

then

$$\beta = e(p_1,\ q + \bar{p}_1 e\sigma).$$

Herein b is chosen so that all the elements (a, j) in which

$$a = 1,\ 2,\ \ldots,\ n \quad \text{and} \quad j = 1,\ 2,\ \ldots,\ a$$

are zero. A more general form of b, obtained by multiplying the Jacobian function by the exponential of a quadratic function of $u_1, \ldots, u_n$, is obtained by adding Aa to the b, where A is a symmetric matrix of the type (n, n).

In addition to the fact that the matrix $\bar{b}a - \bar{a}b = \Delta$ must consist of integers, it is necessary (M.P.F., p. 287) that if z be any row of $2n$ quantities satisfying $az = 0$, and z_0 the row of conjugate complex quantities, that

$$-i\Delta z z_0 > 0.$$

When $a = (e^{-1},\ \sigma)$, if we put $z = (\eta,\ y)$, each of η and y being a row of n quantities, the equation $az = 0$ gives $e^{-1}\eta + \sigma y = 0$, or $\eta = -e\sigma y$, and therefore

$$z = \begin{pmatrix} -e\sigma \\ 1 \end{pmatrix} y,$$

where 1 denotes the unit matrix of type (n, n), namely,

$$z = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} y = E\bar{a}y.$$

Hence the inequality becomes

$$-i\Delta E\bar{a}y\,.\,E\bar{a}_0 y_0 > 0,$$

that is,

$$-iaE\Delta E\bar{a}_0 y_0 y > 0.$$

Denoting the real skew symmetrical matrix $E\Delta E$ by N, and putting $\sigma = \sigma_1 + i\sigma_2$, where σ_1, σ_2 are real, and also

$$A = (e^{-1},\ \sigma_1) N \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}, \qquad V = (0,\ \sigma_2) N \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}, \qquad W = (e^{-1},\ \sigma_1) N \begin{pmatrix} e^{-1} \\ \sigma_1 \end{pmatrix},$$

the identity
$$(e^{-1}, \sigma) N \binom{e^{-1}}{\sigma} = 0$$
is equivalent with
$$W + iA - i\bar{A} - V = 0,$$
so that
$$W = V, \quad A = \bar{A},$$
or A is a symmetrical matrix, while V is obviously skew symmetrical.

Hence
$$ia N \bar{a}_0 = i(e^{-1}, \sigma) N \binom{e^{-1}}{\sigma_0} = iW + A + \bar{A} + iV,$$
and, putting $y = x + i\xi$, $y_0 = x - i\xi$, so that each of x, ξ is a row of n real quantities, the inequality
$$-iaE\Delta E\bar{a}_0 y_0 y > 0$$
is the same as
$$(A + iV)(x - i\xi)(x + i\xi) < 0.$$
Since
$$A(x\xi - \xi x) = 0, \quad Vx^2 = V\xi^2 = 0;$$
the left side of this is
$$A(x^2 + \xi^2) + V(\xi x - x\xi),$$
namely,
$$\begin{pmatrix} A & V \\ -V & A \end{pmatrix} (x, \xi)(x, \xi),$$
and is a real quadratic form in $2n$ variables; while
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} A & V \\ -V & A \end{pmatrix} - \lambda \right] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A - \lambda & V \\ -V & A - \lambda \end{pmatrix} = \begin{pmatrix} V & -A + \lambda \\ A - \lambda & V \end{pmatrix}$$
is a skew symmetrical matrix whose roots are all double. The real quadratic form in $2n$ variables is thus capable of a form
$$\lambda_1 (t_1^2 + u_1^2) + \ldots + \lambda_n (t_n^2 + u_n^2),$$
wherein $\lambda_1, \ldots, \lambda_n$ are n real quantities, and $t_1, u_1, \ldots, t_n, u_n$ are $2n$ real linear functions of $x_1, \ldots, \xi_n$; and the condition for the form to be constantly negative is that each of $\lambda_1, \ldots, \lambda_n$ be negative.

The reduction to this form can be made in steps as follows: let
$$A = (a_{pq}), \quad V = (v_{pq}), \quad b_{pq} = \frac{(a_{p1} + iv_{p1})(a_{1q} + iv_{1q})}{a_{11}}, \quad B = (b_{pq}),$$
the denominator a_{11} not being zero in virtue of the condition; thence
$$b_{1q} = a_{1q} + iv_{1q}, \quad b_{p1} = a_{p1} + iv_{p1},$$
and
$$B = A' + iV' = (a'_{pq}) + i(v'_{pq}),$$
the matrix A' being symmetrical, and V' skew symmetrical. We at once

find that

$$(A'+iV')(x-i\xi)(x+i\xi)$$

$$=\frac{1}{a_{11}}\left\{(a_{11}x_1+\ldots+a_{1n}x_n+v_{12}\xi_2+\ldots+v_{1n}\xi_n)^2 + (a_{11}\xi_1+\ldots+a_{1n}\xi_n-v_{12}x_2-\ldots-v_{1n}x_n)^2\right\}$$

$$=\frac{1}{a_{11}}(t_1^2+u_1^2),$$

and can write

$$(A+iV)(x-i\xi)(x+i\xi)$$

$$=\frac{1}{a_{11}}(t_1^2+u_1^2)+[(A-A')+i(V-V')](x-i\xi)(x+i\xi),$$

where on the right the form at the end does not contain x_1 or ξ_1, but is of the same form as the original in $x_2, \ldots, x_n, \xi_2, \ldots, \xi_n$. To it the same reduction can then be applied.

If we put, as above, $\Delta=\begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix}$,

the condition to be satisfied is

$$\begin{pmatrix} \sigma_2 e(Q+Pe\sigma_1) & -\sigma_2 ePe\sigma_2 \\ \sigma_2 ePe\sigma_2 & \sigma_2 e(Q+Pe\sigma_1) \end{pmatrix}(x, \xi)(x, \xi)<0.$$

Putting $\Delta=rE^{-1}+k\nabla, \quad P=kp, \quad Q=-re^{-1}+kq$ (see p. 369)

and taking $p=0$, the condition breaks down into

$$\sigma_2(keq-r)\,x^2<0.$$

When $n=2$, the conditions for this are that the coefficient of x_1^2 should be negative, and that the determinant of the form in x_1, x_2 should be positive. If

$$q=\begin{pmatrix} 0 & -A \\ C & -B \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} \xi & \eta \\ \eta & \zeta \end{pmatrix},$$

we have $keq-r=\begin{pmatrix} -r & -Ae_1k \\ Ce_2k & -r-Be_2k \end{pmatrix}$,

and the former of these conditions is

$$-\xi r+Ce_2k\eta<0,$$

while the latter is $|\sigma_2|\,|keq-r|>0$, which, in virtue of $|\sigma_2| > 0$, reduces to

$$|keq-r| > 0,$$

that is,
$$r^2+rkBe_2+ACe_1e_2k^2 > 0.$$

For this case, and with $e_1=1$, $e_2=1$, the conditions are given by Humbert, *Liouville*, 1899, pp. 259, 260.

4. *The Number and Sum of the Common Zeros of n Jacobian Functions of n Variables.*

We consider now a set of n Jacobian functions of n variables $u_1, \ldots, u_n$, say

$$f_1(u-v^{(1)}),\ \ldots,\ f_n(u-v^{(n)}),$$

where $f_r(u)$ is subject to equations of the form

$$f_r(u+am) = e^{2\pi i H_r} f_r(u),$$

in which
$$H_r = b_r m(u+\tfrac{1}{2}am)+c_{r,m},$$

m denoting a row of $2n$ integers, and $c_{r,m}$ being independent of $u_1, \ldots, u_n$; we represent by Δ_r the matrix of integers $\bar{b}_r a-\bar{a}b_r$, by K a matrix of integers of type $(2n, 2n)$ such that the universal relation connecting the periods is

$$aK\bar{a} = 0$$

(see M.P.F., p. 286), and we put $D_r = \Delta_r K$. Also we put (cf. p. 357)

$$\delta_r = \tfrac{1}{2}\sum_{l=1}^{2n}(D_r)_{l,l}, \qquad \delta_{12} = \frac{1}{2(2!)}\sum_{l=1}^{2n}(D_1D_2+D_2D_1),$$

and so on, so that, for instance, δ_{123} is one half the sum, divided by (3!) of the diagonal elements in the matrix which is the sum of all the (3!) products of D_1, D_2, D_3. Then we put

$$\phi_r = \delta_r, \qquad \phi_{12} = \delta_1\delta_2-\delta_{12}, \qquad \phi_{123} = \delta_1\delta_2\delta_3-\delta_1\delta_{23}-\delta_2\delta_{31}-\delta_3\delta_{12}+2\delta_{123},$$

$$\phi_{1234} = \delta_1\delta_2\delta_3\delta_4-\Sigma\delta_1\delta_2\delta_{34}+\Sigma\delta_{12}\delta_{34}+2\Sigma\delta_1\delta_{234}-6\delta_{1234},$$

and so on, the general form being obtainable by the equation (see above, p. 359)

$$k\phi_{12\ldots k} = \sum_1 \delta_1\phi_{23\ldots k}-(2!)\sum_2 \delta_{12}\phi_{34\ldots k}+(3!)\sum_3 \delta_{123}\phi_{45\ldots k}-\ldots.$$

Finally, we denote by H the positive integer which is the square root of the determinant of the skew symmetrical matrix K.

Then, if $v^{(m)}$ denote a set of n constants, we prove that the n equations

$$f_1(u-v^{(1)}) = 0,\ \ldots,\ f_n(u-v^{(n)}) = 0,$$

have a number of sets of solutions for $u_1, \ldots, u_n$, no two of which are congruent in regard to the periods a, given by

$$\frac{1}{H}\,\phi_{12\ldots n},$$

and that the sum of the values of u_a at these solutions, save for quantities independent of $v^{(1)}, \ldots, v^{(n)}$, is equal to the a-th of the quantities expressed by

$$\frac{1}{H}\,[aK\psi_{23\ldots n}\,\bar{b}_1\,v^{(1)}+aK\psi_{13\ldots n}\,\bar{b}_2\,v^{(2)}+\ldots+aK\psi_{12\ldots(n-1)}\,\bar{b}_n\,v^{(n)}],$$

where $\psi_{12\ldots k}$, as before (p. 360), denotes the matrix which can be symbolically denoted by

$$(\phi_1'-D_1')\ldots(\phi_k'-D_k'),$$

where, after multiplication, we are to replace ϕ_1' by ϕ_1, $\phi_1'\phi_2'$ by ϕ_{12}, $\phi_1'\phi_2'\phi_3'$ by ϕ_{123}, and so on, and are to replace D_1' by D_1, $D_1'D_2'$ by $(D')_{12}$, that is, $D_1D_2+D_2D_1$, and $D_1'D_2'D_3'$ by $(D')_{123}$, that is, by

$$D_1D_2D_3+D_1D_3D_2+D_2D_3D_1+D_2D_1D_3+D_3D_1D_2+D_3D_2D_1,$$

and so on. Thus the term in $v^{(m)}$ in the sum is symbolically

$$\frac{1}{H}\,aK\,\frac{(\phi_1'-D_1')\ldots(\phi_n'-D_n')}{\phi_m'-D_m'}\,\bar{b}_m\,v^{(m)}.$$

As already remarked (p. 359), when the functions $f_1(u), \ldots, f_n(u)$ are the same, and therefore the matrices $D_1, \ldots, D_n$ all reduce to the same matrix D, the function $\phi_{12\ldots n}$ reduces to $(n!)$ times the square root of the determinant of the matrix D. In general $\phi_{12\ldots n}$ is the coefficient of $h_1h_2\ldots h_n$ in the square root of the determinant of the matrix

$$h_1D_1+\ldots+h_nD_n;$$

this is the matrix $(\bar{B}a-\bar{a}B)K$, where B is the matrix arising by adding periods to the arguments for the function

$$F(u)=[f_1(u)]^{h_1}[f_2(u)]^{h_2}\ldots[f_n(u)]^{h_n},$$

which gives $F(u+am)/F(u)=\exp 2\pi i\,\{Bm(u+\tfrac{1}{2}am)+C\}$,

C being independent of u.

If any difference $u-v^{(m)}$ be written in the form

$$u-\frac{1}{n}[v^{(1)}+\ldots+v^{(n)}]-w^{(m)},$$

we can similarly obtain a formula for the sum of the incongruent values of

$$w = u - \frac{1}{n}\left[v^{(1)} + \ldots + v^{(n)}\right],$$

which satisfy the equations

$$f_1(w - w^{(1)}) = 0, \; \ldots, \; f_n(w - w^{(n)}) = 0,$$

which, so far as it depends upon $v^{(1)}, \ldots, v^{(n)}$, is linear in $w^{(1)}, \ldots, w^{(n)}$. Each of the quantities

$$w^{(m)} = v^{(m)} - \frac{1}{n}\left[v^{(1)} + \ldots + v^{(n)}\right]$$

vanishes, however, when we put

$$v^{(1)} = v^{(2)} = \ldots = v^{(n)}.$$

The number of values of w being $\phi_{12\ldots n}/H$, we thus infer that, for arbitrary v,

$$\frac{1}{H}\phi_{12\ldots n}\, v = \frac{1}{H}\sum_{i=1}^{n} aKP_i\bar{b}_i v,$$

where $$P_i \text{ (symbolically)} = \frac{(\phi_1' - D_1') \ldots (\phi_n' - D_n')}{\phi_i' - D_i'}.$$

Now we know that any n quantities v can be written in the form $a\xi$, where ξ is a row of $2n$ real quantities; we have only to equate real and imaginary parts, and utilise the known properties of the matrix a (M.P.F., p. 225); also we have $aK\bar{a} = 0$, $aKD\bar{a} = 0$, $aKD_1D_2\bar{a} = 0$ (see above, p. 367), and so on, and therefore, as will abundantly appear,

$$aKP_i\bar{a} = 0;$$

hence we can write

$$aKP_i\bar{b}_i v = aKP_i\bar{b}_i a\xi = aKP_i(\Delta_i + \bar{a}b_i)\xi = aKP_i\Delta_i\xi,$$

and obtain $$a\left(\phi_{12\ldots n} - \sum_{i=1}^{n} KP_i\Delta_i\right)\xi = 0;$$

but, if η be a set of $2n$ real quantities, an equation $a\eta = 0$, involving $a_0\eta = 0$, involves, in virtue of the inequality,

$$-iaK\bar{a}_0x_0x > 0$$

(M.P.F., p. 286), that $\eta = 0$; thus the equation here leads to

$$\phi_{12\ldots n} - \sum_{i=1}^{n} KP_i\Delta_i = 0; \qquad \text{(P)}$$

now, for instance, P_1 is $\psi_{23\ldots n}$, namely,

$$\phi_{23\ldots n}-D_2\phi_{34\ldots n}-D_3\phi_{24\ldots n}-\ldots+(D_2D_3+D_3D_2)\phi_{4\ldots n}+\ldots,$$

and the matrix obtained by transposing $KP_1\Delta_1$ is thus

$$\begin{aligned}\Delta_1\bar{P}_1K &= \Delta_1\{\phi_{23\ldots n}-K\Delta_2\phi_{34\ldots n}-K\Delta_3\phi_{24\ldots n}-\ldots\\ &\qquad\qquad+(K\Delta_3K\Delta_2+K\Delta_2K\Delta_3)\phi_{4\ldots n}+\ldots\}K\\ &= D_1\phi_{23\ldots n}-D_1D_2\phi_{34\ldots n}-D_1D_3\phi_{24\ldots n}-\ldots\\ &\qquad\qquad+D_1(D_3D_2+D_2D_3)\phi_{4\ldots n}+\ldots\\ &= D_1\psi_{23\ldots n},\end{aligned}$$

and so on ; thus the equation (P) here leads, after transposition, to

$$\phi_{12\ldots n}-D_1\psi_{23\ldots n}-D_2\psi_{134\ldots n}-\ldots-D_n\psi_{12\ldots(n-1)}=0,$$

that is (see p. 360) to $\qquad\psi_{12\ldots n}=0,$

the fundamental equation connecting the matrices $D_1, \ldots, D_n$.

It thus appears that the formula for the sum of the incongruent solutions includes the formula for the number of solutions.

In order to prove these results, we utilise the theory of a Riemann surface. The Jacobian functions $f(u)$ give rise, by taking the second partial logarithmic differential coefficients, to functions of $u_1, \ldots, u_n$, with no essential singularities for finite values of these, which are periodic with the system of periods a. There can be constructed then a Riemann surface with p ($\geqslant n$) everywhere finite integrals upon which $u_1, \ldots, u_n$ may be regarded as a defective system of everywhere finite integrals (M.P.F., Chap. VII). The period system of the integrals u upon this Riemann surface, a matrix Π of n rows and $2p$ columns, is then capable of the form ak, where a is the period system here employed and k a matrix of integers of type $(2n, 2p)$. (Cf. M.P.F., p. 286.)

For this Riemann surface we require a preliminary lemma. Let ϕ be a function of the n integrals of the first kind, $u_1(x), \ldots, u_n(x)$, which is single valued on the dissected Riemann surface, and is capable of expression about any place of this as a series of integral powers of the parameter of the place, consisting either of only positive powers, or of such a series multiplied by a finite negative power ; which, further, when (x) passes from a point on the right side of the j-th period loop to the corresponding point on the left side—in which case u is increased by $ak^{(j)}$, where $k^{(j)}$ denotes the j-th column of the matrix k—is multiplied by $\exp(2\pi i H_j)$, where

$$H_j = bk^{(j)}u+ck^{(j)}v+d_j,$$

b and c denoting matrices of type $(n, 2n)$, so that $bk^{(j)}$, $ck^{(j)}$ are each a row of n quantities, v (like u) being a row of n quantities, and d_j a single quantity depending on j, but independent of (x) and v; then

(1) The difference between the number of places (x) where the function ϕ under consideration vanishes to the first order, and the number of places (x) where it has a pole of the first order, or the difference between the aggregate number of zeros and poles when these are multiple, is

$$\tfrac{1}{2} \sum_{l=1}^{2n} \{(\bar{b}a - \bar{a}b)K\}_{l,l},$$

where $K = k\epsilon_{2p}\bar{k}$, ϵ_{2p} denoting the matrix of type $(2p, 2p)$ of which the elements $(j, j+p)$ are each -1 for $j \leqslant p$, and the elements $(j+p, j)$ are each $+1$.

(2) The difference between the sum of the values of the α-th of the integrals $u_1, \ldots, u_n$ at these zeros and at these poles is the α-th of the quantities

$$-aK\bar{c}v,$$

save for an additive quantity not depending on $v_1, \ldots, v_n$.

It is unnecessary to repeat the details of the demonstration (see M.P.F., p. 290). The first result is to be obtained by the integral

$$\frac{1}{2\pi i} \int d \log \phi$$

taken round the period loops of the Riemann surface, and is equal to

$$\sum_{\alpha=1}^{n} \frac{1}{2\pi i} \int \frac{1}{\phi} \frac{\partial \phi}{\partial u_\alpha}\, du_\alpha = \frac{1}{2\pi i} \sum_{\alpha=1}^{n} \sum_{j=1}^{p} \left[(bk)_{\alpha j}(ak)_{\alpha, j+p} - (ak)_{\alpha j}(bk)_{\alpha, j+p}\right]$$

$$= \frac{1}{2\pi i} \sum_{\alpha=1}^{n} \sum_{j=1}^{p} \sum_{l, m}^{1 \ldots 2n} b_{\alpha l} a_{\alpha m} \left[k_{lj} k_{m, j+p} - k_{mj} k_{l, j+p}\right]$$

$$= \frac{1}{2\pi i} \sum_{\alpha=1}^{n} \sum_{j=1}^{p} \sum_{l\, m}^{1 \ldots 2n} b_{\alpha l} a_{\alpha m} K_{m, l},$$

$$(\alpha = 1, \ldots, n;\ j = 1, \ldots, p),$$

which immediately gives the result above.

We may notice incidentally that another form for the result is

$$\sum_{\alpha=1}^{n} (aK\bar{b})_{\alpha, \alpha}.$$

To find the difference of the sum of the values of the α-th of the integrals $u(x)$ at the zeros and poles in question we are to evaluate the integral

$$-\frac{1}{2\pi i} \int du_\alpha \log \phi$$

taken round the period loops. Now for the two sides of the j-th loop

$(j \leqslant p)$ the values of $\log \phi$ differ, save possibly for a part of the form $2\pi i N_j$, where N_j is an integer, by

$$2\pi i \left[bk^{(j)} u + ck^{(j)} v + d_j\right],$$

and the contribution to the integral arising from the two sides of this loop is thus

$$-\int du_\alpha \left[bk^{(j)} u + ck^{(j)} v + d_j\right]$$

taken along the positive side of this loop, that is, from the right to the left side of the loop $(j+p)$; a similar statement can be made for the $(j+p)$-th loop, but the integral obtained will be taken along the positive (or left) side of the loop $(j+p)$, from the left to the right side of the j-th loop. Altogether the loop pair $(j, j+p)$ gives rise to a contribution in which the part containing v is

$$-\left[(ak)_{\alpha, j+p}\, ck^{(j)} v - (ak)_{\alpha, j}\, ck^{(j+p)} v\right],$$

which is

$$(ak)_{\alpha, j} \sum_{\beta=1}^{n} (ck)_{\beta, j+p} v_\beta - (ak)_{\alpha, j+p} \sum_{\beta=1}^{n} (ck)_{\beta, j} v_\beta,$$

or

$$\sum_{\beta=1}^{n} \left[(ak)_{\alpha j} (ck)_{\beta, j+p} - (ak)_{\alpha, j+p} (ck)_{\beta, j}\right] v_\beta,$$

or

$$-\sum_{\beta=1}^{n} \left[(ak)_{\alpha j} \epsilon_{j, j+p} (ck)_{\beta, j+p} + (ak)_{\alpha, j+p} \epsilon_{j+p, j} (ck)_{\beta, j}\right] v_\beta,$$

or

$$-\sum_{\beta=1}^{n} \left[(ak)_{\alpha j} (\epsilon \overline{ck})_{j, \beta} + (ak)_{\alpha, j+p} (\epsilon \overline{ck})_{j+p, \beta}\right] v_\beta.$$

The sum of these for all the loop pairs is thus

$$-\sum_{\beta=1}^{n} (ak\epsilon \bar{k} \bar{c})_{\alpha\beta} v_\beta$$

or

$$-(aK\bar{c}v)_\alpha,$$

as was stated.

When the function ϕ is a single Jacobian function $f(u-v)$, wherein u represents the integrals of the first kind on the Riemann surface, these formulæ give the number of places (x), and the sum of the values of $u(x)$, at the zeros of $f(u-v)$, in the respective forms

$$\tfrac{1}{2} \sum_{l=1}^{2n} \left\{(\bar{b}a - \bar{a}b)K\right\}_{l, l}$$

and

$$-aK\bar{b}v,$$

save, in the latter case, for additive quantities independent of v.

With the help of this lemma we proceed to the proof of the general result stated at the beginning of this section. The Riemann surface being obtained, as explained, if $u_1, \ldots, u_n$ denote arbitrary values, the number of sets of positions of n places (x_1), (x_2), ..., (x_{n-1}), (x) which satisfy the n congruences

$$u_i(x_1)+\ldots+u_i(x_{n-1})+u_i(x) \equiv u_i \quad (i = 1, \ldots, n)$$

is known (M.P.F., p. 250) to be $H = \sqrt{|\bar{K}|}$. Consider, then, the n equations

$$f_1[u(x_1)+\ldots+u(x_{n-1})+u(x)-v^{(1)}] = 0, \quad \ldots,$$
$$f_n[u(x_1)+\ldots+u(x_{n-1})+u(x)-v^{(n)}] = 0; \quad \text{(A)}$$

since to each value of u_i there correspond H sets of positions for (x_1), ..., (x_{n-1}), (x), and to each such set of positions there correspond, by the symmetry among (x_1), ..., (x_{n-1}), (x), n positions for x, it follows from the general result enunciated at the beginning of this section that the n equations (A) are satisfied, with appropriate positions for (x_1), (x_2), ..., (x_{n-1}), by $n\phi_{12\ldots n}$ positions for x; and, conversely, the establishment of this result will prove the general result as to the number of solutions of the equations

$$f_1(u-v^{(1)}) = 0, \quad \ldots, \quad f_n(u-v^{(n)}) = 0. \quad \text{(B)}$$

Again, if $(x_1^{(\lambda)})$, ..., $(x_{n-1}^{(\lambda)})$, $(x^{(\lambda)})$ denote any one of the sets of solutions of equations (A), the sum of the values of $u_a(x^{(\lambda)})$, at the various positions of (x), will be the same as the sum of the quantities

$$u_a(x_1^{(\lambda)})+u_a(x_2^{(\lambda)})+\ldots+u_a(x_{n-1}^{(\lambda)})+u_a(x^{(\lambda)}),$$

and therefore the same as the sum of the values of u_a at the solutions of the equations (B), when each such solution is counted as many times over as it arises for different sets of solutions of the equations

$$u_i(x_1)+\ldots+u_i(x_{n-1})+u_i(x) \equiv u_i \quad (i = 1, \ldots, n),$$

that is, H times over. In other words, the sum of the values $u_a(x^{(\lambda)})$ for the solutions (x) of the equations (A) is, in accordance with the general theorem enunciated above, given by the a-th element in

$$S_{n,0} = \sum_{i=1}^{n} aKP_i\bar{b}_i v^{(i)},$$

save for quantities independent of $v^{(1)}$, ..., $v^{(n)}$; and, conversely, if this is proved, the general theorem enunciated is also proved as regards the sum of the values of u satisfying the equations (B).

We consider then the equations (A), but first of all under a somewhat generalised form; namely, denoting $u(x_1)$, $u(x_2)$, ..., $u(x_{n-1})$, $u(x)$ respectively by $u_1, u_2, \ldots, u_{n-1}, u$, we consider, as equations for (x) upon the Riemann surface, the s equations

$$f_1(u_1+\ldots+u_{s-1}-v^{(1)}) = 0, \quad \ldots, \quad f_r(u_1+\ldots+u_{s-1}-v^{(r)}) = 0,$$
$$f_{r+1}(u_1+\ldots+u_{s-1}+u-v^{(r+1)}) = 0. \quad \ldots, \quad f_s(u_1+\ldots+u_{s-1}+u-v^{(s)}) = 0,$$

in which r has one of the values 0, 1, 2, ..., $(s-1)$. These equations will be said to form a case (s, r), and we denote the number of positions (x) satisfying them by $N_{s,r}$, and the sum of the corresponding values of u by $S_{s,r}$, this latter being a row of n quantities. We have seen that we desire to prove that

$$N_{n,0} = n\phi_{12\ldots n}, \qquad S_{n,0} = \sum_{i=1}^{n} aKP_i\bar{b}_i v^{(i)};$$

we obtain an explicit formula for

$$N_{s,r}-N_{s,r+1}, \qquad S_{s,r}-S_{s,r+1},$$

where, if $r+1 = s$, both $N_{s,r+1}$ and $S_{s,r+1}$ are to be replaced by zero; this enables us then to find $N_{s,0}$ and $S_{s,0}$.

For this consider first the equations

$$f_1(u_1+\ldots+u_{s-1}-v^{(1)}) = 0, \quad \ldots, \quad f_r(u_1+\ldots+u_{s-1}-v^{(r)}) = 0,$$
$$f_{r+2}(u_1+\ldots+u_{s-1}+u-v^{(r+2)}) = 0, \quad \ldots, \quad f_s(u_1+\ldots+u_{s-1}+u-v^{(s)}) = 0,$$

obtained from the set above by omission of the $(r+1)$-th of them; regard these last as equations for $(x_1), \ldots, (x_{s-1})$ in terms of (x); the supposed substitution of their values in the omitted equation

$$f_{r+1}(u_1+\ldots+u_{s-1}+u-v^{(r+1)}) = 0$$

leaves an equation for (x) only. The $(s-1)$ equations for $(x_1), \ldots, (x_{s-1})$ form a case $(s-1, 0)$, in which, however, $v^{(r+1)}, \ldots, v^{(s-1)}$ are replaced by $v^{(r+2)}-u, \ldots, v^{(s)}-u$ respectively. The number of positions for any one of the places $(x_1), \ldots, (x_{s-1})$ which satisfy these equations being $N_{s-1,0}$, the number of sets of positions for $(x_1), \ldots, (x_{s-1})$ is $\frac{1}{s-1}N_{s-1,0}$, and the sum of the values of u at these sets of positions, or

$$\sum_{(\lambda)} (u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}),$$

is $S_{s-1,0}$, it being remembered that the $v^{(r+1)}, \ldots, v^{(s-1)}$ of the general case $(s-1, 0)$ are here to be replaced by $v^{(r+2)}-u, \ldots, v^{(s)}-u$. We shall assume, as the formula for $S_{s-1,0}$ in the general case $(s-1, 0)$, an expression of the form

$$aKP_1\bar{b}_1v^{(1)}+\ldots+aKP_{s-1}\bar{b}_{s-1}v^{(s-1)},$$

where $P_1, \ldots, P_{s-1}$ are matrices depending only on the functions

$f_1, \ldots, f_{s-1}$ involved in that case; this is correct for $s-1=1$, and, when we have thence deduced the formula for the case $(s, 0)$, will be justified in general by induction. In the particular case $(s-1, 0)$ now under consideration, instead of the functions $f_{r+1}, \ldots, f_{s-1}$ we have the functions $f_{r+2}, \ldots, f_s$, and we shall denote the matrices consequently replacing $P_1, \ldots, P_{s-1}$ respectively by $Q_1, \ldots, Q_r, Q_{r+2}, \ldots, Q_s$; similarly we shall denote the corresponding number $N_{s-1,0}$ by $M_{s-1,0}$. The sum of the solutions for our particular case $(s-1, 0)$ will then be

$$aKQ_1\bar{b}_1v^{(1)}+\ldots+aKQ_r\bar{b}_rv^{(r)}+aKQ_{r+2}\bar{b}_{r+2}v^{(r+2)}+\ldots+aKQ_s\bar{b}_sv^{(s)}$$
$$-(aKQ_{r+2}\bar{b}_{r+2}+\ldots+aKQ_s\bar{b}_s)u.$$

The places $(x_1), \ldots, (x_{s-1})$ being thus supposed determined in terms of (x), consider as a function of (x) the quotient

$$\prod_{\lambda} f_{r+1}[u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}+u-v^{(r+1)}]/f_{r+1}[u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}-v^{(r+1)}],$$

referring to the function f_{r+1} omitted in the determination of $(x_1), \ldots, (x_{s-1})$ in terms of (x), there being as many factors in the numerator and denominator as sets $(x_1), \ldots, (x_{s-1})$ corresponding to a given position of (x), namely, $\dfrac{1}{s-1}M_{s-1,0}$.

When (x) changes its position from any point on the right of the j-th period loop to the corresponding place on the left side of this loop, the integral u will increase by the quantity $ak^{(j)}$, where $k^{(j)}$ denotes the j-th column of the matrix k; that is, the integral u_a will increase by the a-th of the n quantities $ak^{(j)}$; the sum

$$u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}$$

will change in consequence into a quantity of the form

$$u_1^{(\mu)}+\ldots+u_{s-1}^{(\mu)}+akt_\mu,$$

where μ refers to another set of positions for $(x_1), \ldots, (x_{s-1})$ belonging to the same position for (x), and t_μ denotes a set of $2p$ integers remaining constant along the period loop, that is, the a-th of the n quantities $u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}$ will change into the a-th of the new quantities. In consequence of this, the single factor

$$f_{r+1}[(u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}+u-v^{(r+1)}]/f_{r+1}[u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}-v^{(r+1)}]$$

will be multiplied by $e^{2\pi ih}$, where

$$h=[b_{r+1}kt_\mu+b_{r+1}k^{(j)}][u_1^{(\mu)}+\ldots+u_{s-1}^{(\mu)}+u-v^{(r+1)}]$$
$$-b_{r+1}kt_\mu[u_1^{(\mu)}+\ldots+u_{s-1}^{(\mu)}-v^{(r+1)}]+D_j,$$

D_j being independent of (x) and $v^{(1)}, \ldots, v^{(s)}$. Hence the product of such factors will be affected with a multiplier $e^{2\pi i H}$, where, save for quantities independent of (x) and $v^{(1)}, \ldots, v^{(s)}$,

$$H = \Big(\sum_{(\lambda)} b_{r+1} k t_\mu\Big) u + b_{r+1} k^{(j)} \sum_{(\lambda)} [u_1^{(\mu)} + \ldots + u_{s-1}^{(\mu)}] + \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} [u - v^{(r+1)}].$$

Now from the formula above for the sum of the solutions, in our case $(s-1, 0)$, we can write down the sum of the expressions

$$u_1^{(\lambda)} + \ldots + u_{s-1}^{(\lambda)},$$

and also the sum of the expressions

$$u_1^{(\mu)} + \ldots + u_{s-1}^{(\mu)} + akt_\mu;$$

by subtraction we thence obtain

$$ak \sum_{(\lambda)} t_\mu = -(aKQ_{r+2}\bar{b}_{r+2} + \ldots + aKQ_s\bar{b}_s)\, ak^{(j)};$$

we have previously remarked that $aK\bar{a} = 0$; it will be seen in the course of this argument that, if we assume

$$aKP_i\bar{a} = 0 \quad (i = 1, \ldots, s-1),$$

the matrices P_i being, as above explained, those that arise in the solution of the case $(s-1, 0)$, then it will also follow, for the case $(s, 0)$, that

$$aKP_i^{(s)}\bar{a} = 0 \quad (i = 1, \ldots, s),$$

we shall then be justified in making the former assumption; this gives, however,

$$aKP_i\bar{b}_i a = aKP_i\bar{a}b_i + aKP_i\Delta_i = aKP_i\Delta_i,$$

and therefore, considering our particular case $(s-1, 0)$, the formula above, for $ak \sum_{(\lambda)} t_\mu$, gives

$$a\Big[k \sum_{(\lambda)} t_\mu + (KQ_{r+2}\Delta_{r+2} + \ldots + KQ_s\Delta_s)\, k^{(j)}\Big] = 0,$$

which is of the form $a\xi = 0$, wherein the $2n$ quantities of the row ξ are all real, and must therefore be separately zero. Thus we can infer

$$b_{r+1} k \sum_{(\lambda)} t_\mu = -b_{r+1}[KQ_{r+2}\Delta_{r+2} + \ldots + KQ_s\Delta_s]\, k^{(j)},$$

and so rewrite the formula for H in the form

$$\begin{aligned} H = {} & -b_{r+1}[KQ_{r+2}\Delta_{r+2} + \ldots + KQ_s\Delta_s]\, k^{(j)} u + \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} [u - v^{(r+1)}] \\ & + b_{r+1} k^{(j)} [aKQ_1\bar{b}_1 v^{(1)} + \ldots + aKQ_r\bar{b}_r v^{(r)} + aKQ_{r+2}\bar{b}_{r+2} v^{(r+2)} + \ldots \\ & \qquad + aKQ_s\bar{b}_s v^{(s)} - (aKQ_{r+2}\bar{b}_{r+2} + \ldots + aKQ_s\bar{b}_s)\, u]. \end{aligned}$$

Now if μ, ν be matrices of types respectively $(n, 2n)$, (n, n), and x, y be rows respectively of $2n$ and n quantities, we have

$$\mu x . \nu y = \bar{\mu} \nu y x = \bar{\nu} \mu x y ;$$

thus
$$b_{r+1} k^{(j)} . a K Q_{r+2} \bar{b}_{r+2} u = - b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} k^{(j)} u,$$

and so on ; thus the coefficient of u in H is $B k^{(j)} u$, where

$$\begin{aligned} B = \quad & b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} + \ldots + b_s \bar{Q}_s K \bar{a} b_{r+1} \\ & - b_{r+1} K Q_{r+2} \Delta_{r+2} - \ldots - b_{r+1} K Q_s \Delta_s + \frac{M_{s-1,0}}{s-1} b_{r+1}, \end{aligned}$$

and the terms independent of u in H are

$$\begin{aligned} & - b_1 \bar{Q}_1 K \bar{a} b_{r+1} k^{(j)} v^{(1)} - \ldots - b_r \bar{Q}_r K \bar{a} b_{r+1} k^{(j)} v^{(r)} \\ & \qquad - b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} k^{(j)} v^{(r+2)} - \ldots - b_s \bar{Q}_s K \bar{a} b_{r+1} k^{(j)} v^{(s)} \\ & - \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} v^{(r+1)}. \end{aligned}$$

Here in B a representative pair of terms is

$$b_m \bar{Q}_m K \bar{a} b_{r+1} - b_{r+1} K Q_m \Delta_m \quad (m = r+2, \ldots, s) ;$$

if we form the matrix $\bar{B} a - \bar{a} B$, the corresponding terms are

$$\begin{aligned} & - \bar{b}_{r+1} a K Q_m \bar{b}_m a - \Delta_m \bar{Q}_m K \bar{b}_{r+1} a \\ & + \bar{a} b_{r+1} K Q_m \Delta_m - \bar{a} b_m \bar{Q}_m K \bar{a} b_{r+1} ; \end{aligned}$$

replacing $\bar{b}_m a$ by $\bar{a} b_m + \Delta_m$, $\bar{a} b_m$ by $\bar{b}_m a - \Delta_m$, and recalling

$$a K P_m \bar{a} = 0,$$

these terms become
$$\begin{aligned} & - \bar{b}_{r+1} a K Q_m \Delta_m - \Delta_m \bar{Q}_m K \bar{b}_{r+1} a \\ & + \bar{a} b_{r+1} K Q_m \Delta_m + \Delta_m \bar{Q}_m K \bar{a} b_{r+1}, \end{aligned}$$

or
$$- \Delta_{r+1} K Q_m \Delta_m - \Delta_m \bar{Q}_m K \Delta_{r+1}.$$

Thus by the lemma proved near the beginning of this section (p. 376), the difference between the number of zeros and poles of the function of (x) expressed by

$$\prod_{(\lambda)} f_{r+1} [u_1^{(\lambda)} + \ldots + u_{s-1}^{(\lambda)} + u - v^{(r+1)}] / f_{r+1} [u_1^{(\lambda)} + \ldots + u_{s-1}^{(\lambda)} - v^{(r+1)}],$$

is given by

$$\frac{M_{s-1,0}}{s-1} \tfrac{1}{2} \sum_{l=1}^{2n} (\Delta_{r+1} K)_{l,l} - \tfrac{1}{2} \sum_{m=r+2}^{s} \left\{ \sum_{l=1}^{2n} (\Delta_{r+1} K Q_m \Delta_m K + \Delta_m \bar{Q}_m K \Delta_{r+1} K)_{l,l} \right\} .$$

Also, by the same lemma, the difference between the sum of the values of $u_a(x)$ at the zeros and poles in question is the a-th element of

$$-aK[\bar{b}_{r+1}aKQ_1\bar{b}_1v^{(1)}+\ldots+\bar{b}_{r+1}aKQ_r\bar{b}_rv^{(r)}]$$

$$-aK[\bar{b}_{r+1}aKQ_{r+2}\bar{b}_{r+2}v^{(r+2)}+\ldots+\bar{b}_{r+1}aKQ_s\bar{b}_sv^{(s)}]+\frac{M_{s-1,0}}{s-1}aK\bar{b}_{r+1}v^{(r+1)},$$

which itself, in virtue of $aK\bar{a}=0$, can be written in the form

$$-aK[\Delta_{r+1}KQ_1\bar{b}_1v^{(1)}+\ldots+\Delta_{r+1}KQ_r\bar{b}_rv^{(r)}]$$

$$-aK[\Delta_{r+1}KQ_{r+2}\bar{b}_{r+2}v^{(r+2)}+\ldots+\Delta_{r+1}KQ_s\bar{b}_sv^{(s)}]+\frac{M_{s-1,0}}{s-1}aK\bar{b}_{r+1}v^{(r+1)}.$$

We can, however, introduce both the number of the zeros of the denominator function

$$\prod_{(\lambda)} f_{r+1}[u_1^{(\lambda)}+\ldots+u_{s-1}^{(\lambda)}-v^{(r+1)}],$$

and the sum of the values of $u(x)$ at these zeros, if we suppose that the case $(s, r+1)$ has been considered before the case (s, r) now under consideration. For these are the zeros of the set of equations

$$f_1[u_1+\ldots+u_{s-1}-v^{(1)}]=0,\quad\ldots,\quad f_r[u_1+\ldots+u_{s-1}-v^{(r)}]=0,$$

$$f_{r+1}[u_1+\ldots+u_{s-1}-v^{(r+1)}]=0,$$

$$f_{r+2}[u_1+\ldots+u_{s-1}+u-v^{(r+2)}]=0,\quad\ldots,\quad f_s[u_1+\ldots+u_{s-1}+u-v^{(s)}]=0,$$

of which the number is $N_{s,r+1}$, and the sum is $S_{s,r+1}$. When r is $s-1$, or $r+1$ is s, this last set of equations does not contain (x) at all, and the numbers $N_{s,s}$, $S_{s,s}$ may be replaced by zeros. In general we have, adding the number and sum of the solutions of these last equations to the number and sum found above for the quotient function

$$\left.\begin{aligned}N_{s,r}-N_{s,r+1}&=\frac{M_{s-1,0}}{s-1}\tfrac{1}{2}\sum_{l=1}^{2n}\{\Delta_{r+1}K\}_{l,l}\\&\quad-\tfrac{1}{2}\sum_{m=r+2}^{s}\left\{\sum_{l=1}^{2n}(\Delta_{r+1}KQ_m\Delta_mK+\Delta_m\bar{Q}_mK\Delta_{r+1}K)_{l,l}\right\}\\S_{s,r}-S_{s,r+1}&=\frac{M_{s-1,0}}{s-1}aK\bar{b}_{r+1}v^{(r+1)}-aK\Delta_{r+1}K\sum_{m=1}^{s}{}'\,Q_m\bar{b}_mv^{(m)}\end{aligned}\right\},\quad\text{(Y)}$$

where in $\sum\limits_{m=1}^{s}{}'$ the value $m=r+1$ is to be omitted, and

$$M_{s-1,0},\ Q_1,\ \ldots,\ Q_r,\ Q_{r+2},\ \ldots,\ Q_s$$

are obtained, it will be remembered, from

$$N_{s-1,0},\ P_1,\ \ldots,\ P_r,\ P_{r+1},\ \ldots,\ P_{s-1},$$

by changing the functions $f_{r+1}, \ldots, f_{s-1}$, and the matrices deduced from them, into the functions $f_{r+2}, \ldots, f_s$. In obtaining these formulæ, moreover, it has been assumed that

$$S_{s-1,0} = aKP_1\bar{b}_1v^{(1)} + \ldots + aKP_{s-1}\bar{b}_{s-1}v^{(s-1)},$$

and also that $$aKP_i\bar{a} = 0 \quad [i = 1, \ldots, (s-1)].$$

By examining the formulæ for the first few cases we shall arrive at an explicit form for $P_1, \ldots, P_{s-1}$, which can then be justified by induction. For $s = 2$, $r = 1$, the formulæ give

$$N_{2,1} = \frac{M_{1,0}}{1}\,\tfrac{1}{2}\sum_{l=1}^{2n}\{\Delta_2K\}_{l,l},$$

$$S_{2,1} = \frac{M_{1,0}}{1}\,aK\bar{b}_2v^{(2)} - aK\Delta_2KQ_1\bar{b}_1v^{(1)},$$

where the Q_1 refers to the corresponding case $(s-1, 0)$ or $(1, 0)$. For this, however, by the lemma (p. 376),

$$N_{1,0} = \tfrac{1}{2}\sum_{l=1}^{2n}\{\Delta_1K\}_{l,l}, \qquad S_{1,0} = aK\bar{b}_1v^{(1)},$$

and the substitutions to be made, of the suffixes $r+1, \ldots, s-1$ into $r+2, \ldots, s$, are nugatory. Hence we obtain

$$N_{2,1} = \delta_1\delta_2, \qquad S_{2,1} = \delta_1 . aK\bar{b}_2v^{(2)} - aK\Delta_2K\bar{b}_1v^{(1)},$$

where, if $D_i = \Delta_iK$, $$\delta_i = \tfrac{1}{2}\sum_{l=1}^{2n}(D_i)_{l,l}.$$

Next, for $s = 2$, $r = 0$, we have, by the induction formulæ,

$$N_{2,0} - N_{2,1} = \frac{M_{1,0}}{1}\,\delta_1 - \tfrac{1}{2}\sum_{l=1}^{2n}\{\Delta_1KQ_2\Delta_2K + \Delta_2\bar{Q}_2K\Delta_1K\}_{l,l}.$$

$$S_{2,0} - S_{2,1} = \frac{M_{1,0}}{1}\,aK\bar{b}_1v^{(1)} - aK\Delta_1KQ_2\bar{b}_2v^{(2)},$$

where, as before, Q_2 is derived from the formula relating to the case $(1, 0)$, and is unity, and the substitution to obtain $M_{1,0}$ and Q_2 from $N_{1,0}$ and P_1 is that of changing the suffix 1 into 2; thus $Q_2 = P_1 = 1$, $M_{1,0} = \delta_2$, and we obtain

$$N_{2,0} - N_{2,1} = \delta_2\delta_1 - \tfrac{1}{2}\sum_{l=1}^{2n}\{D_1D_2 + D_2D_1\}_{l,l} = \delta_1\delta_2 - 2\delta_{12},$$

$$N_{2,0} = 2(\delta_1\delta_2 - \delta_{12}),$$

δ_{12} being as defined before (p. 372); also we obtain

$$S_{2,0}-S_{2,1}=\delta_2 . aK\bar{b}_1 v^{(1)}-aK\Delta_1 K\bar{b}_2 v^{(2)},$$

and hence $S_{2,0}=(\delta_2 . aK\bar{b}_1-aK\Delta_2 K\bar{b}_1)\,v^{(1)}+(\delta_1 . aK\bar{b}_2-aK\Delta_1 K\bar{b}_2)v^{(2)}$

$$=aK(\delta_2-D_2)\,\bar{b}_1 v^{(1)}+aK(\delta_1-D_1)\,\bar{b}_2 v^{(2)},$$

which is in accordance with the assumption expressed by

$$S_{s-1,0}=aKP_1\bar{b}_1 v^{(1)}+\dots+aKP_{s-1}\bar{b}_{s-1}v^{(s-1)},$$

with $$P_1=\delta_2-D_2, \quad P_2=\delta_1-D_1,$$

so that $aKP_1\bar{a}=-aK\Delta_2 K\bar{a}=0, \quad aKP_2\bar{a}=-aK\Delta_1 K\bar{a}=0.$

The results are then in accord with those originally stated in this section (p. 379), namely, for $s-1=1$ or 2, we have shewn that

$$\frac{N_{s-1,0}}{s-1}=\phi_{1,2,\dots,(s-1)},$$

$$P_m=\psi_{1,2,\dots,m-1,m+1,\dots,(s-1)}=(\text{symbolically})\,\frac{(\phi_1'-D_1')\dots(\phi_{s-1}'-D_{s-1}')}{\phi_m'-D_m'}$$

$$[m=1,2,\dots,(s-1)],$$

the notation being that explained before (pp. 373, 375). Our process of induction taking the cases in the order

$[(1,0)],\ [(2,1),\ (2,0)],\ [(3,2),\ (3,1),\ (3,0)],\ \dots,$

$[(s-1,s-2),\dots,(s-1,0)],\ [(s,s-1),\dots,(s,r+1),\ (s,r),\dots,(s,0)],\ \dots,$

and the substitution to be made to pass from $N_{s-1,0}$, $P_1, \dots, P_{s-1}$ to $M_{s-1,0}$, $Q_1, \dots, Q_s$, being that of $r+1, \dots, s-1$ into $r+2, \dots, s$, we can hence assume, in the formulæ of induction (Y),

$$\frac{M_{s-1,0}}{s-1}=\phi_{1,2,\dots,r,r+2,\dots,s},$$

$$Q_i=\frac{(\phi_1'-D_1')\dots(\phi_r'-D_r')(\phi_{r+2}'-D_{r+2}')\dots(\phi_s'-D_s')}{\phi_i'-D_i'},$$

where i is in turn equal to $1, 2, \dots, r, r+2, \dots, s$, the symbol Q_{r+1} not occurring; that is,

$$Q_i=\frac{(\phi_1'-D_1')\dots(\phi_s'-D_s')}{(\phi_i'-D_i')(\phi_{r+1}'-D_{r+1}')}.$$

Further, we have the identities

$$\Delta_m K \Delta_{r+1} K = D_m D_{r+1},$$

$$\Delta_m \bar{D}_i K \Delta_{r+1} K = \Delta_m K \Delta_i K \Delta_{r+1} K = D_m (D_i) D_{r+1},$$

$$\Delta_m \overline{D_i} \overline{D_j} K \Delta_{r+1} K = \Delta_m \bar{D}_j \bar{D}_i K \Delta_{r+1} K = D_m (D_j D_i) D_{r+1},$$

and so on; thus, as in Q_m, every term $D_i D_j$ or $D_i D_j D_k$ occurs associated with $D_j D_i$ or $D_k D_j D_i$, and so on, we see that

$$\Delta_m \bar{Q}_m K \Delta_{r+1} K = D_m Q_m D_{r+1}.$$

Thus the induction formulæ (Y) can be written

$$\left.\begin{aligned} & N_{s,r} - N_{s,r+1} \\ & \quad = \delta_{r+1} \phi_{1,2,\ldots,r,r+2,\ldots,s} - \tfrac{1}{2} \sum_{m=r+2}^{s} \left\{ \sum_{l=1}^{2n} (D_{r+1} Q_m D_m + D_m Q_m D_{r+1})_{l,l} \right\} \\ & S_{s,r} - S_{s,r+1} \\ & \quad = \phi_{1,2,\ldots,r,r+2,\ldots,s} \, aK \bar{b}_{r+1} v^{(r+1)} - aK D_{r+1} \sum_{m=1}^{s}{}' \, Q_m \bar{b}_m v^{(m)} . \end{aligned}\right\}, \quad (Z)$$

where $\sum_{m=1}^{s}{}'$ omits the term for $m = r+1$, and symbolically

$$Q_m = \frac{(\phi_1' - D_1') \ldots (\phi_s' - D_s')}{(\phi_m' - D_m')(\phi_{r+1}' - D_{r+1}')}.$$

These are the final formulæ of induction, from which the results for all the cases can be deduced. We illustrate them by obtaining explicit results for all the cases (3, 2), (3, 1), (3, 0) for $s = 3$; and then we obtain the general result of this section for any case (s, 0).

For the case (3, 2), $s = 3$, $r+1 = 3$,

$$N_{s,r+1} = 0, \qquad S_{s,r+1} = 0, \qquad \sum_{m=r+2}^{s} = 0,$$

and

$$N_{3,2} = \delta_3 . \phi_{1,2},$$

$$S_{3,2} = \phi_{1,2} . aK \bar{b}_3 v^{(2)} - aK D_3 [(\phi_2 - D_2) \bar{b}_1 v^{(1)} + (\phi_1 - D_1) \bar{b}_2 v^{(2)}].$$

For the case (3, 1), $s = 3$, $r+1 = 2$,

$$N_{3,1} - N_{3,2} = \delta_2 . \phi_{13} - \tfrac{1}{2} \sum_{l=1}^{2n} \{ D_2 (\phi_1 - D_1) D_3 + D_3 (\phi_1 - D_1) D_2 \}_{l},$$

$$= \delta_2 . \phi_{13} - 2\phi_1 \delta_{23} + \tfrac{1}{2} \sum_{l=1}^{2n} \{ D_2 D_1 D_3 + D_3 D_1 D_2 \}_{l,l},$$

so that, by the previous case,

$$N_{3,1} = \delta_2\phi_{13}+\delta_3\phi_{12}-2\phi_1\delta_{23}+\tfrac{1}{2}\sum_{l=1}^{2n}\{D_2D_1D_3+D_3D_1D_2\}_{l,l},$$

while $S_{3,1}-S_{3,2} = \phi_{13}aK\bar{b}_2v^{(2)}-aKD_2[(\phi_3-D_3)\bar{b}_1v^{(1)}+(\phi_1-D_1)\bar{b}_3v^{(3)}]$,

so that, by the previous case,

$$S_{3,1} = \phi_{12}aK\bar{b}_3v^{(3)}+\phi_{13}aK\bar{b}_2v^{(2)}-aKD_2[(\phi_3-D_3)\bar{b}_1v^{(1)}+(\phi_1-D_1)\bar{b}_3v^{(3)}]$$
$$-aKD_3[(\phi_2-D_2)\bar{b}_1v^{(1)}+(\phi_1-D_1)\bar{b}_2v^{(2)}].$$

Lastly, for the case (3, 0), $s = 3$, $r+1 = 1$, we have

$$N_{3,0}-N_{3,1} = \delta_1\phi_{23}-\tfrac{1}{2}\sum_{l=1}^{2n}\{D_1Q_2D_2+D_2Q_2D_1+D_1Q_3D_3+D_3Q_3D_1\}_{l,l},$$

$$S_{3,0}-S_{3,1} = \phi_{23}aK\bar{b}_1v^{(1)}-aKD_1[Q_2\bar{b}_2v^{(2)}+Q_3\bar{b}_3v^{(3)}],$$

where $$Q_2 = \phi_3-D_3, \qquad Q_3 = \phi_2-D_2,$$

so that

$$N_{3,0}-N_{3,1} = \delta_1\phi_{23}-2\phi_3\delta_{12}-2\phi_2\delta_{13}$$
$$+\tfrac{1}{2}\sum_{l=1}^{2n}\{D_1D_3D_2+D_2D_3D_1+D_1D_2D_3+D_3D_2D_1\}_{l,l},$$

and hence, by the previous case,

$$\begin{aligned} N_{3,0} &= \delta_2\phi_{13}+\delta_3\phi_{12}+\delta_1\phi_{23}+6\delta_{123}-2\phi_1\delta_{23}-2\phi_2\delta_{13}-2\phi_3\delta_{12} \\ &= 3(\delta_1\delta_2\delta_3-\delta_1\delta_{23}-\delta_2\delta_{31}-\delta_3\delta_{12}+2\delta_{123}) \\ &= 3\phi_{123}, \end{aligned}$$

as was previously stated; while

$$S_{3,0}-S_{3,1} = \phi_{23}aK\bar{b}_1v^{(1)}-\phi_3aKD_1\bar{b}_2v^{(2)}-\phi_2aKD_1\bar{b}_3v^{(3)}$$
$$+aKD_1D_3\bar{b}_2v^{(2)}+aKD_1D_2\bar{b}_3v^{(3)};$$

and therefore, by the previous case,

$$\begin{aligned} S_{3,0} = {} & aK(\phi_{23}-\phi_2D_3-\phi_3D_2+D_2D_3+D_3D_2)\bar{b}_1v^{(1} \\ & +aK(\phi_{31}-\phi_3D_1-\phi_1D_3+D_3D_1+D_1D_3)\bar{b}_2v^{(2)} \\ & +aK(\phi_{12}-\phi_1D_2-\phi_2D_1+D_1D_2+D_2D_1)\bar{b}_3v^{(3)}, \end{aligned}$$

as also was previously stated (p. 373).

Taking now from the general induction formula (Z) (p. 386)

$$S_{s,r}-S_{s,r+1}=\phi_{12\ldots r,r+2,\ldots,s}\,aK\bar{b}_{r+1}v^{(r+1)}-aKD_{r+1}\sum_{m=1}^{s}{}'\,Q_m\bar{b}_m v^{(m)},$$

where in Σ' the value $m=r+1$ is to be omitted, and adding these equations for different values of r, we obtain

$$S_{s,0}=\sum_{r+1}^{s}\phi_{12\ldots r,r+2,\ldots,s}\,aK\bar{b}_{r+1}v^{(r+1)}-aK\sum_{r+1=1}^{s}D_{r+1}\sum_{m=1}^{s}{}'\,Q_m\bar{b}_m v^{(m)};$$

the right side is a sum of terms such as $aKT_m\bar{b}_m v^{(m)}$, where

$$T_m=\phi_{12\ldots m-1,m+1,\ldots,s}-D_1Q_{m,1}-D_2Q_{m,2}-\ldots-D_{m-1}Q_{m,m-1}$$
$$-D_{m+1}Q_{m,m+1}-\ldots-D_sQ_{m,s},$$

$Q_{m,i}$ being symbolically

$$\frac{(\phi_1'-D_1')\ldots(\phi_s'-D_s')}{(\phi_i'-D_i')(\phi_m'-D_m')},=\psi_{1,\ldots,i-1,i+1,\ldots,m-1,m+1,\ldots,s},$$

in accordance with the notation of p. 360. From the identity there remarked we have thence

$$T_m=\psi_{1,2,\ldots,m-1,m+1,\ldots,s}=\frac{(\phi_1'-D_1')\ldots(\phi_s'-D_s')}{\phi_m'-D_m'},$$

and so, finally,

$$S_{s,0}=\sum_{m=1}^{s}aK\psi_{12,\ldots,m-1,m+1,\ldots,s}\,\bar{b}_m v^{(m)},$$

which is of the same form for s as was that originally assumed for $s-1$; the general result of this section, so far as regards the sum of the solutions, is thus established by induction.

We have already proved that the formula for $N_{s,0}$ is a consequence of the formula stated for $S_{s,0}$ (p. 375), and it is therefore unnecessary to shew that this also follows from the formula (Z).

5. *Some Particular Cases.*

(I) If, beside the universal bilinear relation connecting the periods expressed by $aK\bar{a}=0$, or

$$\sum_{i,j}K_{i,j}\,a_{p,i}\,a_{q,j}=0\quad(p,q=1,\ldots,2n),$$

there exist no other bilinear relations, then the various forms we have obtained

$$aK\bar{a} = 0, \quad aK\Delta K\bar{a} = 0, \quad \ldots, \quad a\Delta^{-1}\bar{a} = 0,$$

must be the same; and therefore

$$K\Delta = \Delta K = r,$$

where r is an integer ($= Rm$ in the notation of M.P.F., p. 289). This would give, from the definitions (p. 372),

$$D_i = r_i, \quad \delta_i = nr_i, \quad \delta_{ij} = nr_i r_j, \quad \delta_{ijk} = nr_i r_j r_k, \quad \ldots,$$

and

$$\phi_1 = nr_1, \quad \phi_{12} = n^2 r_1 r_2 - nr_1 r_2 = n(n-1)\, r_1 r_2,$$

$$\phi_{123} = n^3 r_1 r_2 r_3 - 3n^2 r_1 r_2 r_3 + 2nr_1 r_2 r_3 = n(n-1)(n-2) r_1 r_2 r_3.$$

If we assume, in accordance with these, that for $m < k$,

$$\phi_{12\ldots m} = \frac{n!}{(n-m)!} r_1 r_2 \ldots r_m,$$

the formula (p. 359)

$$k\phi_{12\ldots k} = \sum_1 \delta_1 \phi_{23\ldots k} - 2! \sum_2 \delta_{12} \phi_{34\ldots k} + 3! \sum_3 \delta_{123} \phi_{4\ldots k} - \ldots$$

gives

$$k\phi_{12\ldots k} = n \left\{ k \frac{n!}{(n-k+1)!} - 2! \frac{k!}{2!\,(k-2)!} \frac{n!}{(n-k+2)!} \right.$$

$$\left. + 3! \frac{k!}{3!\,(k-3)!} \frac{n!}{(n-k+3)!} - \ldots \right\} r_1 r_2 \ldots r_k$$

$$= n\,[(k)!] \left\{ \binom{n}{k-1} - \binom{n}{k-2} + \binom{n}{k-3} - \ldots \right\} r_1 r_2 \ldots r_k$$

$$= k \frac{n!}{(n-k)!} r_1 \ldots r_k,$$

and justifies the assumption in general.

Also $\psi_{12} = \phi_{12} - \phi_1 D_2 - \phi_2 D_1 + D_1 D_2 + D_2 D_1$

$$= \frac{n!}{(n-2)!} r_1 r_2 - nr_1 r_2 - nr_2 r_1 + 2r_1 r_2 = (n-1)(n-2)\, r_1 r_2,$$

$$\psi_{123} = \phi_{123} - \phi_{23} D_1 - \ldots + \phi_1 (D')_{23} + \ldots - (D')_{123}$$

$$= \left\{ \frac{n!}{(n-3)!} - 3 \frac{n!}{(n-2)!} + 3 \frac{n!}{(n-1)!} 2! - 3! \right\} r_1 r_2 r_3$$

$$= (n-1)(n-2)(n-3) r_1 r_2 r_3,$$

and, in general,

$$\psi_{12\ldots k} = \frac{n!}{(n-k)!} - \frac{n!}{(n-k+1)!}\,\frac{k!}{1!\,(k-1)!} + \frac{n!}{(n-k+2)!}\,2!\,\frac{k!}{2!\,(k-2)!}$$
$$- \frac{n!}{(n-k+3)!}\,3!\,\frac{k!}{3!\,(k-3)!} + \ldots,$$

multiplied by $r_1 r_2 \ldots r_k$, which is

$$= (k!) \left\{ \binom{n}{k} - \binom{n}{k-1} + \binom{n}{k-2} - \binom{n}{k-3} + \ldots \right\} r_1 r_2 \ldots r_k$$
$$= (n-1)(n-2)\ldots(n-k)\, r_1 \ldots r_k.$$

Thus the formula for the sum of the solutions in the case $(n, 0)$

$$\sum_{i=1}^{n} aKP_i \bar{b}_i v^{(i)}$$

becomes $$[(n-1)!]\, r_1 \ldots r_n \sum_{i=1}^{n} \frac{1}{r_i}\, aK\bar{b}_i v^{(i)},$$

wherein $$aK\bar{b}_i = ar_i \Delta_i^{-1} \bar{b}_i = r_i$$

(see p. 368, for the proof of $a\Delta_i^{-1}\bar{b}_i = 1$), and so becomes

$$\Sigma v^{(i)} [(n-1)!]\, r_1 \ldots r_n.$$

As, with the particular periods considered in § 3, we have

$$H = \sqrt{|K|} = (e_1 e_2 \ldots e_n)^{-1},$$

our result is thus that for the n equations

$$f_1(u - v^{(1)}) = 0,\ \ldots,\ f_n(u - v^{(n)}) = 0,$$

in the most general case, the number and sum of the solutions are respectively

$$(n!)\, r_1 r_2 \ldots r_n / e_1 \ldots e_n \quad \text{and} \quad \{[(n-1)!]\, r_1 \ldots r_n / e_1 \ldots e_n\}\, \Sigma v^{(i)},$$

agreeing with Wirtinger's result (M.P.F., p. 293).

(II) If, next, we take the case of other bilinear relations than $aK\bar{a} = 0$ connecting the periods, namely (p. 369, § 3), with

$$a = (e^{-1}, \sigma), \qquad K = E,$$

suppose
$$b = k\beta - r(0,\ 1),$$
$$\Delta = k\nabla + rE^{-1},$$
where
$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & \rho \end{pmatrix},$$
$$p = \bar{p}_1 - p_1, \qquad \beta = e(p_1,\ q + \bar{p}_1 e\sigma),$$
then, the bilinear relations connecting the periods
$$aE\bar{a} = 0, \qquad aE\nabla E\bar{a} = 0,$$
being the same for all, the different Jacobian functions will be distinguished by the values of the integers k, r; and
$$b_i = k_i\beta - r_i(0,\ 1),$$
$$\Delta_i = k_i\nabla + r_i E^{-1}.$$
This gives
$$D_i = \Delta_i E = k_i \nabla E + r_i,$$
and hence
$$\delta_i = k_i\sigma_1 + r_i n,$$
where
$$\sigma_1 = \tfrac{1}{2}\sum_{l=1}^{2n} (\nabla E)_{l,\,l}.$$
Also
$$\delta_{ij} = \frac{1}{2(2!)}\sum_{l=1}^{2n}(D_i D_j + D_j D_i)_{l,\,l}$$
$$= k_i k_j \sigma_2 + (k_i r_j + k_j r_i)\,\sigma_1 + r_i r_j n,$$
where
$$\sigma_2 = \tfrac{1}{2}\sum_{l=1}^{2n}\{(\nabla E)^2\}_{l,\,l}.$$
Also
$$\delta_{ijk} = k_i k_j k_h \sigma_3 + (k_i k_j r_h + \ldots)\,\sigma_2 + (k_i r_j r_h + \ldots)\,\sigma_1 + r_i r_j r_h n,$$
where
$$\sigma_3 = \tfrac{1}{2}\sum_{l=1}^{2n}\{(\nabla E)^3\}_{l,\,l}.$$
Hence
$$\phi_{12} = \delta_1\delta_2 - \delta_{12} = 2k_1k_2P_2 + (k_1r_2 + k_2r_1)P_1(n-1) + r_1r_2n(n-1),$$
where
$$P_1 = \sigma_1, \qquad P_2 = \frac{1}{2\,!}(\sigma_1^2 - \sigma_2),$$
and
$$\phi_{123} = \delta_1\delta_2\delta_3 - \delta_1\delta_{23} - \ldots + 2\delta_{123}$$
$$= 3\,!\,k_1k_2k_3P_3 + (k_2k_3r_1 + \ldots)\,2\,!\,P_2(n-2)$$
$$+ (k_1r_2r_3 + \ldots)P_1(n-1)(n-2) + r_1r_2r_3n(n-1)(n-2),$$

where $$P_3 = \frac{1}{3!}(\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3),$$

and so on; from these the general law for $\phi_{12\ldots k}$ is clear, and hence the formula for the number of solutions of the Jacobian equations can be put down.

Also $$\psi_i = \phi_i - D_i = k_i(\sigma_1 - M) + r_i(n-1),$$

where $$M = \nabla E,$$

and $$\begin{aligned}\psi_{ij} &= \phi_{ij} - \phi_i D_j - \phi_j D_i + D_i D_j + D_j D_i \\ &= 2k_i k_j P_2 + (k_i r_j + k_j r_i)\, P_1(n-2) + r_i r_j\,(n-1)(n-2) \\ &\qquad - M[2k_i k_j P_1 + (k_i r_j + k_j r_i)(n-2)] + 2M^2 k_i k_j,\end{aligned}$$

and so on.

In particular, when $n = 2$, we have for the number of solutions of the equations
$$f_1(u - v^{(1)}) = 0, \quad f_2(u - v^{(2)}) = 0,$$

$$\frac{\phi_{12}}{e_1 e_2} = \frac{1}{e_1 e_2}\{k_1 k_2(\sigma_1^2 - \sigma_2) + (k_1 r_2 + k_2 r_1)\sigma_1 + 2r_1 r_2\},$$

while the sum of the solutions, save for quantities independent of $v^{(1)}$ and $v^{(2)}$, is
$$\frac{1}{e_1 e_2}\{aE[k_2(\sigma_1 - M) + r_2]\bar{b}_1 v^{(1)} + aE[k_1(\sigma_1 - M) + r_1]\bar{b}_2 v^{(2)}\}.$$

We proceed to find the explicit forms for these in the notation adopted by Humbert, *Liouville*, 1899, pp. 254, 270, and *Liouville*, 1900, p. 313, footnote. Here
$$\nabla = \begin{Bmatrix} 0 & D & 0 & A \\ -D & 0 & -C & B \\ 0 & C & 0 & -E \\ -A & -B & E & 0 \end{Bmatrix},$$

in which each element is an integer, and the element (1, 3) is taken to be zero. If, with our notation

$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & r \end{pmatrix},$$

then
$$p = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \qquad p_1 = \begin{pmatrix} 0 & -D \\ 0 & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} 0 & -A \\ C & -B \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

and hence
$$M = \nabla E = \begin{pmatrix} p & -q \\ \bar{q} & r \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} = \begin{pmatrix} -qe & -pe \\ re & -\bar{q}e \end{pmatrix},$$

which, with
$$e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

give
$$pe = \begin{pmatrix} 0 & De_2 \\ -De_1 & 0 \end{pmatrix}, \qquad qe = \begin{pmatrix} 0 & -Ae_2 \\ Ce_1 & -Be_2 \end{pmatrix},$$

$$\bar{q}e = \begin{pmatrix} 0 & Ce_2 \\ -Ae_1 & -Be_2 \end{pmatrix}, \qquad re = \begin{pmatrix} 0 & -Ee_2 \\ Ee_1 & 0 \end{pmatrix},$$

so that
$$M = \begin{pmatrix} 0 & Ae_2 & 0 & -De_2 \\ -Ce_1 & Be_2 & De_1 & 0 \\ 0 & -Ee_2 & 0 & -Ce_2 \\ Ee_1 & 0 & Ae_1 & Be_2 \end{pmatrix}$$

and
$$\sigma_1 = Be_2,$$

and
$$\sigma_2 = \tfrac{1}{2} \sum_{l=1}^{2n} \{M^2\}_{l,l}$$
$$= -ACe_1e_2 - DEe_1e_2 - ACe_1e_2 + B^2e_2^2 - DEe_1e_2,$$

which give
$$\sigma_1^2 - \sigma_2 = 2(AC+DE)e_1e_2.$$

The function $\phi_{12} = k_1k_2(\sigma_1^2-\sigma_2)+(k_1r_2+k_2r_1)\sigma_1+2r_1r_2$ thus becomes
$$\phi_{12} = 2k_1k_2e_1e_2(AC+DE)+(k_1r_2+k_2r_1)Be_2+2r_1r_2,$$
and the ϕ_{12}/e_1e_2 agrees with the result given by Humbert for the number of solutions of
$$f_1(u-v^{(1)}) = 0, \quad f_2(u-v^{(2)}) = 0$$
when $e_1 = 1 = e_2$. The function ϕ_{12} is the coefficient of $\lambda_1\lambda_2$ in
$$(\lambda_1k_1+\lambda_2k_2)^2e_1e_2(AC+DE)+(\lambda_1k_1+\lambda_2k_2)(\lambda_1r_1+\lambda_2r_2)Be_2+(\lambda_1r_1+\lambda_2r_2)^2,$$
which, as is easily verified, is the square root of the determinant of
$$\lambda_1(k_1M+r_1)+\lambda_2(k_2M+r_2) \quad \text{or} \quad (\lambda_1k_1+\lambda_2k_2)M+\lambda_1r_1+\lambda_2r_2.$$
This is an example of a general remark made above (p. 359).

To find the sum of the solutions of the two equations we have to evaluate two matrices of the form

$$aE\left[k_2(\sigma_1-M)+r_2\right]\bar{b}_1.$$

Now here

$$\begin{aligned}
\beta &= e(p_1,\ q+\bar{p}_1 e\sigma) \\
&= e\left[\begin{pmatrix}0 & -D\\ 0 & 0\end{pmatrix},\ \begin{pmatrix}0 & -A\\ C & -B\end{pmatrix}+\begin{pmatrix}0 & 0\\ -D & 0\end{pmatrix}\begin{pmatrix}e_1\sigma_{11} & e_1\sigma_{12}\\ e_2\sigma_{12} & e_2\sigma_{22}\end{pmatrix}\right] \\
&= e\begin{pmatrix}0 & -D & 0 & -A\\ 0 & 0 & C-De_1\sigma_{11} & -B-De_1\sigma_{12}\end{pmatrix} \\
&= \begin{pmatrix}0 & -De_1 & 0 & -Ae_1\\ 0 & 0 & Ce_2-De_1e_2\sigma_{11} & -Be_2-De_1e_2\sigma_{12}\end{pmatrix},
\end{aligned}$$

and therefore

$$\begin{aligned}
b_1 &= k_1\beta-r_1(0,\ 1) \\
&= \begin{pmatrix}0 & -k_1De_1 & -r_1 & -k_1Ae_1\\ 0 & 0 & Ck_1e_2-Dk_1e_1e_2\sigma_{11} & -Bk_1e_2-Dk_1e_1e_2\sigma_{12}-r_1\end{pmatrix},
\end{aligned}$$

Also

$$k_2(\sigma_1-M)+r_2 = \begin{Bmatrix}k_2Be_2+r_2 & -Ak_2e_2 & 0 & Dk_2e_2\\ Ck_2e_1 & r_2 & -Dk_2e_1 & 0\\ 0 & Ek_2e_2 & k_2Be_2+r_2 & Ck_2e_2\\ -Ek_2e_1 & 0 & -Ak_2e_1 & r_2\end{Bmatrix}$$

and

$$aE = (e^{-1},\ \sigma)\begin{pmatrix}0 & -e\\ e & 0\end{pmatrix} = (\sigma e,\ -1)$$

$$= \begin{pmatrix}\sigma_{11}e_1 & \sigma_{12}e_2 & -1 & 0\\ \sigma_{21}e_1 & \sigma_{22}e_2 & 0 & -1\end{pmatrix},$$

so that, if $aE\left[k_2(\sigma_1-M)+r_2\right]$ be written

$$\begin{pmatrix}\mu_{11} & \mu_{12} & \mu_{13} & \mu_{14}\\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24}\end{pmatrix},$$

we have

$$\begin{aligned}
\mu_{11} &= \quad (k_2Be_2+r_2)\sigma_{11}e_1+Ck_2e_1\sigma_{12}e_2, \\
\mu_{12} &= -Ak_2e_2\sigma_{11}e_1+r_2\sigma_{12}e_2-Ek_2e_2, \\
\mu_{13} &= -Dk_2e_1\sigma_{12}e_2-(k_2Be_2+r_2) \\
\mu_{14} &= \quad Dk_2e_2\sigma_{11}e_1-Ck_2e_2, \\
\mu_{21} &= \quad (k_2Be_2+r_2)\sigma_{21}e_1+Ck_2e_1\sigma_{22}e_2+Ek_2e_1, \\
\mu_{22} &= -Ak_2e_2\sigma_{21}e_1+r_2\sigma_{22}e_2, \\
\mu_{23} &= -Dk_2e_1\sigma_{22}e_2+Ak_2e_1, \\
\mu_{24} &= \quad Dk_2e_2\sigma_{21}e_1-r_2.
\end{aligned}$$

The matrix $$aE\left[k_2(\sigma_1-M)+r_2\right]\bar{b}_1$$

is then $$\begin{pmatrix}(1,1) & (1,2)\\ (2,1) & (2,2)\end{pmatrix},$$

where
$$\begin{aligned}
(1,1) &= -k_1 De_1.\mu_{12}-r_1.\mu_{13}-k_1 Ae_1.\mu_{14},\\
(1,2) &= (Ck_1e_2-Dk_1e_1e_2\sigma_{11})\mu_{13}-(Bk_1e_2+Dk_1e_1e_2\sigma_{12}+r_1)\mu_{14},\\
(2,1) &= -k_1 De_1.\mu_{22}-r_1.\mu_{23}-k_1 Ae_1.\mu_{24},\\
(2,2) &= (Ck_1e_2-Dk_1e_1e_2\sigma_{11})\mu_{23}-(Bk_1e_2+Dk_1e_1e_2\sigma_{12}+r_1)\mu_{24}.
\end{aligned}$$

With these forms, if we take account of the relation $aE\nabla E\bar{a}=0$, which in this case can be calculated to be the single relation

$$Ae_1\sigma_{11}+Be_2\sigma_{12}+Ce_2\sigma_{22}+De_1e_2(\sigma_{12}^2-\sigma_{11}\sigma_{22})+E=0,$$

in order to simplify (2, 2), we find that the formula for the sum of the solutions of the equations $f_1(u-v^{(1)})=0$, $f_2(u-v^{(2)})=0$ reduces to

$$\begin{aligned}
&\tfrac{1}{2}\phi_{12}(v^{(1)}+v^{(2)})\\
&\qquad+(k_2r_1-k_1r_2)\begin{pmatrix}\tfrac{1}{2}Be_2+D\sigma_{12}e_1e_2 & Ce_2-D\sigma_{11}e_1e_2\\ -Ae_1+D\sigma_{22}e_1e_2 & -\tfrac{1}{2}Be_2-D\sigma_{12}e_1e_2\end{pmatrix}(v^{(1)}-v^{(2)}).
\end{aligned}$$

The matrix multiplying $v^{(1)}-v^{(2)}$ is

$$\tfrac{1}{2}Be_2+(\bar{q}-\sigma ep)e.$$

ON THE MULTIPLICATION OF DIRICHLET'S SERIES

By G. H. HARDY.

[Received May 18th, 1911.—Read June 8th, 1911.]

1. In this paper I propose to generalise some results communicated to the Society in 1908.*

As in my former communication, I denote by A and B the series

$$a_1+a_2+a_3+\dots,\qquad b_1+b_2+b_3+\dots,$$

and by C the series $$c_1+c_2+c_3+\dots,$$

where c_n is a function of the a's and b's, to be defined more precisely in a moment. I shall also use the letters A, B, C to denote the sums of the series, when they are convergent.

I shall denote by $$A_x$$

the finite sum $$\sum_{n\leqslant x} a_n;$$

here x is not restricted to be integral. Similarly I define B_x, C_x.

2. When the series C is the product-series of A and B, formed in accordance with Cauchy's rule, we have

$$c_p = a_1 b_p + a_2 b_{p-1} + \dots + a_p b_1 = \sum_{(m+n=p+1)} a_m b_n.$$

This was the only case that I considered in my former paper.

Cauchy's rule for multiplication is, however, only one among an infinity. We are led to it by arranging the formal product of the power series

$$\Sigma a_m x^m,\qquad \Sigma b_n x^n$$

in ascending powers of x, and then putting $x=1$. It is the same thing to say that we arrange the formal product of the Dirichlet's series

$$\Sigma a_m e^{-ms},\qquad \Sigma b_n e^{-ns}$$

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* "The Multiplication of Conditionally Convergent Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410.

according to the ascending order of the sums

$$m+n,$$

associating together all the terms for which $m+n$ has the same value, and then put $s=0$. It is clear that we arrive at a generalisation of our conception of multiplication by considering the more general Dirichlet's series

$$\Sigma a_m e^{-\lambda_m s}, \quad \Sigma b_n e^{-\lambda_n s},^*$$

and arranging their formal product according to the ascending order of the sums

$$\lambda_m+\lambda_n.^\dagger$$

Let

$$\nu_1,\ \nu_2,\ \ldots,\ \nu_p,\ \ldots$$

be the ascending sequence defined by the possible values of $\lambda_m+\lambda_n$. Then the Dirichlet's product of the series A, B, according to the rule defined by the sequence λ_n, is

$$c_1+c_2+c_3+\ldots,$$

where

$$c_p = \sum_{(\lambda_m+\lambda_n=\nu_p)} a_m b_n.$$

Thus, if $\lambda_m = \log m$, so that the Dirichlet's series are ordinary Dirichlet's series,

$$\nu_p = \log p,$$

and

$$c_p = \sum_{mn=p} a_m b_n = \sum_{(d)} a_d b_{p/d},$$

the summation being extended to all the divisors of p.

3. The three classical theorems relating to ordinary multiplication have their analogues for the general form of Dirichlet's multiplication.

(1) **Analogue of Abel's Theorem.**—*If all three series are convergent, then* $C = AB$.

(2) **Analogue of Cauchy's Theorem.**—*If* A *and* B *are absolutely convergent, then* C *is absolutely convergent.*

(3) **Analogue of Mertens' Theorem.**—*If* A *is absolutely and* B *conditionally convergent, then* C *is convergent.*

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[*] Here, of course, (λ_m) is any increasing sequence whose limit is infinity.

[†] For a general account of the theory, see Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd. 2, S. 750. Landau remarks that there is no loss of generality in adopting the same sequence (λ_m) in both series: for if we had two sequences (λ_m), (μ_n), we could combine them into one, regarding some of the a's and b's as zero.

Of these results (2) is almost obvious, while (3) was deduced by Landau as a corollary of a more general theorem of Stieltjes.* The analogue of Abel's theorem was also proved by Landau, by considerations drawn from the theory of analytic functions. An elementary proof was afterwards found independently by Phragmen, Bohr, and Riesz,† who deduce it from the following theorem :—

(4) **Analogue of Cesàro's Theorem.**—*If A and B are convergent, then C is summable $(R, 1, \nu)$ to sum AB: that is to say,*

$$\frac{(\nu_2-\nu_1)\,C_1+(\nu_3-\nu_2)\,C_2+\ldots+(\nu_p-\nu_{p-1})\,C_{p-1}}{\nu_p}\rightarrow AB,$$

as $p \rightarrow \infty$.

4. In my former paper I proved the following theorems (for multiplication by Cauchy's rule).

(5) *If A and B are convergent, and*

$$na_n \rightarrow 0, \qquad nb_n \rightarrow 0,$$

then C is convergent.

(6) *The same result holds under the more general conditions*

$$|\,na_n\,| < K, \qquad |\,nb_n\,| < K.$$

I propose now to establish the analogues of these theorems for the general form of Dirichlet's multiplication. It might be thought that, as (5) is a special case of (6), it would not be worth while to prove it independently, as in my previous paper. This view would, I think, be mistaken. Theorem (5) above, and its generalisation, can be proved by a very much simpler argument than seems to be called for by (6) and its generalisation; and the simpler proof of the less general theorem affords a good deal more information about the behaviour of the product series than can be obtained in the more general case. It therefore seems worth while to keep the two distinct.‡

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* Stieltjes, *Nouvelles Annales*, Sér. 3, t. VI, p. 210; Landau, *Rendiconti di Palermo*, t. XXIV, p. 81; *Handbuch*, S. 752.

† Landau, *Handbuch*, S. 762 and 904; Riesz, *Comptes Rendus*, July 9, 1909; Bohr, *Nachrichten der König. Gesellschaft der Wiss. zu Göttingen*, 1909, S. 247.

‡ These theorems may be compared with Tauber's theorem (the converse of Abel's theorem on the continuity of power series) and its extension given recently in these *Proceedings* by Mr. Littlewood—a similar distinction presents itself in the case of the two latter theorems.

5. **Theorem 1.**—*If A and B are convergent, and*

$$\frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} \to 0, \qquad \frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}} \to 0,$$

then the product series C, formed by the rule of Dirichlet's multiplication corresponding to the sequence (λ_n), is convergent.

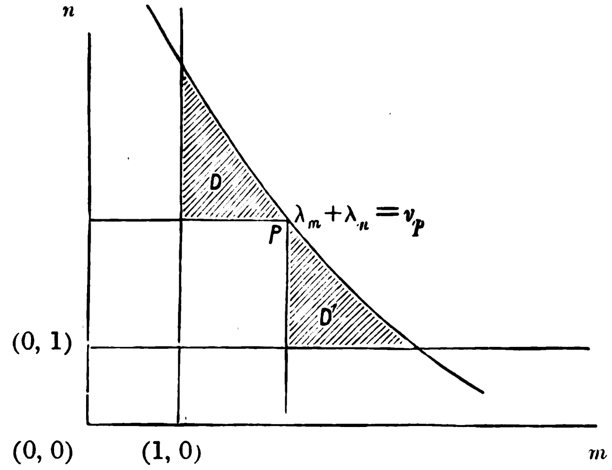
We have
$$c_p = \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n,$$

and
$$C_p = \Sigma a_m b_n,$$

the summation being bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

I shall suppose that $\lambda_1 = 0$; this hypothesis in no way affects the generality of the result, and simplifies the expression of the proof a little.



Let us draw the curve $\lambda_m + \lambda_n = \nu_p$

in the (m, n) plane (see the figure), and on it take the point P whose coordinates are

$$m_p = n_p = \bar{\lambda}(\tfrac{1}{2}\nu_p),$$

where $\bar{\lambda}$ is the function inverse to λ. Then

$$C_p - A_{m_p} B_{n_p} = \Sigma a_m b_n,$$

where the summation extends to all points (m, n) inside the regions D, D' shaded in the figure, including those on the curved (but not on the straight) boundaries of those regions. Now

$$\sum_{(D)} a_m b_n = \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \lambda(\nu_p)} b_n A_{\bar{\lambda}(\nu_p - \lambda_n)}.$$

There is a constant K such that

$$|A_x| < K$$

for all values of x. Moreover we can choose p so that

$$|b_n| < \epsilon(\lambda_n - \lambda_{n-1})/\lambda_n$$

for $n \geqslant \lambda(\frac{1}{2}\nu_p)$. Then

$$\begin{aligned} \Big|\sum_{(D)} a_m b_n\Big| &< \epsilon K \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \bar{\lambda}(\nu_p)} \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \\ &< \epsilon K \left\{ 1 + \sum_{\bar{\lambda}(\frac{1}{2}\nu_p)+1 < n \leqslant \lambda(\nu_p)} \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right) \right\}^* \\ &< \epsilon K \left\{ 1 + \log \frac{\lambda\bar{\lambda}(\nu_p)}{\lambda\bar{\lambda}(\frac{1}{2}\nu_p)} \right\} \\ &= \epsilon K (1 + \log 2). \end{aligned}$$

It follows that $$\sum_{(D)} a_m b_n \to 0$$

as $p \to \infty$. Similarly we can shew that the sum of the terms inside (D') tends to zero. Hence

$$C_p - A_{m_p} B_{n_p} \to 0, \tag{1}$$

and the theorem follows.

It should be observed that the same argument proves that

$$C_p - A_{\bar{\lambda}(\alpha\nu_p)} B_{\bar{\lambda}(\beta\nu_p)} \to 0, \tag{2}$$

if α and β are any positive numbers such that $\alpha + \beta = 1$. Moreover the truth of (1) and (2) depends *only* on the existence of an upper limit for $|A_x|$ and $|B_x|$, and not on the actual convergence of A and B. Thus we have in reality proved more than is actually contained in the enunciation of the theorem.

6. I shall now proceed to the proof of the generalised form of Theorem 6 of § 4. Here we find it necessary to pursue an entirely

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[*] For, if $u = (\lambda_n - \lambda_{n-1})/\lambda_n$, we have $0 < u < 1$, and

$$u < \log\left(\frac{1}{1-u}\right) = \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right).$$

different and much less direct form of argument, and the touch of extra precision, pointed out at the end of the last paragraph, cannot be obtained. We shall also find it necessary to subject the absolute generality of the sequence (λ_n) to a restriction, viz., that

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0.^*$$

7. THEOREM II.—*If* $$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0,$$

then the result of Theorem I is still valid when we assume only that

$$\left| \frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} \right| < K, \qquad \left| \frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}} \right| < K.$$

We know, by Theorem 4 of § 4, that

$$\frac{(\nu_2 - \nu_1) C_1 + (\nu_3 - \nu_2) C_2 + \ldots + (\nu_p - \nu_{p-1}) C_{p-1}}{\nu_p}$$

tends to the limit AB. This expression may also be written in the form

$$C_p - \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p}.$$

Hence the necessary and sufficient condition that the product series should be convergent is that

$$\frac{\bar{C}_p}{\nu_p} = \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p} \to 0.$$

Now $$\bar{C}_p = \Sigma\, \nu_p \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n = \Sigma (\lambda_m + \lambda_n) a_m b_n,$$

where the summation is bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

That is to say,

$$\begin{aligned} \bar{C}_p &= \sum_{\lambda_n \leqslant \nu_p} \lambda_n b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} a_m + \sum_{\lambda_n \leqslant \nu_p} b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} \lambda_m a_m \\ &= {}_1\Gamma_p + {}_2\Gamma_p, \end{aligned}$$

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[*] I return to the subject of this restriction later on. It is interesting to observe that Mr. Littlewood has found it necessary to make a similar restriction in one of his theorems concerning the converse of Abel's theorem.

say.* I shall first prove that

$$(1)\qquad {}_1\Gamma_p/\nu_p \to 0.$$

Let m_n be the largest integral value of m for which

$$\lambda_m+\lambda_n \leqslant \nu_p,$$

and q the largest value of n for which $\lambda_n \leqslant \nu_p$, so that $m_1 = q$. Then

$${}_1\Gamma_p = \sum_1^q \lambda_n b_n A_{m_n}.$$

Since A is convergent, we may write

$$A_x = A+\epsilon_x \quad (\epsilon_x \to 0),$$

$$\frac{{}_1\Gamma_p}{\nu_p} = \Delta_p+\Delta'_p,$$

where

$$\Delta_p = \frac{A}{\nu_p}\sum_1^q \lambda_n b_n \to 0,\dagger$$

and

$$(2)\qquad \Delta'_p < \frac{K}{\nu_p}\sum_1^q (\lambda_n-\lambda_{n-1})\,|\epsilon_{m_n}|.\ddagger$$

Choose M so that

$$|\epsilon_x| < \epsilon \quad (x \geqslant M).$$

Then we shall have $m_n \geqslant M$, if

$$\lambda_M+\lambda_n \leqslant \nu_p,$$

or if

$$n \leqslant \bar{\lambda}(\nu_p-\lambda_M).$$

Suppose that this condition is satisfied for $n = 1, 2, \ldots, N$, but that

$$\lambda_{N+1} > \nu_p-\lambda_M.$$

It is plain that, when M has been fixed, $N \to \infty$ with p.

Then the right-hand side of (2) is less than

$$\frac{\epsilon K}{\nu_p}\sum_1^N (\lambda_n-\lambda_{n-1})+K\left(\frac{\lambda_{N+1}-\lambda_N}{\nu_p}\right)+\frac{K'}{\nu_p}\sum_{N+2}^q (\lambda_n-\lambda_{n-1})$$

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[*] In the special case when

$$\lambda_m = m, \quad \lambda_n = n, \quad \nu_p = p+1,$$

${}_1\Gamma_p$ and ${}_2\Gamma_p$ reduce to the expressions X and Y of my former paper (*l.c.*, pp. 415, 416).

[†] Since B is convergent, so that $(\lambda_1 b_1+\lambda_2 b_2+\ldots+\lambda_q b_q)/\lambda_q \to 0$, and $\lambda_q \leqslant \nu_p$.

[‡] It is convenient to agree that $\lambda_0 = \lambda_1 = 0$.

(where K' is any number not less than the product of K by the greatest value of $|\epsilon_x|$)

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'}{\nu_p}(\lambda_q - \lambda_{N+1})$$

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'\lambda_M}{\nu_p},$$

since $$\lambda_q \leqslant \nu_p, \quad \lambda_{N+1} > \nu_p - \lambda_M.$$

But when M has been fixed, each of the last two terms tends to zero as $p \to \infty$; and so

$$\Delta'_p \to 0,$$

which establishes the truth of the assertion (1).

It remains to prove that

(3) $$_2\Gamma_p/\nu_p \to 0.$$

Now $$_2\Gamma_p = \sum_{\lambda_n \leqslant \nu_p} b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} \lambda_m a_m$$

$$= b_1 \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m + b_2 \sum_{\lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots + b_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m$$

$$= B_1 \sum_{\nu_p - \lambda_2 < \lambda_m \leqslant \nu_p} \lambda_m a_m + B_2 \sum_{\nu_p - \lambda_3 < \lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots$$

$$+ B_{q-1} \sum_{\nu_p - \lambda_q < \lambda_m \leqslant \nu_p - \lambda_{q-1}} \lambda_m a_m + B_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m.$$

If in this equation we write

$$B_x = B + \epsilon_x,$$

so that $\epsilon_x \to 0$, we obtain $$\frac{_2\Gamma_p}{\nu_p} = \Delta_p + \Delta'_p,$$

where $$\Delta_p = \frac{B}{\nu_p} \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m \to 0,$$

and $$\nu_p\,|\Delta'_p| < K\,|\epsilon_1| \sum_{\nu - \lambda_2 < \lambda_m \leqslant \nu_p} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K\,|\epsilon_n| \sum_{\nu_p - \lambda_{n+1} < \lambda_m \leqslant \nu_p - \lambda_n} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K\,|\epsilon_q| \sum_{\lambda_m \leqslant \nu_p - \lambda_q} (\lambda_m - \lambda_{m-1}).$$

Choose N so that $$|\epsilon_x| < \epsilon \quad (x \geqslant N).$$

Then

$$|\Delta_p'| < \frac{K'}{\nu_p} \sum_{\nu_p-\lambda_N<\lambda_m\leqslant\nu_p} (\lambda_m-\lambda_{m-1}) + \frac{\epsilon K}{\nu_p} \sum_{\lambda_m\leqslant\nu_p-\lambda_N} (\lambda_m-\lambda_{m-1}),$$

where K' is defined as before.

Let M be the largest value of m for which

$$\lambda_m+\lambda_N \leqslant \nu_p,$$

so that

$$\lambda_{M+1} > \nu_p-\lambda_N.$$

Then

$$|\Delta_p'| < \epsilon K + \frac{K'}{\nu_p}(\lambda_{M+1}-\lambda_M) + \frac{K'}{\nu_p}\sum_{M+2}^{q}(\lambda_m-\lambda_{m-1})$$

$$< \epsilon K + K'\left(\frac{\lambda_{M+1}-\lambda_M}{\lambda_{M+1}}\right) + \frac{K'\lambda_N}{\nu_p};$$

and it follows, as in our previous discussion of ${}_1\Gamma_p$, that $\Delta_p' \to 0$ and that the assertion (3) is true. Thus the proof of Theorem II is completed.

8. It will be observed that the condition that $(\lambda_n-\lambda_{n-1})/\lambda_n \to 0$ is only used twice in the above proof, and then in a way that rather suggests the possibility of avoiding it. But I have not been able to free Theorem II of this condition. Nor does the point seem to be of importance.

When the sequence (λ_n) does not satisfy this condition, the conditions

$$\left|\frac{\lambda_n a_n}{\lambda_n-\lambda_{n-1}}\right| < K, \ \ldots$$

tell us nothing (in any interesting case) except that $|a_n|$ and $|b_n|$ have finite upper limits. But this is, of course, already involved in the fact of the convergence of A and B: we know, in fact, that a_n and b_n tend to zero, so that the conditions of Theorem I will be satisfied. Thus there appears to be no particular purpose to be served by attempting a proof of the more general theorem.[*]

If $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to a limit other than zero, Theorem I suffices to tell us that *any* two convergent series may be multiplied by the corresponding rule of Dirichlet's multiplication. It is interesting to verify this conclusion in a particular case. Let

$$\lambda_n = 2^n.$$

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[*] In all cases of interest $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to zero or to some limit (obviously not greater than unity). This limit is zero if

$$\lambda_n \prec e^{\delta n},$$

positive but less than unity if λ_n is (roughly) of increase $e^{\Delta n}$, unity if

$$\lambda_n \succ e^{\Delta n}.$$

Here δ and Δ denote respectively arbitrarily small and arbitrarily large positive numbers.

Then
$$\lambda_m+\lambda_n > \lambda_\mu+\lambda_\nu,$$
if
$$m \geqslant n, \quad \mu \geqslant \nu, \quad \text{and} \quad m > \mu.$$
The Dirichlet's product of A and B is

$$a_1 b_1+(a_1 b_2+a_2 b_1)+a_2 b_2+(a_1 b_3+a_3 b_1)+(a_2 b_3+a_3 b_2)+a_3 b_3$$
$$+(a_1 b_4+a_4 b_1)+\ldots,$$

the principle of the method being that a suffix m does not appear at all until all possible combinations of two lesser suffixes are exhausted.[*] It will easily be verified that the mere convergence of A and B is enough to ensure the convergence of the product series.[†]

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[*] The rule is exactly the same for

$$\lambda = 3^n,\ 4^n,\ \ldots,\ 2^{2^n},\ \ldots .$$

[†] It is perhaps worth pointing out that the reasons which make Theorem II (as an addition to Theorem I) trivial in the case of very "high" indices λ_n, do not apply to the problem of completing Mr. Littlewood's results concerning "Tauber's theorem" in the case of such high indices.

THE INVARIANTS OF THE LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES

By J. E. CAMPBELL.

[Received September 23rd, 1911.—Read November 9th, 1911.]

1. Let the equation be written in the form

$$az_{11}+2hz_{12}+bz_{22}+2gz_1+2fz_2+cz = 0, \tag{1}$$

where a, b, c, f, g, h are functions of the independent variables x and y, the suffixes denoting that the functions to which they are attached are partial derivatives (1 with respect to x and 2 with respect to y) of the function with the suffixes deleted.

The equation (1) does not alter its form when multiplied by any function of the independent variables nor when subjected to any transformations of the infinite group

$$x' = \phi(x, y), \quad y' = \psi(x, y), \quad z' = z\chi(x, y). \tag{2}$$

The object of this paper is to obtain the criteria which will decide whether any two assigned equations, each of the form (1), are transformable into one another by the operations of the group (2).

The necessary and sufficient conditions for the transformability are that all the functions, of a, b, c, f, g, h and their partial derivatives up to any order, which are invariant under the operations of the infinite group (2), should be same for the two equations. That is, that the series of equations connecting x, y and x' y', obtained by equating the corresponding series of invariants, should be consistent.

By multiplying the equation (1) by a properly chosen factor, we can bring it to such a form that h^2-ab will be unity. We therefore always suppose that our equations are in this standard form.

Any function of the coefficients a, b, c, f, g, h and their derivatives, which is unaltered, save for a power of the Jacobian $\dfrac{\partial(x', y')}{\partial(x, y)}$ multiplying it, for all transformations of the group (2), will be called as usual an invariant; when the power is zero, the invariant will be called absolute. It is the absolute invariants that we require for our criteria.

Let

$$\begin{aligned}
2X &\equiv b(a_1+h_2-2g)-h(h_1+b_2-2f),\\
2Y &\equiv a(h_1+b_2-2f)-h(a_1+h_2-2g),\\
I &\equiv aX_1+h(X_2+Y_1)+bY_2+(a_1+h_2)X+(h_1+b_2)Y\\
&\qquad\qquad +aX^2+2hXY+bY^2-c,\\
J &\equiv X_2-Y_1.
\end{aligned}$$

We shall prove that I and J are invariants, and that J/I is an absolute invariant.

Let Δ denote the operator

$$I^{-1}\left[\frac{\partial}{\partial x}\left(a\frac{\partial}{\partial x}+h\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)\right].$$

We shall see that this operator is absolutely invariant.

Let $$C \equiv I^{-1}\left(a_{11}+2h_{12}+b_{22}+\frac{a_1b_2-a_2b_1}{2h}\right),$$

then $\Delta \log I+C$ is another absolute invariant.

Let these two absolute invariants be denoted by α and β respectively. Clearly any function of a system of absolute invariants is itself an absolute invariant. Also, if γ is any absolute invariant, $\Delta\gamma$ is another absolute invariant. We prove that there is no absolute invariant which cannot be obtained by repeated application of these two rules.

Let $$\begin{aligned}
ID(\alpha) &\equiv a\alpha_1^2+2h\alpha_1\alpha_2+b\alpha_2^2,\\
ID(\beta) &\equiv a\beta_1^2+2h\beta_1\beta_2+b\beta_2^2,\\
ID(\alpha,\beta) &= a\alpha_1\beta_1+h(\alpha_1\beta_2+\alpha_2\beta_1)+b\alpha_2\beta_2,
\end{aligned}$$

we shall see that $D(\alpha)$, $D(\beta)$, and $D(\alpha,\beta)$

are absolute invariants.

In general, the necessary and sufficient condition that the equation

$$a'z'_{11}+2h'z'_{12}+b'z'_{22}+2g'z'_1+2f'z'_2+c'z'=0,$$

in the independent variables x', y', should be transformable into (1) by an operation of the group (2) is that the equations in x, y, x', y' given by

$$\alpha'=\alpha,\quad \beta'=\beta,\quad D'(\alpha')=D(\alpha),\quad D'(\beta')=D(\beta),\quad D'(\alpha',\beta')=D(\alpha,\beta)$$

should be consistent.

There is a special case, however, when all the invariants are functions of one argument.

Leaving this case aside, we see that the conditions are that $D'(\alpha')$, $D'(\beta')$, and $D'(\alpha',\beta')$ must be respectively (when expressed in terms of x', y') the same functions of α' and β', that $D(\alpha)$, $D(\beta)$, and $D(\alpha,\beta)$ are of α and β.

When the equation (1) is given in the Laplacian form

$$z_{12}+az_1+bz_2+cz=0,$$

$\frac{1}{2}(I-J)$ and $\frac{1}{2}(I+J)$ are Laplace's invariants.

Symbolical expressions for these invariants (not absolute invariants it will be noticed) for the general equation (1) are found by Imschenetsky in his "Memoir on Partial Differential Equations" (*Grinert Archiv*, t. LIV, 1872, p. 209–360).

In the *American Journal of Mathematics*, Vol. XXVI, 1904, p. 319, Wilczynski considers the "Invariants of a System of Partial Differential Equations." His method, like that of the present paper, is an application of Lie's general theory of infinitesimal transformations. The problem to which his method is applied is closely related to the problem here considered.

I would also refer to papers by Forsyth, on "Differential Invariants," in the *Transactions of the Royal Society*, A339 and A353, and to a paper by Tresse (*Acta Math.*, t. XVIII, 1894), "Sur les invariants différentiels des groupes continus de transformations."*

I do not know if the results I have given are explicitly stated anywhere; if not, they may be of some interest and justify some rather laborious calculation.

2. In order to find the invariants of the equation (1), it is only necessary to consider infinitesimal operations, viz., the point transformations

$$x'=x+t\xi,\quad y'=y+t\eta,\quad z'=z-t\zeta z,$$

and multiplication by the factor $1+t(\rho-\zeta)$, in order to preserve the invariance of h^2-ab. The quantities ξ, η, ζ, and ρ are functions of x and y, and t is an infinitesimal constant.

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* I am indebted to Dr. Baker, F.R.S., and to H. T. Gerrans, Esq., for most of the references I have given.

Remembering that we are always to neglect powers of t above the first, we get

$$\left.\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x} - t\left(\xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}\right)\\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} - t\left(\xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}\right)\\
\frac{\partial^2}{\partial x'^2} &= \frac{\partial^2}{\partial x^2} - 2t\left(\xi_1 \frac{\partial^2}{\partial x^2} + \eta_1 \frac{\partial^2}{\partial x\,\partial y}\right) - t\left(\xi_{11} \frac{\partial}{\partial x} + \eta_{11} \frac{\partial}{\partial y}\right)\\
\frac{\partial^2}{\partial x'\,\partial y'} &= \frac{\partial^2}{\partial x\,\partial y} - t\left(\xi_2 \frac{\partial^2}{\partial x^2} + (\xi_1+\eta_2) \frac{\partial^2}{\partial x\,\partial y} + \eta_1 \frac{\partial^2}{\partial y^2}\right) - t\left(\xi_{12} \frac{\partial}{\partial x} + \eta_{12} \frac{\partial}{\partial y}\right)\\
\frac{\partial^2}{\partial y'^2} &= \frac{\partial^2}{\partial y^2} - 2t\left(\xi_2 \frac{\partial^2}{\partial x\,\partial y} + \eta_2 \frac{\partial^2}{\partial y^2}\right) - t\left(\xi_{22} \frac{\partial}{\partial x} + \eta_{22} \frac{\partial}{\partial y}\right)
\end{aligned}\right\}. \quad (3)$$

We can now express the derivatives of z', with respect to x' and y', in terms of t, of ξ, η, ζ and their derivatives with respect to x and y, and of the derivatives of z with respect to x and y. Let

$$a' \frac{\partial^2 z'}{\partial x'^2} + 2h' \frac{\partial^2 z'}{\partial x'\,\partial y'} + b' \frac{\partial^2 z'}{\partial y'^2} + 2g' \frac{\partial z'}{\partial x'} + 2f' \frac{\partial z'}{\partial y'} + c'z'$$

$$\equiv [1+t(\rho-\zeta)]\,(az_{11}+2hz_{12}+bz_{22}+2gz_1+2fz_2+cz),$$

then we have
$$\begin{aligned}
a' &= a+2t(a\xi_1+h\xi_2)+at\rho,\\
h' &= h+t[a\eta_1+h(\xi_1+\eta_2)+b\xi_2]+ht\rho,\\
b' &= b+2t(h\eta_1+b\eta_2)+bt\rho.
\end{aligned}$$

Since $h'^2-a'b' = h^2-ab$, we deduce that

$$\xi_1+\eta_2+\rho = 0.$$

If we write $\dfrac{\partial a}{\partial t}$ for $\dfrac{(a'-a)}{t}$, and similarly with regard to the other coefficients, we have

$$\left.\begin{aligned}
\frac{\partial a}{\partial t} &= a(\xi_1-\eta_2)+2h\xi_2\\
\frac{\partial h}{\partial t} &= a\eta_1+b\xi_2\\
\frac{\partial b}{\partial t} &= 2h\eta_1+b(\eta_2-\xi_1)\\
2\frac{\partial g}{\partial t} &= a\xi_{11}+2h\xi_{12}+b\xi_{22}-2g\eta_2+2f\xi_2+2a\zeta_1+2h\zeta_2\\
2\frac{\partial f}{\partial t} &= a\eta_{11}+2h\eta_{12}+b\eta_{22}+2g\eta_1-2f\xi_1+2h\zeta_1+2b\zeta_2\\
\frac{\partial c}{\partial t} &= a\zeta_{11}+2h\zeta_{12}+b\zeta_{22}+2g_1\zeta+2f\zeta_2-c(\xi_1+\eta_2)
\end{aligned}\right\}. \qquad (4)$$

The expressions on the right may be called the rate of increase of the corresponding coefficients.

If we write $\frac{\partial a_1}{\partial t}$ for $\left(\frac{\partial a'}{\partial x'}-\frac{\partial a}{\partial x}\right)\Big/ t$, and use a similar notation for the rate of increase of any derivative, and if F is any function of the coefficients and their derivatives, and F' the same function of the coefficients a', b', c', f', g', h' and their derivatives, with respect to x' and y', so that in our notation

$$F' = F+t\frac{\partial F}{\partial t},$$

then

$$\frac{\partial F'}{\partial x'} = \left[\frac{\partial}{\partial x}-t\left(\xi_1\frac{\partial}{\partial x}+\eta_1\frac{\partial}{\partial y}\right)\right]\left(F+t\frac{\partial F}{\partial t}\right)$$

$$= \frac{\partial F}{\partial x}+t\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial t}-\xi_1\frac{\partial F}{\partial x}-\eta_1\frac{\partial F}{\partial y}\right).$$

We now have

$$\left.\begin{aligned}\frac{\partial}{\partial t}\frac{\partial F}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial F}{\partial t}-\xi_1\frac{\partial F}{\partial x}-\eta_1\frac{\partial F}{\partial y}\\ \frac{\partial}{\partial t}\frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial F}{\partial t}-\xi_2\frac{\partial F}{\partial x}-\eta_2\frac{\partial F}{\partial y}\end{aligned}\right\}. \tag{5}$$

We thus obtain, for the rate of increase of the first derivatives of a, h, and b,

$$\left.\begin{aligned}\frac{\partial a_1}{\partial t} &= a(\xi_{11}-\eta_{12})+2h\xi_{12}-a_1\eta_2+2h_1\xi_2-a_2\eta_1\\ \frac{\partial a_2}{\partial t} &= a(\xi_{12}-\eta_{22})+2h\xi_{22}+a_2(\xi_1-2\eta_2)+2h_2\xi_2-a_1\xi_2\\ \frac{\partial b_1}{\partial t} &= 2h\eta_{11}+b(\eta_{12}-\xi_{11})+b_1(\eta_2-2\xi_1)+2h_1\eta_1-b_2\eta_1\\ \frac{\partial b_2}{\partial t} &= 2h\eta_{12}+b(\eta_{22}-\xi_{12})-b_2\xi_1+2h_2\eta_1-b_1\xi_2\\ \frac{\partial h_1}{\partial t} &= a\eta_{11}+b\xi_{12}+a_1\eta_1+b_1\xi_2-h_1\xi_1-h_2\eta_1\\ \frac{\partial h_2}{\partial t} &= a\eta_{12}+b\xi_{22}+a_2\eta_1+b_2\xi_2-h_1\xi_2-h_2\eta_2\end{aligned}\right\}. \tag{6}$$

3. The invariants of the equation (1) must be obtained by eliminating ξ, η, and ζ from the equations (4) and equations derived from them, like (6), by the rule (5).

Suppose now we have three equations of the forms

$$\alpha = a\xi_1 + h\xi_2, \qquad \beta = h\eta_1 + b\eta_2, \qquad \gamma = \xi_1 + \eta_2,$$

and it is required to find the relations between α, β, and γ that they may be consistent.

There are three equations connecting the four derivatives ξ_1, ξ_2, η_1, and η_2. By differentiating these equations we obtain six equations connecting the six derivatives of the second order, and nine equations connecting the eight derivatives of the third order. In general, on eliminating the derivatives of the second and third orders, we obtain an equation of the first order, and this, together with the three assigned equations, will enable us to determine the first derivatives ξ_1, ξ_2, η_1, and η_2. But, if a certain equation connects α, β, γ, a, b, and h and their derivatives, we cannot determine the derivatives ξ_1, ξ_2, η_1, and η_2, and the assigned equations will be consistent; and no other equation connecting α, β, γ, a, b, and h can be deduced, except by differentiation of this one.

Consider now the equations

$$\frac{\partial a}{\partial t} = a(\xi_1 - \eta_2) + 2h\xi_2,$$

$$\frac{\partial h}{\partial t} = a\eta_1 + b\xi_2$$

$$\frac{\partial b}{\partial t} = 2h\eta_1 + b(\eta_2 - \xi_1),$$

and notice that the equations (6) are consequences of these three.

Again, since
$$a\frac{\partial b}{\partial t} + b\frac{\partial a}{\partial t} = 2h\frac{\partial h}{\partial t},$$

we see that the second one is a mere consequence of the other two, and need not therefore be regarded, except for temporary convenience.

If we take a new variable M defined by

$$M \equiv \xi_1 + \eta_2,$$

we can take the two independent equations to be

$$\frac{\partial a}{\partial t} + aM = 2a\xi_1 + 2h\xi_2,$$

$$\frac{\partial b}{\partial t} + bM = 2h\eta_1 + 2b\eta_2\,;$$

and we require the equation connecting a, h, b, M, $\frac{\partial a}{\partial t}$, $\frac{\partial b}{\partial t}$ and their derivatives that

$$\left.\begin{aligned}\frac{\partial a}{\partial t}+aM &= 2a\xi_1+2h\xi_2\\ \frac{\partial b}{\partial t}+bM &= 2h\eta_1+2b\eta_2\\ M &= \xi_1+\eta_2\end{aligned}\right\} \tag{7}$$

may be consistent.

From the equations (6) we deduce

$$a_1\frac{\partial b_2}{\partial t}+b_2\frac{\partial a_1}{\partial t}-a_2\frac{\partial b_1}{\partial t}-b_1\frac{\partial a_2}{\partial t}$$

is equal to

$$\begin{aligned}(a_1b+ab_1)(\eta_{22}-\xi_{12})+(a_2b+ab_2)(\xi_{11}-\eta_{12})+2(a_1h_2-a_2h_1)\eta_1+2(b_2h_1-b_1h_2)\xi_2\\ -(a_1b_2-a_2b_1)M+2h(a_1\eta_{12}+b_2\xi_{12}-a_2\eta_{11}-b_1\xi_{22}).\end{aligned}$$

From this equation we deduce, by aid of the following equations,

$$h^2-ab=1,\quad 2hh_1=ab_1+a_1b,\quad 2hh_2=ab_2+a_2b,$$

$$2h(a_1h_2-a_2h_1)=a(a_1b_2-a_2b_1),$$

$$2h(b_2h_1-b_1h_2)=b(a_1b_2-a_2b_1),$$

$$\frac{\partial h}{\partial t}=a\eta_1+b\xi_2,$$

that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{a_1b_2-a_2b_1}{2h}\right)=(a_1+h_2)M_1+(h_1+b_2)M_2-M\,\frac{a_1b_2-a_2b_1}{2h}\\ -a_1\xi_{11}-a_2\eta_{11}-b_1\xi_{22}-b_2\eta_{22}-2h_1\xi_{12}-2h_2\eta_{12}.\end{aligned} \tag{8}$$

From the equations (6) we also have

$$\left.\begin{aligned}\frac{\partial}{\partial t}(a_1+h_2) &= a\xi_{11}+2h\xi_{12}+b\xi_{22}-(a_1+h_2)\eta_2+(h_1+b_2)\xi_2\\ \frac{\partial}{\partial t}(h_1+b_2) &= a\eta_{11}+2h\eta_{12}+b\eta_{22}+(a_1+h_2)\eta_1-(h_1+b_2)\xi_1\end{aligned}\right\}, \tag{9}$$

and therefore, by rule (5),

$$\begin{aligned}\frac{\partial}{\partial t}(a_{11}+b_{22}+2h_{12})=aM_{11}+2hM_{12}+bM_{22}-M(a_{11}+b_{22}+2h_{12})\\ +a_1\xi_{11}+a_2\eta_{11}+b_1\xi_{22}+b_2\eta_{22}+2h_1\xi_{12}+2h_2\eta_{12}.\end{aligned} \tag{10}$$

Let $$\Omega \equiv a\frac{\partial^2}{\partial x^2}+2h\frac{\partial^2}{\partial x\,\partial y}+b\frac{\partial^2}{\partial y^2}+(a_1+h_2)\frac{\partial}{\partial x}+(h_1+b_2)\frac{\partial}{\partial y},$$

and let $$E \equiv a_{11}+2h_{12}+b_{22}+\frac{a_1b_2-a_2b_1}{2h},$$

then, adding (8) and (10), we see that the only equation necessary for the consistency of (7) is

$$\frac{\partial}{\partial t}(E) = \Omega M - ME. \tag{11}$$

4. Let

$$\left.\begin{aligned} 2X &\equiv b(a_1+h_2-2g)-h(h_1+b_2-2f) \\ 2Y &\equiv a(h_1+b_2-2f)-h(a_1+h_2-2g) \end{aligned}\right\}. \tag{12}$$

From the equations (4) and (9) we deduce, by simple calculation, that

$$\left.\begin{aligned} \frac{\partial X}{\partial t} &= \zeta_1-\xi_1X-\eta_1Y \\ \frac{\partial Y}{\partial t} &= \zeta_2-\xi_2X-\eta_2Y \end{aligned}\right\}, \tag{13}$$

and hence we see that

$$\left.\begin{aligned} \frac{\partial X_1}{\partial t} &= \zeta_{11}-\xi_{11}X-\eta_{11}Y-2\xi_1X_1-\eta_1(X_2+Y_1) \\ \frac{\partial X_2}{\partial t} &= \zeta_{12}-\xi_{12}X-\eta_{12}Y-\xi_1X_2-\eta_1Y_2-\xi_2X_1-\eta_2X_2 \\ \frac{\partial Y_1}{\partial t} &= \zeta_{12}-\xi_{12}X-\eta_{12}Y-\xi_1Y_1-\eta_1Y_2-\xi_2X_1-\eta_2Y_1 \\ \frac{\partial Y_2}{\partial t} &= \zeta_{22}-\xi_{22}X-\eta_{22}Y-2\eta_2Y_2-\xi_2(X_2+Y_1) \end{aligned}\right\}. \tag{14}$$

Let $$J \equiv X_2 - Y_1,$$

then the fourth and fifth equations of (4), after the elimination of ζ, may be replaced by

$$\frac{\partial J}{\partial t} = -MJ. \tag{15}$$

Let

$$I \equiv aX_1+h(X_2+Y_1)+bY_2+(a_1+h_2)X+(h_1+b_2)Y+aX^2+2hXY+bY^2-c,$$

then we see, from (13) and (14), by simple calculation and use of (6), that the sixth equation of (4), after elimination of ζ, may be replaced by

$$\frac{\partial I}{\partial t} = -MI. \tag{16}$$

It follows that $$\frac{\partial}{\partial t}\left(\frac{I}{J}\right)=0,$$
so that $\frac{J}{I}$ is an absolute invariant.

From (11) we also deduce that
$$\frac{\partial}{\partial t}\left(\frac{E}{I}\right)=\frac{\Omega}{I}M,$$
or, if $$E\equiv IC \quad\text{and}\quad I\Delta\equiv\Omega,$$
$$\frac{\partial}{\partial t}C=\Delta M. \tag{17}$$

From (3) and (6) we may easily verify that, Ω' denoting the operator
$$\frac{\partial}{\partial x'}\left(a'\frac{\partial}{\partial x'}+h'\frac{\partial}{\partial y'}\right)+\frac{\partial}{\partial y'}\left(h'\frac{\partial}{\partial x'}+b'\frac{\partial}{\partial y'}\right),$$
$$\Omega'=\Omega-tM\Omega;$$
and therefore, as $$I'=I-tMI,$$
we see that $$\Delta'=\Delta. \tag{18}$$

We also have $$\log I'=\log I-tM,$$
and therefore $$\Delta'\log I'=\Delta\log I-t\Delta M$$
or $$\frac{\partial}{\partial t}(\Delta\log I)=-\Delta M;$$
and thus, by (17), $$\frac{\partial}{\partial t}(\Delta\log I+C)=0$$
or $$\Delta\log I+C$$
is a second absolute invariant.

Denote the two invariants we have now obtained, viz.,
$$\frac{J}{I}\quad\text{and}\quad\Delta\log I+C,$$
by α and β respectively. Clearly their product $\alpha\beta$ will be an absolute invariant and therefore, from (18), so will $\Delta(\alpha\beta)$.

Let
$$ID(\alpha)\equiv a\alpha_1^2+2h\alpha_1\alpha_2+b\alpha_2^2,$$
$$ID(\alpha,\beta)\equiv a\alpha_1\beta_1+h(\alpha_1\beta_2+\alpha_2\beta_1)+b\alpha_2\beta_2,$$
$$ID(\beta)\equiv a\beta_1^2+2h\beta_1\beta_2+b\beta_2^2.$$

Now $$\Delta\alpha\beta=\alpha\Delta\beta+\beta\Delta\alpha+2D(\alpha,\beta),$$
and therefore we see that $D(\alpha,\beta)$ is an absolute invariant.

Similarly, we conclude that, since $\Delta\alpha^2$ and $\Delta\beta^2$ are absolute invariants, $D(\alpha)$ and $D(\beta)$ are also.

We thus see how an infinity of invariants can be obtained by the rules given in § 1.

5. We have now to prove that there cannot be any other invariants.

The equations (4), after the elimination of ζ, can be replaced by

$$\left.\begin{aligned}
\frac{\partial a}{\partial t} &= a(\xi_1-\eta_2)+2h\xi_2\\
\frac{\partial h}{\partial t} &= a\eta_1+b\xi_2\\
\frac{\partial b}{\partial t} &= 2h\eta_1+b(\eta_2-\xi_1)\\
\frac{\partial I}{\partial t} &= -I(\xi_1+\eta_2)\\
\frac{\partial \alpha}{\partial t} &= 0\\
\frac{\partial \beta}{\partial t} &= 0
\end{aligned}\right\}. \tag{19}$$

From the fifth equation, by application of rule (5), we see that

$$\left.\begin{aligned}
\frac{\partial \alpha_1}{\partial t}+\alpha_1\xi_1+\alpha_2\eta_1 &= 0\\
\frac{\partial \alpha_2}{\partial t}+\alpha_1\xi_2+\alpha_2\eta_2 &= 0
\end{aligned}\right\}. \tag{20}$$

We have also seen that

$$\frac{\partial}{\partial t}\left(\frac{a\alpha_1^2+2h\alpha_1\alpha_2+b\alpha_2^2}{I}\right)=0,$$

and therefore, from (20) and the first three equations of (19), we deduce that

$$D(\alpha)\left(\frac{\partial I}{\partial t}+I(\xi_1+\eta_2)\right)=0.$$

The fourth equation of (19), being a consequence of the others and of the invariance of $D(\alpha)$, need not be considered as possibly leading to new invariants.

If we define A, B, H by

$$a=AI, \quad h=HI, \quad b=BI,$$

the first three equations of (19) may be replaced by

$$\left.\begin{aligned}\frac{\partial A}{\partial t} &= 2A\xi_1+2H\xi_2\\ \frac{\partial H}{\partial t} &= A\eta_1+H(\xi_1+\eta_2)+B\xi_2\\ \frac{\partial B}{\partial t} &= 2H\eta_1+2B\eta_2\end{aligned}\right\}. \tag{21}$$

From the sixth equation of (19) we deduce, by rule (5),

$$\left.\begin{aligned}\frac{\partial\beta_1}{\partial t}+\beta_1\xi_1+\beta_2\eta_1 &= 0\\ \frac{\partial\beta_2}{\partial t}+\beta_1\xi_2+\beta_2\eta_2 &= 0\end{aligned}\right\}. \tag{22}$$

Now $$\frac{\partial}{\partial t}\left[A\alpha_1\beta_1+H(\alpha_1\beta_2+\alpha_2\beta_1+B\alpha_2\beta_2\right] = 0,$$

and, from (20) and (21), we deduce that

$$\frac{\partial}{\partial t}(A\alpha_1+H\alpha_2) = (A\xi_1+H\xi_2)\alpha_1+(H\xi_1+B\xi_2)\alpha_2,$$

$$\frac{\partial}{\partial t}(H\alpha_1+B\alpha_2) = (A\eta_1+H\eta_2)\alpha_1+(H\eta_1+B\eta_2)\alpha_2;$$

it therefore follows that

$$(A\alpha_1+H\alpha_2)\left(\frac{\partial\beta_1}{\partial t}+\beta_1\xi_1+\beta_2\eta_1\right)+(H\alpha_1+B\alpha_2)\left(\frac{\partial\beta_2}{\partial t}+\beta_1\xi_2+\beta_2\eta_2\right) = 0,$$

Similarly, from $\frac{\partial}{\partial t}(A\beta_1^2+2H\beta_1\beta_2+B\beta_2^2) = 0,$
we deduce that

$$(A\beta_1+H\beta_2)\left(\frac{\partial\beta_1}{\partial t}+\beta_1\xi_1+\beta_2\eta_1\right)+(H\beta_1+B\beta_2)\left(\frac{\partial\beta_2}{\partial t}+\beta_1\xi_2+\beta_2\eta_2\right) = 0.$$

Now, unless $\begin{vmatrix} A\alpha_1+H\alpha_2, & H\alpha_1+B\alpha_2\\ A\beta_1+H\beta_2, & H\beta_1+B\beta_2\end{vmatrix} = 0$

[in which case, since H^2-AB cannot be zero, β is a function of α, and therefore (22) is a mere consequence of (20)] we must have

$$\frac{\partial\beta_1}{\partial t}+\beta_1\xi_1+\beta_2\eta_1 = 0, \qquad \frac{\partial\beta_2}{\partial t}+\beta_1\xi_2+\beta_2\eta_2 = 0.$$

It therefore follows that, in considering the possibility of new invariants, we need not regard (22).

We have therefore only to consider whether new invariants can be deduced from the equations

$$\left.\begin{aligned}
\frac{\partial a}{\partial t} &= a(\xi_1-\eta_2)+2h\xi_2\\
\frac{\partial h}{\partial t} &= a\eta_1+b\xi_2\\
\frac{\partial b}{\partial t} &= 2h\eta_1+b(\eta_2-\xi_1)\\
\frac{\partial a_1}{\partial t} &= -a_1\xi_1-a_2\eta_1\\
\frac{\partial a_2}{\partial t} &= -a_1\xi_2-a_2\eta_2
\end{aligned}\right\}. \tag{23}$$

From the last two equations of (23) we deduce, by rule (5), three new equations containing derivatives of the second order of ξ and η, viz.,

$$\left.\begin{aligned}
\frac{\partial a_{11}}{\partial t} &= -a_1\xi_{11}-a_2\eta_{11}-2a_{11}\xi_1-2a_{12}\eta_1\\
\frac{\partial a_{12}}{\partial t} &= -a_1\xi_{12}-a_2\eta_{12}-a_{11}\xi_2-a_{12}M-a_{22}\eta_1\\
\frac{\partial a_{22}}{\partial t} &= -a_1\xi_{22}-a_2\eta_{22}-2a_{12}\xi_2-2a_{22}\eta_2
\end{aligned}\right\}. \tag{24}$$

From these three equations and the four—there cannot be more—independent equations amongst (6), we must eliminate the derivatives of the second order in ξ and η.

From (23) and (24) we see that

$$\frac{\partial}{\partial t}(aa_{11}+2ha_{12}+ba_{22})$$
$$= -M(aa_{11}+2ha_{12}+ba_{22})-(a\xi_{11}+2h\xi_{12}+b\xi_{22})a_1-(a\eta_{11}+2h\eta_{12}+b\eta_{22})a_2.$$

With the help of (9) and (23) we deduce that

$$\frac{\partial}{\partial t}[aa_{11}+2ha_{12}+ba_{22}+(a_1+h_2)a_1+(h_1+b_2)a_2]$$
$$= -M(aa_{11}+2ha_{12}+ba_2)+(a_1+h_2)(a_2\eta_1-a_1\eta_2-a_1\xi_1-a_2\eta_1)$$
$$+(h_1+b_2)(a_1\xi_2-a_2\xi_1-a_1\xi_2-a_2\eta_2):$$

or $$\frac{\partial}{\partial t}\Omega a = -M\Omega a.$$

But, as $$\frac{\partial I}{\partial t} = -MI,$$

this is merely $$\frac{\partial}{\partial t}(\Delta a) = 0,$$

so that we cannot obtain any new invariant.

6. Suppose now that we are given two linear partial differential equations of the second order in two independent variables, and we want to know whether they are transformable into one another by an operation of the group (1), we proceed as follows.

Bring both equations to such a form that for each h^2-ab is unity. Form the invariants α and β and assume that they are functionally unconnected. Deduce the invariants

$$D(\alpha), \quad D(\beta), \quad D(\alpha, \beta),$$

then each of these invariants must be the same functions of α and β for the two equations.

Further, these necessary conditions are sufficient. To prove this take α and β as the independent variables, and we have

$$a = ID(\alpha), \qquad h = ID(\alpha, \beta), \qquad b = ID(\beta);$$

then, as h^2-ab is unity,

$$I^{-2} = D(\alpha)\,D(\beta)-[D(\alpha, \beta)]^2.$$

Similarly we take $\quad x' = \alpha' = \alpha, \quad y' = \beta' = \beta,$

and we thus see that, as Δ, I, a, h, and b will now be the same for both equations, all the invariants will be the same, and therefore the relations

$$\alpha' = \alpha, \qquad \beta' = \beta$$

will transform the given equations into one another.

Unless all the invariants of one of the two assigned equations are functionally connected in pairs, we can choose one pair and proceed as before.

We have therefore only to consider the case where there is but one invariant for one of the given equations, so that $\Delta(\alpha)$ and $D(\alpha)$ are both functions of α, and β also such a function.

For the second equation it is then necessary that $\Delta'(\alpha')$, $D'(\alpha')$ and β' should be respectively the same functions of α'. This case is characterised by the property of having only one equation

$$\alpha' = \alpha,$$

connecting the variables x, y, x' and y', though there will be other equations connecting these quantities and the derivatives of one pair of variables, with respect to the others.

Equations with this property will be characterised by the property of admitting a group of transformations.

In the special case, where α and β are both constants, then α' and β', the corresponding invariants for the other equation, must be the same constants, all other invariants in this case will be zero.

7. Equations which have two functionally independent invariants cannot admit the transformations of any continuous group, for the relations

$$\alpha = \alpha', \qquad \beta = \beta'$$

do not allow of continuous variations of x' and y'.

In the case where there is one invariant which is not a constant, and all the other invariants are functions of that one, we may take it as the variable x; and therefore in any possible continuous transformation admitted by the equation we must have ξ zero.

Since we now have

$$\frac{\partial a}{\partial t} = a_2\eta, \qquad \frac{\partial h}{\partial t} = h_2\eta, \qquad \frac{\partial b}{\partial t} = b_2\eta,$$

the equations (19) give
$$a_2\eta = -a\eta_2,$$
$$h_2\eta = a\eta_1.$$

We thus see that
$$a\eta = f(x),$$

and from the second equation it follows that

$$a_1 + h_2 = \frac{f'(x)}{f(x)}a.$$

We see that η involves one arbitrary constant only, so that the group has only one parameter.

We must now consider the case where all the invariants are constants. It is well known that every equation of the form (1) can be brought, by a transformation of the group (2), to the Laplacian form

$$z_{12} + az_1 + bz_2 + cz = 0.$$

For this form
$$J = b_2 - a_1, \qquad I = b_2 + a_1 + 2ab - c,$$
$$\alpha = \frac{J}{I}, \qquad \beta = \frac{II_{12} - I_1I_2}{I^3}.$$

Consider now Euler's equation for which

$$a = \frac{m}{x-y}, \qquad b = \frac{n}{x-y}, \qquad c = \frac{p}{(x-y)^2},$$

where m, n and p are constants; and therefore

$$J = \frac{m+n}{(x-y)^2}, \qquad I = \frac{n-m-2mn-p}{(x-y)^2},$$

$$\alpha = \frac{m+n}{n-m-2mn-p}, \qquad \beta = \frac{2}{2mn+p+m-n}.$$

For Euler's equation therefore the absolute invariants are constants; and therefore, in general, any equation whose absolute invariants are constants can be transformed into Euler's equation, which may therefore be taken as its canonical form.

There is only one exception to this general statement: when β is zero, we cannot choose the constants m, n and p, so as to make the β of Euler's equation zero. I do not discuss the canonical form for such an equation, but merely notice that if α is also zero, the equation can be brought to one of the forms

$$z_{12} = z, \qquad z_{12} = 0.$$

It is well known (Darboux, *Theory of Surfaces*, Vol. IV, p. 61) that Euler's equation admits the three infinitesimal transformations given by

$$x' = x+t\xi, \qquad y' = y+t\eta, \qquad z' = z-t\zeta z,$$

where $\xi = a+bx+cx^2, \qquad \eta = a+by+cy^2, \qquad \zeta = c\,(mx-ny),$

a, b, and c being arbitrary constants, a result easily verified by the methods of this paper.

We conclude that in general an equation whose two absolute invariants are constants admits a group of the third order.

8. When the invariant J vanishes, and the equation is given in the Laplacian form, we have

$$b_2 = a_1,$$

and therefore we may take

$$a = \frac{\gamma_2}{\gamma}, \qquad b = \frac{\gamma_1}{\gamma},$$

where γ is some function of the independent variables.

If we take $$z' = \gamma z,$$

the equation now becomes $$z'_{12} - c'z' = 0,$$

where $$(c'+c)\gamma = \gamma_{12}.$$

The canonical form of an equation for which J is zero is therefore

$$z_{12}-cz = 0.$$

This is, of course, well known.

Another form to which the equation (1) can be reduced is

$$az_{11}+2hz_{12}+bz_{22} = 0.$$

For, if $$z = p, \quad z = q, \quad z = r$$

are any three particular integrals of (1), p/r and q/r being functionally independent, the transformation

$$rx' = p, \quad ry' = q, \quad rz' = z$$

will clearly reduce (1) to the above form.

Suppose now that J is zero, we have

$$X_2 = Y_1;$$

and therefore may take a function ϕ such that

$$2X\phi-\phi_1 = 0, \quad 2Y\phi-\phi_2 = 0.$$

From (12) we now have

$$\frac{\phi_1}{\phi} = b(a_1+h_2)-h(h_1+b_2),$$

$$\frac{\phi_2}{\phi} = a(h_1+b_2)-h(a_1+h_2);$$

and therefore $$a\phi_1+h\phi_2+\phi(a_1+h_2) = 0,$$
$$h\phi_1+b\phi_2+\phi(h_1+b_2) = 0.$$

If we now take $$a' = a\phi, \quad h' = h\phi, \quad b' = b\phi,$$
we have $$a_1'+h_2' = 0, \quad h_1'+b_2' = 0;$$

and therefore there is a function ψ of the independent variables, such that

$$a' = \psi_{22}, \quad h' = -\psi_{12}, \quad b' = \psi_{11}.$$

Another canonical form for the equation (1) when J is zero is therefore

$$\psi_{22}z_{11}-2\psi_{12}z_{12}+\psi_{11}z_{22} = 0.$$

If f is any integral of this equation, then the two surfaces

$$z = f(x, y), \quad z = \psi(x, y),$$

correspond orthogonally (Darboux, *Surfaces*, Vol. IV, p. 10).

We thus see how what Darboux calls the first and second solutions of the deformation problem correspond to the two different canonical forms of equation (1), when J is zero (compare Darboux, Vol. IV, §§ 857 and 869).

A simple example of the application of this invariant theory would be to find the condition that (1) has an integral of the form

$$z = f(p) + \phi(q),$$

when p and q are any pair of independent functions, and f and ϕ are arbitrary functional symbols.

Take $$x' = p, \qquad y' = q,$$

and (1) is transformed to an equation with the integral

$$z = f(x') + \phi(y');$$

and therefore into $$\frac{\partial^2 z}{\partial x' \partial y'} = 0.$$

The invariants (not absolute) I and J must thus vanish if (1) is to admit an integral of the form proposed.

ON INVARIANTS OF A CANONICAL SUBSTITUTION

By HAROLD HILTON.

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1. It is proposed in this paper to discuss the properties of homogeneous invariants of the first and second degrees of a homogeneous linear substitution, and to deduce certain properties of orthogonal substitutions.

Suppose A is the substitution

$$x'_t = a_{t1}x_1+a_{t2}x_2+\ldots+a_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

and let $f(x_1, x_2, \ldots, x_m)$ be an invariant, so that

$$f(x_1, x_2, \ldots, x_m)$$
$$\equiv f(a_{11}x_1+a_{12}x_2+\ldots+a_{1m}x_m,\ a_{21}x_2+\ldots+a_{2m}x_m,\ \ldots,\ a_{m1}x_1+\ldots+a_{mm}x_m).$$

If B is the substitution

$$x'_t = b_{t1}x_1+b_{t2}x_2+\ldots+b_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

the corresponding invariant of BAB^{-1} is

$$f(b_{11}x_1+b_{12}x_2+\ldots+b_{1m}x_m,\ b_{21}x_1+\ldots+b_{2m}x_m,\ \ldots,\ b_{m1}x_1+\ldots+b_{mm}x_m).^*$$

It follows that, in order to obtain the invariants of A, it is sufficient to discuss the invariants of any "canonical form" into which A can be transformed.

The canonical form N we shall take is

$$\begin{cases} x'_1 = \alpha x_1+x_2,\ x'_2 = \alpha x_2+x_3,\ \ldots,\ x'_{r-1} = \alpha x_{r-1}+x_r,\ x'_r = \alpha x_r\,; \\ y'_1 = \beta y_1+y_2,\ \ldots,\ y'_{s-1} = \beta y_{s-1}+y_s,\ y'_s = \beta y_s\,; \\ z_1 = \gamma z_1+z_2,\ \ldots,\ z'_{t-1} = \beta z_{t-1}+z_t,\ z_t = \gamma z_t\,; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

where $r+s+t+\ldots = m$, and the quantities $\alpha, \beta, \gamma, \ldots$ are not necessarily all distinct.†

---

* This is readily proved by using the theorem: "The transform of a substitution S by a substitution T is found by expressing S in terms of new variables defined by T." See Hilton's *Finite Groups*, ch. III, § 3.

† For a proof that any substitution is transformable into this type see, for instance, *Messenger of Mathematics* (1909), p. 24.

The invariant-factors (*Elementartheiler*) of this form are evidently

$$(\lambda-\alpha)^r,\ (\lambda-\beta)^s,\ (\lambda-\gamma)^t,\ \dots.$$

The inverse of this form is

$$\left\{\begin{array}{ll} x_1' = \alpha^{-1}x_1-\alpha^{-2}x_2+\alpha^{-3}x_3-\alpha^{-4}x_4+\dots, & x_2' = \alpha^{-1}x_2-\alpha^{-2}x_3+\alpha^{-3}x_4-\dots, \\ & x_3' = \alpha^{-1}x_3-\alpha^{-2}x_4+\dots,\ \dots; \\ y_1' = \beta^{-1}y_1-\beta^{-2}y_2+\beta^{-3}y_3-\beta^{-4}y_4+\dots, & y_2' = \beta^{-1}y_2-\beta^{-2}y_3+\beta^{-3}y_4-\dots, \\ & y_3' = \beta^{-1}y_3-\beta^{-2}y_4+\dots,\ \dots; \\ z_1' = \gamma^{-1}z_1-\gamma^{-2}z_2+\gamma^{-3}z_3-\gamma^{-4}z_4+\dots, & z_2' = \gamma^{-1}z_2-\gamma^{-2}z_3+\gamma^{-3}z_4-\dots, \\ & z_3' = \gamma^{-1}z_3-\gamma^{-2}z_4+\dots,\ \dots; \\ \dots\ \dots\ \dots\ \dots\ \dots & \dots\ \dots\ \dots\ \dots\ \dots \end{array}\right.$$

whose invariant-factors are

$$(\lambda-\alpha^{-1})^r,\ (\lambda-\beta^{-1})^s,\ (\lambda-\gamma^{-1})^t,\ \dots.$$

2. It is immediately verified that

$$k_1x_1+k_2x_2+\dots+k_rx_r$$

is an invariant of

$$x_1' = \alpha x_1+x_2,\ \dots,\ x_{r-1}' = \alpha x_{r-1}+x_r,\ x_r' = \alpha x_r$$

if, and only if, $\alpha = 1,\ k_1 = k_2 = \dots = k_{r-1} = 0.$

Hence we have the result:—

*A substitution has no homogeneous linear invariant unless it has a unit characteristic-root.** *If it has such a unit characteristic-root with k corresponding invariant-factors, it has k independent homogeneous linear invariants.*

We proceed now in §§ 3 and 4 to obtain the homogeneous *quadratic* invariants of the canonical substitution.

3. Suppose $\Sigma e_{ij}y_ix_j \quad (i = 1, 2, \dots, s;\ j = 1, 2, \dots, r)$

is an invariant of

$$\left.\begin{array}{l} x_1' = \alpha x_1+x_2,\ \dots,\ x_{r-1}' = \alpha x_{r-1}+x_r,\ x_r' = \alpha x_r \\ y_1' = \beta y_1+y_2,\ \dots,\ y_{s-1}' = \beta y_{s-1}+y_s,\ y_s' = \beta y_s \end{array}\right\} \quad (r \geqslant s).$$

---

* The "characteristic-roots" of A are the roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{m1} \\ a_{21} & a_{22}-\lambda & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm}-\lambda \end{vmatrix} = 0$$

of A.

Comparing the coefficients in

$$\Sigma e_{ij} y_i x_j \quad \text{and} \quad \Sigma e_{ij} (\beta y_i + y_{i+1})(\alpha x_j + x_{j+1}),$$

we see that the following quantities vanish

$$\begin{array}{lllll}
(\alpha\beta-1)e_{11} & (\alpha\beta-1)e_{12}+\beta e_{11} & (\alpha\beta-1)e_{13}+\beta e_{12} & \dots & (\alpha\beta-1)e_{1r}+\beta e_{1\,r-1} \\
(\alpha\beta-1)e_{21} & (\alpha\beta-1)e_{22}+\beta e_{21} & (\alpha\beta-1)e_{23}+\beta e_{22} & & (\alpha\beta-1)e_{2r}+\beta e_{2\,r-1} \\
\quad +\alpha e_{11} & \quad +\alpha e_{12}+\ e_{11} & \quad +\alpha e_{13}+\ e_{12} & \dots & \quad +\alpha e_{1r}+\ e_{1\,r-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(\alpha\beta-1)e_{s1} & (\alpha\beta-1)e_{s2}+\beta e_{s1} & (\alpha\beta-1)e_{s3}+\beta e_{s2} & & (\alpha\beta-1)e_{sr}+\beta e_{s\,r-1} \\
\quad +\alpha e_{s-1\,1} & \quad +\alpha e_{s-1\,2}+e_{s-1\,1} & \quad +\alpha e_{s-1\,3}+e_{s-1\,2} & \dots & \quad +\alpha e_{s-1\,r}+e_{s-1\,r-1}
\end{array}$$

A glance at these equations shows that $e_{ij} = 0$ for each value of i and j unless $\alpha\beta = 1$.

If $\alpha\beta = 1$, the equations give

$$\begin{array}{llllllll}
e_{11} = & e_{12} = \dots = & e_{1\,r-s} = \dots = & e_{1\,r-2} = & e_{1\,r-1} & = 0, \\
e_{21} = & e_{22} = \dots = & e_{2\,r-s} = \dots = & e_{2\,r-2} & & = 0, \\
\vdots & \vdots & \vdots & \vdots & & \\
e_{s1} = & e_{s2} = \dots = & e_{s\,r-s} & & & = 0,
\end{array}$$

together with $\frac{1}{2}s(s-1)$ relations of the type

$$\left\{\begin{array}{llll}
\alpha e_{1r}+\beta e_{2\,r-1} = 0, & \alpha e_{2\,r-1}+\beta e_{3\,r-2} = 0, & \dots, & \alpha e_{s-1\,r-s+2}+\beta e_{s\,r-s+1} = 0; \\
\beta e_{3\,r-1}+\alpha e_{2r} \quad +e_{2\,r-1} = 0, & \beta e_{4\,r-1}+\alpha e_{3r} \quad +e_{3\,r-1} = 0, & & \\
 & \beta e_{5\,r-1}+\alpha e_{4r} \quad +e_{4\,r-1} = 0, & \dots; & \\
\beta e_{4\,r-2}+\alpha e_{3\,r-1}+e_{3\,r-2} = 0, & \beta e_{5\,r-2}+\alpha e_{4\,r-1}+e_{4\,r-2} = 0, & & \\
 & \beta e_{6\,r-2}+\alpha e_{5\,r-1}+e_{5\,r-2} = 0, & \dots; & \\
\dots & \dots & \dots & \dots
\end{array}\right.$$

between the remaining $\frac{1}{2}s(s+1)$ e's.

This suggests that there are s independent quadratic invariants.

Taking as an illustration the cases in which $s \leqslant 6$, the reader will have no difficulty in proving that this is really the case. These simple illustrations also suggest that these invariants are the expressions

$$f_1(x, y), \; f_2(x, y), \; \dots, \; f_s(x, y)$$

defined as follows:—When $r = s = 2n$,

$$
\begin{aligned}
&(-1)^{n-1} f_1(x, y)\\
&= \{\; x_n\;\, [a y_{n+1} - a^2 y_{n+2} + a^3 y_{n+3} - a^4 y_{n+4} + \dots + (-1)^{n-1} a^n y_{2n}]\\
&\quad + x_{n-1}[-a^3 y_{n+2} + {}^2C_1 a^4 y_{n+3} - {}^3C_1 a^5 y_{n+4} + \dots + (-1)^{n-1}\, {}^{n-1}C_1 a^{n+1} y_{2n}]\\
&\quad + x_{n-2}[a^5 y_{n+3} - {}^3C_2 a^6 y_{n+4} + \dots + (-1)^{n-1}\, {}^{n-1}C_2 a^{n+2} y_{2n}]\\
&\quad + \dots \dots \dots \dots \dots \dots \dots \dots \dots\\
&\quad + x_1\;\, [(-1)^{n-1} a^{2n-1} y_{2n}]\}\\
&- \{ y_n [a^{-1} x_{n+1} - a^{-2} x_{n+2} + a^{-3} x_{n+3} - a^{-4} x_{n+4} + \dots + (-1)^{n-1} a^{-n} x_{2n}]\\
&\quad + y_{n-1}[-a^{-3} x_{n+2} + {}^2C_1 a^{-4} x_{n+3} - {}^3C_1 a^{-5} x_{n+4} + \dots + (-1)^{n-1}\, {}^{n-1}C_1 a^{-n-1} x_{2n}]\\
&\quad + y_{n-2}[a^{-5} x_{n+3} - {}^3C_2 a^{-6} x_{n+4} + \dots + (-1)^{n-1}\, {}^{n-1}C_2 a^{-n-2} x_{2n}]\\
&\quad + \dots \dots \dots \dots \dots \dots \dots \dots \dots\\
&\quad + y_1\;\, [(-1)^{n-1} a^{-2n+1} x_{2n}]\}\;;
\end{aligned}
$$

and, when $r = s = 2n+1$,

$$
\begin{aligned}
&(-1)^n f_1(x, y)\\
&= \tfrac{1}{2} x_{n+1} [y_{n+1} - a\, y_{n+2} + a^2\, y_{n+3} - \dots + (-1)^n a^n\, y_{2n+1}]\\
&\quad + \tfrac{1}{2} y_{n+1} [x_{n+1} - a^{-1} x_{n+2} + a^{-2} x_{n+3} - \dots + (-1)^n a^{-n} x_{2n+1}]\\
&\quad + \{\; x_n\;\, [-a^2 y_{n+2} + (1+\tfrac{1}{2}) a^3 y_{n+3} - (1+\tfrac{1}{2}\,{}^2C_1) a^4 y_{n+4} + \dots\\
&\qquad\qquad + (-1)^n\, (1+\tfrac{1}{2}\,{}^{n-1}C_1) a^{n+1} y_{2n+1}]\\
&\qquad + x_{n-1}[a^4 y_{n+3} - ({}^2C_1 + \tfrac{1}{2})\, a^5 y_{n+4} + \dots\\
&\qquad\qquad + (-1)^n ({}^{n-1}C_1 + \tfrac{1}{2}\,{}^{n-1}C_2) a^{n+2} y_{2n+1}]\\
&\qquad + x_{n-2}[-a^6 y_{n+4} + \dots\\
&\qquad\qquad + (-1)^n ({}^{n-1}C_2 + \tfrac{1}{2}\,{}^{n-1}C_3) a^{n+3} y_{2n+1}]\\
&\qquad + \dots \dots \dots \dots \dots \dots \dots \dots\\
&\qquad + x_1\;\, [(-1)^n a^{2n} y_{2n+1}]\}\\
&\quad + \{\; y_n\;\, [-a^{-2} x_{n+2} + (1+\tfrac{1}{2}) a^{-3} x_{n+3} - (1+\tfrac{1}{2}\,{}^2C_1) a^{-4} x_{n+4} + \dots\\
&\qquad\qquad + (-1)^n\, (1+\tfrac{1}{2}\,{}^{n-1}C_1) a^{-n-1} x_{2n+1}]\\
&\qquad + y_{n-1}[a^{-4} x_{n+3} - ({}^2C_1 + \tfrac{1}{2})\, a^{-5} x_{n+4} + \dots\\
&\qquad\qquad + (-1)^n ({}^{n-1}C_1 + \tfrac{1}{2}\,{}^{n-1}C_2) a^{-n-2} x_{2n+1}]\\
&\qquad + y_{n-2}[-a^{-6} x_{n+4} + \dots\\
&\qquad\qquad + (-1)^n ({}^{n-1}C_2 + \tfrac{1}{2}\,{}^{n-1}C_3) a^{-n-3} x_{2n+1}]\\
&\qquad + \dots \dots \dots \dots \dots \dots \dots \dots\\
&\qquad + y_1\;\, [(-1)^n a^{-2n}\, x_{2n+1}]\}\;;
\end{aligned}
$$

$f_2(x, y)$ is the corresponding expression in $x_2, x_3, \ldots, x_s$; $y_2, y_3, \ldots, y_s$; $f_3(x, y)$ is the corresponding expression in $x_3, x_4, \ldots, x_s$; $y_3, y_4, \ldots, y_s$; and so on; $f_s(x, y)$ being $x_s y_s$.

If $r > s$, we replace x_1 by x_{r-s+1}, x_2 by x_{r-s+2}, ..., x_s by x_r in the above expressions.

These results, when once suggested, are easily verified by direct substitution.

The most general invariant is $k_1 f_1 + k_2 f_2 + \ldots + k_s f_s$, where the k's are arbitrary constants.

4. Now suppose $\Sigma e_{ij} x_i x_j \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, r)$,

where $e_{ij} = e_{ji}$, is an invariant of

$$x_1' = ax_1 + x_2, \quad \ldots, \quad x_{r-1}' = ax_{r-1} + x_r, \quad x_r = ax_r.$$

Comparing the coefficients in

$$\Sigma e_{ij} x_i x_j \quad \text{and} \quad \Sigma e_{ij}(ax_i + x_{i+1})(ax_j + x_{j+1}),$$

we see that the following quantities vanish,

$$\begin{array}{lllll}
\tfrac{1}{2}(a^2-1)e_{11} & (a^2-1)e_{12}+ae_{11} & (a^2-1)e_{13}+ae_{12} & \ldots & (a^2-1)e_{1r}+ae_{1\,r-1} \\
 & \tfrac{1}{2}(a^2-1)e_{22} & (a^2-1)e_{23}+ae_{22} & \ldots & (a^2-1)e_{2r}+ae_{2\,r-1} \\
 & \quad +ae_{12}+\tfrac{1}{2}e_{11} & \quad +ae_{13}+e_{12} & & \quad +ae_{1r}+e_{1\,r-1} \\
 & & \tfrac{1}{2}(a^2-1)e_{33} & \ldots & (a^2-1)e_{3r}+ae_{3\,r-1} \\
 & & \quad +ae_{23}+\tfrac{1}{2}e_{22} & & \quad +ae_{2r}+e_{2\,r-1} \\
 & & & \vdots & \vdots \\
 & & & & \tfrac{1}{2}(a^2-1)e_{rr} \\
 & & & & \quad +ae_{r-1\,r}+\tfrac{1}{2}e_{r-1\,r-1}.
\end{array}$$

As in § 3, $e_{ij} = 0$ for each value of i and j unless $a^2 = 1$.

If $a^2 = 1$,

$$\begin{aligned}
e_{11} = e_{12} = e_{13} = \ldots = e_{1\,r-3} = e_{1\,r-2} = e_{1\,r-1} &= 0, \\
e_{22} = e_{23} = \ldots = e_{2\,r-3} = e_{2\,r-2} &= 0, \\
e_{33} = \ldots = e_{3\,r-3} &= 0, \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad &\ldots;
\end{aligned}$$

together with relations of the type

$$e_{1r} + e_{2\,r-1} = 0, \quad ae_{3\,r-1} + ae_{2r} + e_{2\,r-1} = 0, \quad \ldots.$$

The result now obtained is that, when $r = 2n+1$, the independent quadratic invariants are $f_1(x, x)$, $f_3(x, x)$, $f_5(x, x)$, ..., $f_r(x, x)$ defined thus:—When $a = 1$,

$$
\begin{aligned}
&(-1)^n f_1(x, x) \\
&\equiv x_{n+1}[\,x_{n+1}-x_{n+2}+x_{n+3}-\dots+(-1)^n x_{2n+1}] \\
&\quad +x_n\,[-2x_{2n+2}+(2+1)x_{n+3}-(2+{}^2C_1)x_{n+4}+(2+{}^3C_1)x_{n+5}-\dots \\
&\qquad\qquad +(-1)^n(2+{}^{n-1}C_1)\,x_{2n+1}] \\
&\quad +x_{n-1}[\,2x_{n+3}-(2.{}^2C_1+1)x_{n+4}+(2.{}^3C_1+{}^3C_2)\,x_{n+5}-\dots \\
&\qquad\qquad +(-1)^n(2.{}^{n-1}C_1+{}^{n-1}C_2)x_{2n+1}] \\
&\quad +x_{n-2}[\,-2x_{n+4}+(2.{}^3C_2+1)\;x_{n+5}-\dots \\
&\qquad\qquad +(-1)^n(2.{}^{n-1}C_2+{}^{n-1}C_3)x_{2n+1}] \\
&\quad +\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \\
&\quad +x_1\;[\,(-1)^n\,2x_{2n+1}];
\end{aligned}
$$

and, when $a = -1$,

$$
\begin{aligned}
&(-1)^n f_1(x, x) \\
&\equiv x_{n+1}[\,x_{n+1}+x_{n+2}+x_{n+3}+\dots+x_{2n+1}] \\
&\quad -x_n\,[2x_{2n+2}+(2+1)x_{n+3}+(2+{}^2C_1)x_{n+4}+(2+{}^3C_1)x_{n+5}+\dots \\
&\qquad\qquad +(2+{}^{n-1}C_1)x_{2n+1}] \\
&\quad +x_{n-1}[\,2x_{n+3}+(2.{}^2C_1+1)x_{n+4}+(2.{}^3C_1+{}^3C_2)x_{n+5}+\dots \\
&\qquad\qquad +(2.{}^{n-1}C_1+{}^{n-1}C_2)x_{2n+1}] \\
&\quad -x_{n-2}[\,2x_{n+4}+(2.{}^3C_2+1)\;x_{n+5}+\dots \\
&\qquad\qquad +(2.{}^{n-1}C_2+{}^{n-1}C_3)x_{2n+1}] \\
&\quad +\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \\
&\quad +(-1)^n x_1[\,2x_{2n+1}];
\end{aligned}
$$

while $f_3(x, x)$ is the corresponding expression in $x_3, x_4, x_5, \dots, x_r$; $f_5(x, x)$ is the corresponding expression in $x_5, x_6, \dots, x_r$; and so on; $f_r(x, x)$ being x_r^2.

When $r = 2n$ (and $a^2 = 1$), the independent quadratic invariants are the independent invariants of

$$x_2' = ax_2+x_3,\quad x_3' = ax_3+x_4,\quad \dots,\quad x_{r-1}' = ax_{r-1}+x_r,\quad x_r' = ax_r,$$

which involves an *odd* number of variables.

These invariants are obtained if we put x_1 for y_1, x_2 for y_2, x_3 for y_3, ... in the invariants of § 3.

5. The most general quadratic invariant of the canonical substitution of § 1 can now be at once written down by the results of §§ 3, 4. When the determinant formed by the coefficients of such an invariant is written down, it will be readily seen that this determinant must vanish unless to every invariant-factor $(\lambda-\lambda_1)^a$ (where $\lambda_1^2 \neq 1$) of the substitution corresponds another invariant-factor $(\lambda-\lambda_1^{-1})^a$.

This may also be proved as follows :—

If a quadratic invariant of any substitution on $x_1, x_2, \ldots, x_m$ (such as A in § 1) has non-zero determinant, the invariant can be transformed into $x_1^2+x_2^2+\ldots+x_m^2$, and then the substitution A will be orthogonal. Then we shall have $A^{-1}=A'$ (the transposed substitution of A), so that A^{-1} has the same invariant-factors as A. Hence, if A is transformed into the canonical form N (of § 1), N has the same invariant-factors as its inverse, *i.e.*, $(\lambda-\alpha)^r$, $(\lambda-\beta)^s$, $(\lambda-\gamma)^t$, ..., are the same as

$$(\lambda-\alpha^{-1})^r,\ (\lambda-\beta^{-1})^s,\ (\lambda-\gamma^{-1})^t,\ \ldots,$$

in some order or other ; which proves the result.*

In the following we shall confine our attention to substitutions which are such that each invariant-factor $(\lambda-\lambda_1)^a$, $(\lambda_1^2 \neq 1)$ is paired with another invariant-factor $(\lambda-\lambda_1^{-1})^a$.

From the results of §§ 3, 4, we have

The number of independent quadratic invariants in the case of a substitution with invariant-factors

$$(\lambda-\lambda_1)^a,\ (\lambda-\lambda_1^{-1})^a,\ (\lambda-\lambda_1)^b,\ (\lambda-\lambda_1^{-1})^b,\ (\lambda-\lambda_1)^c,\ (\lambda-\lambda_1^{-1})^c,$$
$$(\lambda-\lambda_1)^d,\ (\lambda-\lambda_1^{-1})^d,\ \ldots,$$

where $$a \geqslant b \geqslant c \geqslant d \geqslant \ldots,$$

and with invariant-factors

$$(\lambda-1)^h,\ (\lambda-1)^k,\ (\lambda-1)^l,\ (\lambda-1)^m,\ \ldots,$$

where $$h \geqslant k \geqslant l \geqslant m \geqslant \ldots,$$

and invariant-factors

$$(\lambda+1)^p,\ (\lambda+1)^q,\ (\lambda+1)^r,\ (\lambda+1)^s,\ \ldots,$$

where $$p \geqslant q \geqslant r \geqslant s \geqslant \ldots,$$

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* The result is due to Frobenius, *Crelle*, 84 (1878), p. 41.

is
$$I\left[\frac{h+1}{2}\right]+I\left[\frac{k+1}{2}\right]+I\left[\frac{l+1}{2}\right]+\ldots+(k+2l+3m+\ldots)$$
$$+I\left[\frac{p+1}{2}\right]+I\left[\frac{q+1}{2}\right]+I\left[\frac{r+1}{2}\right]+\ldots+(q+2r+3s+\ldots)$$
$$+\Sigma\,(a+3b+5c+7d+\ldots),^{*}$$

the summation being taken over each distinct pair of reciprocal characteristic-roots of the substitution.

6. Suppose a canonical substitution has exactly k pairs of invariant-factors $(\lambda-a)^r$ and $(\lambda-a^{-1})^r$.

Let the corresponding part of the substitution be

$$\left\{\begin{array}{l} {}_1x'_1 = a\,{}_1x_1 + {}_1x_2,\ \ldots,\ {}_1x'_{r-1} = a\,{}_1x_{r-1} + {}_1x_r,\ {}_1x'_r = a\,{}_1x_r\,; \\ \qquad\qquad\qquad\qquad {}_2x'_1 = a\,{}_2x_1 + {}_2x_2,\ \ldots,\ {}_2x'_r = a\,{}_2x_r\,; \\ {}_1y'_1 = a^{-1}\,{}_1y_1+{}_1y_2,\ \ldots,\ {}_1y'_{r-1} = a^{-1}\,{}_1y_{r-1}+{}_1y_r,\ {}_1y'_r = a^{-1}\,{}_1y_r\,; \\ \qquad\qquad\qquad\qquad {}_2y'_1 = a^{-1}\,{}_2y_1+{}_2y_2,\ \ldots,\ {}_2y'_r = a^{-1}\,{}_2y_r\,; \end{array}\right.$$

$$\left\{\begin{array}{l} \ldots\,;\ {}_kx'_1 = a\,{}_kx_1 + {}_kx_2,\ \ldots,\ {}_kx'_r = a\,{}_kx_r\,; \\ \ldots\,;\ {}_ky'_1 = a^{-1}\,{}_ky_1+{}_ky_2,\ \ldots,\ {}_ky'_r = a^{-1}\,{}_ky_r\,; \end{array}\right.$$

and let the corresponding part of any quadratic invariant be (using the notation of § 3)

$$\sum_{i=1}^{i=k}\{t_{i1}f_1({}_1x,\,{}_iy)+t_{i2}f_1({}_2x,\,{}_iy)+\ldots+t_{ik}f_1({}_kx,\,{}_iy)\}$$
$$+\sum_{i=1}^{i=k}\{s_{i1}f_2({}_1x,\,{}_iy)+s_{i2}f_2({}_2x,\,{}_iy)+\ldots+s_{ik}f_2({}_kx,\,{}_iy)\}$$
$$+\ldots.$$

Suppose, moreover, that the substitution has exactly p invariant-factors $(\lambda-1)^r$, and exactly q invariant-factors $(\lambda+1)^r$.

Let the corresponding parts of the canonical substitution be respectively

$${}_1x'_1 = {}_1x_1+{}_1x_2,\ \ldots,\ {}_1x'_r = {}_1x_r\,;\ {}_2x'_1 = {}_2x_1+{}_2x_2,\ \ldots,\ {}_2x'_r = {}_2x_r\,;\ \ldots\,;$$
$${}_px'_1 = {}_px_1+{}_px_2,\ \ldots,\ {}_px'_r = {}_px_r\,;$$

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* $I[t]$ means "the integral part of t."

and

$$ {}_1X'_1 = -{}_1X_1 + {}_1X_2, \ldots, {}_1X'_r = -{}_1X_r;\quad {}_2X'_1 = -{}_2X_1 + {}_2X_2, \ldots, {}_2X'_r = -{}_2X_r;\ \ldots; $$

$$ {}_qX'_1 = -{}_qX_1 + {}_qX_2, \ldots, {}_qX'_r = -{}_qX_r; $$

and let the corresponding parts of the quadratic invariant be (using the notation of § 4)

$$ \sum_{i=1}^{i=p} \{ h_{ii} f_1({}_ix, {}_ix) + h_{i\,i+1} f_1({}_{i+1}x, {}_ix) + \ldots + h_{ip} f_1({}_px, {}_ix) \} + \ldots, $$

and $$ \sum_{i=1}^{i=q} \{ H_{ii} f_1({}_iX, {}_iX) + H_{i\,i+1} f_1({}_{i+1}X, {}^iX) + \ldots + H_{iq} f_1({}_qX, {}_iX) \} + \ldots, $$

where h_{ii} or H_{ii} is zero if r is even.

Then, taking $h_{ij} = (-1)^{r-1} h_{ji}$ and $H_{ij} = (-1)^{r-1} H_{ji}$, the determinant of the quadratic invariant is

$$ \pm\Pi \begin{vmatrix} h_{11} & \ldots & h_{1p} \\ \vdots & \vdots & \vdots \\ h_{p1} & \ldots & h_{pp} \end{vmatrix}^{r} \times \begin{vmatrix} H_{11} & \ldots & H_{1q} \\ \vdots & \vdots & \vdots \\ H_{q1} & \ldots & H_{qq} \end{vmatrix}^{r} \times \begin{vmatrix} t_{11} & \ldots & t_{1k} \\ \vdots & \vdots & \vdots \\ t_{k1} & \ldots & t_{kk} \end{vmatrix}^{2r}, $$

the product being taken for each distinct value of r and a in the invariant-factors of the substitution.

This will be clear if we consider a simple case. Take, for example, the substitution

$$ x'_1 = ax_1 + x_2,\quad x'_2 = ax_1 + x_2,\quad x'_3 = ax_3; $$

$$ x'_1 = ax_1 + x_2,\quad x'_2 = ax_2 + x_3,\quad x'_3 = ax_3; $$

$$ y'_1 = a^{-1}y_1 + y_2,\quad y'_2 = a^{-1}y_1 + y_2,\quad y'_3 = a^{-1}y_3; $$

$$ \boldsymbol{y}'_1 = a^{-1}\boldsymbol{y}_1 + \boldsymbol{y}_2,\quad \boldsymbol{y}'_2 = a^{-1}\boldsymbol{y}_2 + \boldsymbol{y}_3,\quad \boldsymbol{y}'_3 = a^{-1}\boldsymbol{y}_3; $$

$$ \xi'_1 = a\xi_1 + \xi_2,\quad \xi'_2 = a\xi_2;\qquad \boldsymbol{\xi}'_1 = a\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2,\quad \boldsymbol{\xi}'_2 = a\boldsymbol{\xi}_2; $$

$$ \Xi'_1 = a\Xi_1 + \Xi_2,\quad \Xi'_2 = a\Xi_2; $$

$$ \eta'_1 = a^{-1}\eta_1 + \eta_2,\quad \eta'_2 = a^{-1}\eta_2;\qquad \boldsymbol{\eta}'_1 = a^{-1}\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2,\quad \boldsymbol{\eta}'_2 = a^{-1}\boldsymbol{\eta}_2; $$

$$ \mathrm{H}'_1 = a^{-1}\mathrm{H}_1 + \mathrm{H}_2,\quad \mathrm{H}'_2 = a^{-1}\mathrm{H}_2. $$

Let any quadratic invariant be (with the notation of § 3)

$$ \{ af_1(x, y) + bf_1(x, y) + cf_1(x, y) + df_1(x, y) + lf_1(\xi, \eta) + mf_1(\boldsymbol{\xi}, \eta) + nf_1(\Xi, \eta) $$
$$ + pf_1(\xi, \boldsymbol{\eta}) + qf_1(\boldsymbol{\xi}, \boldsymbol{\eta}) + rf_1(\Xi, \boldsymbol{\eta}) + uf_1(\xi, \mathrm{H}) + vf_1(\boldsymbol{\xi}, \mathrm{H}) + wf_1(\Xi, \mathrm{H}) \} $$
$$ + \{ a'f_2(x, y) + \ldots \} + \{ a''f_3(x, y) + \ldots \}. $$

Then the determinant of the invariant is evidently minus the square of

$$\begin{vmatrix}
0 & 0 & a\alpha^{-2} & 0 & 0 & b\alpha^{-2} & 0 & & 0 & 0 & 0 & 0 \\
0 & -a & * & 0 & -b & * & 0 & * & 0 & * & 0 & * \\
a\alpha^{2} & * & * & b\alpha^{2} & * & * & * & * & * & * & * & * \\
0 & 0 & c\alpha^{-2} & 0 & 0 & d\alpha^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c & * & 0 & -d & * & 0 & * & 0 & * & 0 & * \\
c\alpha^{2} & * & * & d\alpha^{2} & * & * & * & * & * & * & * & * \\
0 & 0 & * & 0 & 0 & * & 0 & -l\alpha^{-1} & 0 & -m\alpha^{-1} & 0 & -n\alpha^{-1} \\
0 & * & * & 0 & * & * & l\alpha & * & m\alpha & * & n\alpha & * \\
0 & 0 & * & 0 & 0 & * & 0 & -p\alpha^{-1} & 0 & -q\alpha^{-1} & 0 & -r\alpha^{-1} \\
0 & * & * & 0 & * & * & p\alpha & * & q\alpha & * & r\alpha & 0 \\
0 & 0 & * & 0 & 0 & * & 0 & -u\alpha^{-1} & 0 & -v\alpha^{-1} & 0 & -w\alpha^{-1} \\
0 & * & * & 0 & * & * & u\alpha & * & v\alpha & * & w\alpha & *
\end{vmatrix};$$

the asterisks denoting certain quantities which are not zero in general. Now rearrange rows and columns, so that the rows and columns now in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, take respectively the orders 5, 3, 1, 6, 4, 2, 10, 7, 11, 8, 12, 9 and 1, 3, 5, 2, 4, 6, 7, 10, 8, 11, 9, 12. Then the determinant last written becomes

$$\begin{vmatrix}
a\alpha^{2} & b\alpha^{2} & * & * & * & * & * & * & * & * & * & * \\
c\alpha^{2} & d\alpha^{2} & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & -a & -b & * & * & 0 & 0 & 0 & * & * & * \\
0 & 0 & -c & -d & * & * & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & a\alpha^{-2} & b\alpha^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c\alpha^{-2} & d\alpha^{-} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & l\alpha & m\alpha & n\alpha & * & * & * \\
0 & 0 & * & * & * & * & p\alpha & q\alpha & r\alpha & * & * & * \\
0 & 0 & * & * & * & * & u\alpha & v\alpha & w\alpha & * & * & * \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & -l\alpha^{-1} & -m\alpha^{-1} & -n\alpha^{-1} \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & -p\alpha^{-1} & -q\alpha^{-1} & -r\alpha^{-1} \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & -u\alpha^{-1} & -v\alpha^{-1} & -w\alpha^{-1}
\end{vmatrix}$$

$$= - \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{3} \times \begin{vmatrix} l & m & n \\ p & q & r \\ u & v & w \end{vmatrix}^{2}.$$

It follows that an orthogonal substitution cannot have an odd number of invariant-factors of the type $(\lambda-1)^{r}$ or $(\lambda+1)^{r}$, where r is a given *even* integer.

7. We now use the preceding results to establish the following theorem:—

Any orthogonal substitution on $x_1, x_2, \ldots, x_m$ *can be transformed by means of an orthogonal substitution into the form*

$$x_1' = \pm x_1, \quad x_2' = \pm x_2, \quad \ldots, \quad x_h' = \pm x_h, \quad x_{h+1}' = \cos\theta_1\, x_{h+1} - \sin\theta_1\, x_{h+2},$$

$$\pm x_{h+2} = \sin\theta_1\, x_{h+1} + \cos\theta_1\, x_{h+2}, \ \ldots, \ x_{h+2r-1}' = \cos\theta_r x_{h+2r-1} - \sin\theta_r x_{h+2r},$$

$$\pm x_{h+2r}' = \sin\theta_r\, x_{h+2r-1} + \cos\theta_r\, x_{h+2r},$$

$$x_t' = g_{t\,h+2r+1}\, x_{h+2r+1} + \ldots + g_{tm} x_m \quad (t = h+2r+1,\ h+2r+2,\ \ldots,\ m),$$

where

$$x_t' = g_{t\,h+2r+1}\, x_{h+2r+1} + \ldots + g_{tm} x_m \quad (t = h+2r+1,\ \ldots,\ m),^{*}$$

is an orthogonal substitution with no simple invariant-factor.[†]

Suppose that in § 1 A is an orthogonal substitution. Then since $x_1^2 + x_2^2 + \ldots + x_m^2$ is an invariant of A, the corresponding invariant of of BAB^{-1} is

$$(b_{11}x_1 + b_{12}x_2 + \ldots + b_{1m}x_m)^2 + (b_{21}x_1 + \ldots + b_{2m}x_m)^2 + \ldots + (b_{m1}x_1 + \ldots + b_{mm}x_m)^2$$
$$\equiv \Sigma e_{ij} x_i x_j \quad (i, j = 1, 2, \ldots, m),$$

where $$e_{ij} = e_{ji} = b_{1i}b_{1j} + b_{2i}b_{2j} + \ldots + b_{mi}b_{mj}.$$

Suppose now that B is chosen so that BAB^{-1} is in canonical form

$$x_1' = ax_1, \ \ldots, \ x_k = ax_k, \ x_{k+1} = a^{-1}x_{k+1}, \ \ldots, \ x_{2k}' = a^{-1}x_{2k},$$
$$x_{2k+1}' = \beta x_{2k+1} + x_{2k+2}, \ \ldots .$$

Then, by § 6, $$\Delta \equiv \begin{vmatrix} e_{1\,k+1} & e_{1\,k+2} & \cdots & e_{1\,2k} \\ e_{2\,k+1} & e_{2\,k+2} & \cdots & e_{2\,2k} \\ \vdots & \vdots & \vdots & \vdots \\ e_{k\,k+1} & e_{k\,k+2} & \cdots & e_{k\,2k} \end{vmatrix}$$

is a factor of the determinant of $\Sigma e_{ij} x_i x_j$, and therefore does not vanish.

Hence Δ contains at least one non-zero element; suppose it is $e_{1\,k+1}$. Since BAB^{-1} has poles $(Y_1, 0, 0, 0, \ldots)$ and $(0, 0, \ldots, 0, Y_{k+1}, 0, 0, \ldots)$ corresponding to its characteristic-roots a and a^{-1}, A has the poles

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[*] The ± signs being taken in any combination whatever.

[†] *I.e.*, an invariant-factor with unit exponent.

$(b_{11}, b_{21}, \ldots, b_{m1})$ and $(b_{1\,k+1}, b_{2\,k+1}, \ldots, b_{m\,k+1})$ corresponding to its characteristic-roots a and a^{-1}.* But

$$e_{1\,k+1} \equiv b_{11} b_{1\,k+1} + b_{21} b_{2\,k+1} + \ldots + b_{m1} b_{m\,k+1} \neq 0.$$

Hence, if A has simple invariant-factors $(\lambda - a)$ and $(\lambda - a^{-1})$, it has corresponding poles $(X_1, X_2, \ldots, X_m)$ and $(\xi_1, \xi_2, \ldots, \xi_m)$, such that

$$X_1 \xi_1 + X_2 \xi_2 + \ldots + X_m \xi_m \neq 0.$$

Conversely, if A has two such poles, they must correspond to simple invariant-factors of the form $(\lambda - a)$ and $(\lambda - a^{-1})$.

In a similar manner we prove that if A has a simple invariant-factor $(\lambda \pm 1)$, it has a corresponding pole $(X_1, X_2, \ldots, X_m)$, such that

$$X_1^2 + X_2^2 + \ldots + X_m^2 \neq 0,$$

and conversely.

8 (a). Suppose now that the orthogonal substitution A has a pole $(X_1, X_2, \ldots, X_m)$, such that

$$X_1^2 + X_2^2 + \ldots + X_m^2 \neq 0.$$

This is the case if A has a simple invariant-factor $(\lambda \pm 1)$.

Since we are only concerned with the ratios of $X_1, X_2, \ldots, X_m$, we may suppose

$$X_1^2 + X_2^2 + \ldots + X_m^2 = 1 \quad \text{and} \quad X_1 + 1 \neq 0.$$

The substitution Q with matrix

$$\left|\begin{array}{ccccc}
X_1 & X_2 & X_3 & \ldots & X_m \\
X_2 & \dfrac{X_2^2}{X_1+1} - 1 & \dfrac{X_2 X_3}{X_1+1} & \ldots & \dfrac{X_2 X_m}{X_1+1} \\
X_3 & \dfrac{X_3 X_2}{X_1+1} & \dfrac{X_3^2}{X_1+1} - 1 & \ldots & \dfrac{X_3 X_m}{X_1+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
X_m & \dfrac{X_m X_2}{X_1+1} & \dfrac{X_m X_3}{X_1+1} & \ldots & \dfrac{X_m^2}{X_1+1} - 1
\end{array}\right|$$

is readily verified to be an orthogonal substitution of order 2.†

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* Using the theorem:—"If we apply T to a pole of S, we obtain a pole of $T^{-1}ST$ corresponding to the same common characteristic-root of S and $T^{-1}ST$." See Hilton's ***Finite Groups***, Ch. III, § 6.

† We verify by squaring that $Q^2 = 1$, since $X_1^2 + X_2^2 + \ldots + X_m^2 = 1$. Then Q is orthogonal, being symmetric and of order 2.

Suppose $Q^{-1}AQ = C$, where C is

$$x_t' = c_{t1}x_1+c_{t2}x_2+\ldots+c_{tm}x_m \quad (t = 1, 2, \ldots, m).$$

Then C has $(1, 0, 0, \ldots, 0)$ as a pole.* Hence

$$c_{21} = c_{31} = \ldots = c_{m1} = 0.$$

But C is orthogonal since Q and A are orthogonal; so that

$$c_{11}^2+c_{21}^2+\ldots+c_{m1}^2 = 1,$$

and
$$c_{11}c_{1t}+c_{21}c_{2t}+\ldots+c_{m1}c_{mt} = 0 \quad (t = 2, 3, \ldots, m).$$

Hence
$$c_{12} = c_{13} = \ldots = c_{1m} = 0 \quad \text{and} \quad c_{11} = \pm 1.$$

Therefore A has been transformed into

$$x_1' = \pm x_1, \quad x_t' = c_{t2}x_2+c_{t3}x_3+\ldots+c_{tm}x_m \quad (t = 2, 3, \ldots, m).$$

Applying now a similar process to

$$x_t' = c_{t2}x_2+c_{t3}x_3+\ldots+c_{tm}x_m \quad (t = 2, 3, \ldots, m),$$

and so on, we finally transform A into the shape

$$x_1' = \pm x_1, \quad x_2' = \pm x_2, \quad \ldots, \quad x_h' = \pm x_h,$$
$$x_t' = f_{t\,h+1}\,x_{h+1}+f_{t\,h+2}\,x_{h+2}+\ldots+f_{tm}x_m \quad (t = h+1, \ldots, m),$$

where there is no pole $(X_{h+1}, X_{h+2}, \ldots, X_m)$ of

$$x_t' = f_{t\,h+1}\,x_{h+1}+f_{t\,h+2}\,x_{h+2}+\ldots+f_{tm}x_m \quad (t = h+1, \ldots, m),$$

such that
$$X_1^2+X_2^2+\ldots+X_m^2 \neq 0.$$

(*b*). Now suppose A has no pole $(X_1, X_2, \ldots, X_m)$ satisfying

$$X_1^2+X_2^2+\ldots+X_m^2 \neq 0,$$

but that A has poles $(X_1, X_2, \ldots, X_m)$ and $(\xi_1, \xi_2, \ldots, \xi_m)$ satisfying

$$X_1\xi_1+X_2\xi_2+\ldots+X_m\xi_m \neq 0.$$

This is the case if A has a pair of simple invariant-factors $(\lambda-a)$ and $(\lambda-a^{-1})$.

Since we are only concerned with the ratios of $X_1, X_2, \ldots, X_m$, and of $\xi_1, \xi_2, \ldots, \xi_m$, we may suppose

$$X_1\xi_1+X_2\xi_2+\ldots+X_m\xi_m = 2;$$

while
$$X_1^2+X_2^2+\ldots+X_m^2 = \xi_1^2+\xi_2^2+\ldots+\xi_m^2 = 0.$$

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* See note *, p. 434.

Let
$$2p_{t1} = X_t+\xi_t, \quad 2ip_{t2} = X_t-\xi_t \quad (t = 1, 2, \ldots, m).^*$$
Then
$$p_{11}^2+p_{21}^2+\ldots+p_{m1}^2 = p_{12}^2+p_{22}^2+\ldots+p_{m2}^2 = 1,$$
$$p_{11}p_{12}+p_{21}p_{22}+\ldots+p_{m1}p_{m2} = 0.$$

Choose $p_{13}, p_{23}, \ldots, p_{m3}$ to satisfy
$$p_{13}^2+p_{23}^2+\ldots+p_{m3}^2 = 1,$$
$$p_{11}p_{13}+p_{21}p_{23}+\ldots+p_{m1}p_{m3} = p_{12}p_{13}+p_{22}p_{23}+\ldots+p_{m2}p_{m3} = 0.$$
Then choose $p_{14}, p_{24}, \ldots, p_{m4}$ to satisfy
$$p_{14}^2+p_{24}^2+\ldots+p_{m4}^2 = 1,$$
$$p_{11}p_{14}+p_{21}p_{24}+\ldots+p_{m1}p_{m4} = p_{12}p_{14}+p_{22}p_{24}+\ldots+p_{m2}p_{m4}$$
$$= p_{13}p_{14}+p_{23}p_{24}+\ldots+p_{m3}p_{m4} = 0,$$
and so on.

The determinant
$$\begin{vmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{vmatrix}$$
is not zero, for its square is 1.

Let P be the substitution
$$x_t' = p_{t1}x_1+p_{t2}x_2+\ldots+p_{tm}x_m \quad (t = 1, 2, \ldots, m).$$
Then P is an orthogonal substitution.

Let $PAP^{-1} = D$, where D is
$$x_t' = d_{t1}x_1+d_{t2}x_2+\ldots+d_{tm}x_m \quad (t = 1, 2, \ldots, m).$$
Then D has $(1, \pm i, 0, 0, \ldots, 0)$ as poles,† hence
$$d_{31} = d_{41} = \ldots = d_{m1} = 0, \qquad d_{32} = d_{42} = \ldots = d_{m2} = 0.$$
But D is orthogonal, since P and A are orthogonal, so that
$$d_{11}^2+d_{21}^2+\ldots+d_{m1}^2 = d_{12}^2+d_{22}^2+\ldots+d_{m2}^2 = 1,$$
$$d_{11}d_{12}+d_{21}d_{22}+\ldots+d_{m1}d_{m2} = 0,$$
$$d_{1t}d_{11}+d_{2t}d_{21}+\ldots+d_{mt}d_{m1} = d_{1t}d_{12}+d_{2t}d_{22}+\ldots+d_{mt}d_{m2} = 0$$
$$(t = 3, 4, \ldots, m).$$

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* $i = \sqrt{-1}$ in this section.

† See note *, p. 434.

Therefore using

$$d_{31} = d_{41} = \ldots = d_{m1} = 0, \quad d_{32} = d_{42} = \ldots = d_{m2} = 0,$$

we have readily

$$d_{13} = d_{14} = \ldots = d_{1m} = 0, \quad d_{23} = d_{24} = \ldots = d_{2m} = 0;$$

$$d_{11} = \cos\theta, \quad d_{12} = -\sin\theta, \quad d_{21} = \pm\sin\theta, \quad d_{22} = \pm\cos\theta.^*$$

Hence A has been transformed into

$$x_1' = \cos\theta\, x_1 - \sin\theta\, x_2, \quad \pm x_2' = \sin\theta\, x_1 + \cos\theta\, x_2;$$

$$x_t' = d_{t3}x_3 + d_{t4}x_4 + \ldots + d_{tm}x_m \quad (t = 3, 4, \ldots, m).$$

If the substitution

$$x_t' = d_{t3}x_3 + d_{t4}x_4 + \ldots + d_{tm}x_m \quad (t = 3, 4, \ldots, m),$$

has poles $(Y_3, Y_4, \ldots, Y_m)$ and $(\eta_3, \eta_4, \ldots, \eta_m)$ satisfying

$$Y_3\eta_3 + Y_4\eta_4 + \ldots + Y_m\eta_m \neq 0,$$

the process may be repeated, and so on.

(*c*). Combining the results of (*a*) and (*b*) we have at once the theorem at the beginning of § 7.

It should be noticed that the method of transformation adopted is a practical one if the characteristic-roots of A are known.

9. If A is a *real* orthogonal substitution and $(X_1, X_2, \ldots, X_m)$ is any pole, we may take $\xi_1 = \bar{X}_1, \xi_2 = \bar{X}_2, \ldots, \xi_m = \bar{X}_m$ in § 8.[†]

Hence $$X_1^2 + X_2^2 + \ldots + X_m^2 \neq 0,$$

if $X_1, X_2, \ldots, X_m$ are real, and otherwise

$$X_1\xi_1 + X_2\xi_2 + \ldots + X_m\xi_m \neq 0.$$

Therefore the process of § 8 gives a practical means of transforming a

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[*] Both the upper or both the lower signs being taken.

[†] $\bar{X}$ being the complex quantity conjugate to X.

real orthogonal substitution by means of another *real* orthogonal substitution into the form

$$x_1' = \pm x_1,\ x_2' = \pm x_2,\ \ldots,\ x_{m-2r}' = \pm x_{m-2r},$$

$$x_{m-2r+1}' = \cos\theta_1\, x_{m-2r+1} - \sin\theta_1\, x_{m-2r+2},$$

$$\pm x_{m-2r+2}' = \sin\theta_1\, x_{m-2r+1} + \cos\theta_1\, x_{m-2r+2},\ \ldots,$$

$$x_{m-1}' = \cos\theta_r\, x_{m-1} - \sin\theta_r\, x_m,\quad \pm x_m' = \sin\theta_r\, x_{m-1} + \cos\theta_r\, x_m.$$[*]

That such a transformation is possible is a well known theorem,[†] of which the result at the beginning of § 7 may be considered an extension.

For a discussion of the general case of a real substitution A with the invariant $x_1^2+x_2^2+\ldots+x_k^2-x_{k+1}^2-\ldots-x_m^2$, the reader may refer to Loewy, *Math. Annalen*, 50 (1898), pp. 557–576.

In this case the equations

$$2p_{t1} = X_t+\xi_t,\qquad 2ip_{t2} = X_t-\xi_t,$$

of § 8 (*b*), may be replaced by

$$2p_{t1} = X_t+\xi_t,\qquad 2p_{t2} = X_t-\xi_t,$$

the corresponding part of the transformed substitution being

$$\pm x_1 = \cosh\theta\, x_1+\sinh\theta\, x_2,\qquad \pm x_2 = \sinh\theta\, x_1+\cosh\theta\, x_2.$$[*]

10. We have shown in § 5 that any substitution A is transformable into its inverse if, and only if, each invariant-factor $(\lambda-a)^r$ of A (where $a^2 \neq 1$) is paired with an invariant-factor $(\lambda-a^{-1})^r$.

If this is the case, we can show that the transformation may be performed by a substitution Q of order 2.

It will follow that A is the product of two substitutions of order 2, either of which transforms A into its inverse.

For, if $$Q^{-1}AQ = A^{-1} \quad\text{and}\quad Q^2 = 1,$$

$$A = (AQ)Q \quad\text{and}\quad (AQ)^2 = AQ.AQ = A.Q^{-1}AQ = 1,$$

while $$(AQ)^{-1}A(AQ) = Q^{-1}AQ = A^{-1}.$$

It is sufficient to establish the result for any form into which A is transformable. For, if

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[*] See note *, p. 433.

[†] See, for instance, Bieberbach, *Math. Annalen*, 70 (1911), p. 301, or Muth's *Elementartheiler*, p. 176.

$$Q^{-1}AQ = A^{-1} \quad \text{and} \quad Q^2 = 1,$$

$$(R^{-1}QR)^{-1}(R^{-1}AR)(R^{-1}QR) = (R^{-1}AR)^{-1}.$$

We may take then A in the canonical form.

Now we easily verify by direct transformation, that the substitution H,

$$\begin{cases} x_1' = ax_1 + x_2, \ \ldots, \ x_{s-1}' = ax_{s-1} + x_s, \ x_s' = ax_s, \\ y_1' = a^{-1}y_1 + y_2, \ \ldots, \ y_{s-1}' = a^{-1}y_{s-1} + y_s, \ y_s' = a^{-1}y_s, \end{cases}$$

is transformable into its inverse H^{-1},

$$\left.\begin{array}{l} x_1' = a^{-1}x_1 - a^{-2}x_2 + \ldots + (-1)^{s-1}a^{-s}x_s \\ x_2' = \qquad\quad a^{-1}x_2 - \ldots + (-1)^{s-2}a^{-s+1}x_s \\ \quad \vdots \qquad\qquad \vdots \\ x_s' = \qquad\qquad\qquad\qquad\qquad a^{-1}x_s \end{array}\right\},$$

$$\left.\begin{array}{l} y_1' = ay_1 - a^2y_2 + \ldots + (-1)^{s-1}a^{s}y_s \\ y_2' = \qquad\quad ay_2 - \ldots + (-1)^{s-2}a^{s-1}y_s \\ \quad \vdots \qquad\qquad \vdots \\ y_s' = \qquad\qquad\qquad\qquad ay_s \end{array}\right\},$$

by the substitution K,

$$\left.\begin{array}{l} x_1' = -a^2y_1 + a^3y_2 - a^4y_3 + a^5y_4 - \ldots + (-1)^s a^{s+1}y_s \\ x_2' = \qquad a^4y_2 - {}^2C_1a^5y_3 + {}^3C_1a^6y_4 - \ldots + (-1)^{s}\,{}^{s-1}C_1a^{s+2}y_s \\ x_3' = \qquad\qquad -a^6y_3 + {}^3C_2a^7y_4 - \ldots + (-1)^{s}\,{}^{s-1}C_2a^{s+3}y_s \\ \quad \vdots \qquad\qquad\qquad \vdots \\ x_s' = \qquad\qquad\qquad\qquad (-1)^s a^{2s}y_s \end{array}\right\},$$

$$\left.\begin{array}{l} y_1' = -a^{-2}x_1 + a^{-3}x_2 - a^{-4}x_3 + a^{-5}x_4 - \ldots + (-1)^s a^{-s-1}x_s \\ y_2' = \qquad a^{-4}x_2 - {}^2C_1a^{-5}x_3 + {}^3C_1a^{-6}x_4 - \ldots + (-1)^{s}\,{}^{s-1}C_1a^{-s-2}x_s \\ y_3' = \qquad\qquad a^{-6}x_3 + {}^3C_2a^{-7}x_4 - \ldots + (-1)^{s}\,{}^{s-1}C_2a^{-s-3}x_s \\ \quad \vdots \qquad\qquad\qquad \vdots \\ y_s' = \qquad\qquad\qquad\qquad (-1)^s a^{-2s}x_s \end{array}\right\},$$

of order 2.

H is the product of two substitutions HK and K of order 2, where HK is

$$\left.\begin{array}{l} x_1' = -a\,y_1 \\ x_2' = a^3y_2-a^4y_3+a^5y_4-a^6y_5+\ldots+(-1)^s\,a^{s+1}y_s \\ x_3' = -a^5y_3+{}^2C_1a^6y_4-{}^3C_1a^7y_5+\ldots+(-1)^s\,{}^{s-2}C_1a^{s+2}y_s \\ x_4' = a^7y_4-{}^3C_2a^8y_5+\ldots+(-1)^s\,{}^{s-2}C_2a^{s+3}y_s \\ \quad\vdots \qquad\qquad\qquad \vdots \\ x_s' = (-1)^s\,a^{2s-1}y_s \end{array}\right\},$$

$$\left.\begin{array}{l} y_1' = -a^{-1}x_1 \\ y_2' = a^{-3}x_2-a^{-4}x_3+a^{-5}x_4-a^{-6}x_5+\ldots+(-1)^s\,a^{-s-1}x_s \\ y_3' = -a^{-5}x_3+{}^2C_1a^{-6}x_4-{}^3C_1a^{-7}x_5+\ldots+(-1)^s\,{}^{s-2}C_1a^{-s-2}x_s \\ y_4' = a^{-7}x_4-{}^3C_2a^{-8}x_5+\ldots+(-1)^s\,{}^{s-2}C_2a^{-s-3}x_s \\ \quad\vdots \qquad\qquad\qquad \vdots \\ y_s' = (-1)^s\,a^{-2s+1}x_s \end{array}\right\}.$$

Both K and HK have s invariant-factors $(\lambda-1)$ and s invariant-factors $(\lambda+1)$.

Similarly, when $a^2=1$, the substitution L

$$x_1' = ax_1+x_2, \quad \ldots, \quad x_{s-1}' = ax_{s-1}+x_s, \quad x_s' = ax_s$$

is transformable into its inverse L^{-1} by the substitution M

$$\left.\begin{array}{l} x_1' = -a^2x_1+a^3x_2-a^4x_3+a^5x_4-\ldots+(-1)^s\,a^{s+1}x_s \\ x_2' = a^4x_2-{}^2C_1a^5x_3+{}^3C_1a^6x_4-\ldots+(-1)^s\,{}^{s-1}C_1a^{s+2}x_s \\ x_3' = -a^6x_3+{}^3C_2a^7x_4-\ldots+(-1)^s\,{}^{s-1}C_2a^{s+3}x_s \\ \quad\vdots \qquad\qquad\qquad \vdots \\ x_s' = (-1)^s\,a^{2s}\,x_s \end{array}\right\}$$

of order 2.

L is the product of two substitutions LM and M of order 2, where LM is

$$\left.\begin{array}{l} x_1' = -a\,x_1 \\ x_2' = a^3x_2-a^4x_3+a^5x_4-a^6x_5+\ldots+(-1)^s\,a^{s+1}x_s \\ x_3' = -a^5x_3+{}^2C_1a^6x_4-{}^3C_1a^7x_5+\ldots+(-1)^s\,{}^{s-2}C_1a^{s+2}x_s \\ x_4' = a^7x_4-{}^3C_2a^8x_5+\ldots+(-1)^s\,{}^{s-2}C_2a^{s+3}x_s \\ \quad\vdots \qquad\qquad\qquad \vdots \\ x_s' = (-1)^s\,a^{2s-1}x_s \end{array}\right\}.$$

When $\alpha = 1$, M and LM have $I\left[\frac{s}{2}\right]$ invariant-factors $(\lambda-1)$

and $I\left[\frac{s+1}{2}\right]$ invariant-factors $(\lambda+1)$;

when $\alpha = -1$, M has $I\left[\frac{s}{2}\right]$ invariant-factors $(\lambda-1)$

and $I\left[\frac{s+1}{2}\right]$ invariant-factors $(\lambda+1)$;

while LM has $I\left[\frac{s+1}{2}\right]$ invariant-factors $(\lambda-1)$

and $I\left[\frac{s}{2}\right]$ invariant-factors $(\lambda+1)$.*

We now have at once the result that:—

If a substitution A transformable into its inverse has invariant-factors $(\lambda-1)^u, \ldots, (\lambda+1)^v, \ldots, (\lambda-a)^w, (\lambda-a^{-1})^w, \ldots,$ *it is the product of two substitutions P and Q of order* 2, *of which Q has*

$$\Sigma I\left[\frac{u}{2}\right]+\Sigma I\left[\frac{v}{2}\right]+\Sigma w$$

invariant-factors $(\lambda\mp 1)$ *and*

$$\Sigma I\left[\frac{u+1}{2}\right]+\Sigma I\left[\frac{v+1}{2}\right]+\Sigma w$$

invariant-factors $(\lambda\pm 1)$, *while P has*

$$\Sigma I\left[\frac{u}{2}\right]+\Sigma I\left[\frac{v+1}{2}\right]+\Sigma w$$

invariant-factors $(\lambda\mp 1)$ *and*

$$\Sigma I\left[\frac{u+1}{2}\right]+\Sigma \left[\frac{v}{2}\right]+\Sigma w$$

invariant-factors $(\lambda\pm 1)$; *and A is transformable into its inverse by either P or Q.*†

This is equivalent to a result obtained by Jackson‡ in a somewhat different manner.

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* See note *, p. 430.

† Taking arbitrarily all the upper signs (which are given by the preceding argument), or all the lower signs which are obtained by replacing P and Q by PS and QS, where S is the substitution changing any variable x_t into $-x_t$.

‡ *Trans. Amer. Math. Soc.*, Vol. x (1909), p. 479.

11. It is of interest to notice that, in § 10, K has as quadratic invariants the quadratic invariants $f_1(x, y)$, $f_2(x, y)$, ..., $f_s(x, y)$ of H obtained in § 8.

This may be verified by direct substitution; but this procedure, though straightforward, is tedious. As an alternative we may proceed as follows.

Assume $f_1(x, y)$, $f_2(x, y)$, ..., $f_s(x, y)$ invariants of the substitution κ,

$$\left.\begin{array}{rl}
(-1)^{t+1}x_1' = & a^{2t+2}y_1 - {}^{t+1}C_t a^{2t+3}y_2 + {}^{t+2}C_t\ a^{2t+4}y_3 - {}^{t+3}C_t\ a^{2t+5}y_4 + \dots \\
(-1)^{t+1}x_2 = & \qquad -a^{2t+4}y_2 + {}^{t+2}C_{t+1}a^{2t+5}y_3 - {}^{t+3}C_{t+1}a^{2t+6}y_4 + \dots \\
(-1)^{t+1}x_3' = & \qquad\qquad a^{2t+6}y_3 - {}^{t+3}C_{t+2}a^{2t+7}y_4 + \dots \\
\vdots & \qquad\qquad \vdots \\
(-1)^{t+1}y_1' = & a^{-2t-2}x_1 - {}^{t+1}C_t a^{-2t-3}x_2 + {}^{t+2}C_t\ a^{-2t-4}x_3 - {}^{t+3}C_t\ a^{-2t-5}x_4 + \dots \\
(-1)^{t+1}y_2' = & \qquad -a^{-2t-4}x_2 + {}^{t+2}C_{t+1}a^{-2t-5}x_3 - {}^{t+3}C_{t+1}a^{-2t-6}x_4 + \dots \\
(-1)^{t+1}y_3' = & \qquad\qquad a^{-2t-6}x_3 - {}^{t+3}C_{t+2}a^{-2t-7}x_4 + \dots \\
\vdots & \qquad\qquad \vdots
\end{array}\right\}$$

for each value of t, and all values of s less than the one considered.

When κ operates on f_1, we obtain an invariant of $\kappa H\kappa^{-1}$, *i.e.*, of H; since H and H^{-1} $(= \kappa H \kappa^{-1})$ have common invariants.

Suppose we obtain $a_1 f_1 + a_2 f_2 + \dots + a_s f_s$.

A consideration of the coefficient of $y_s x_1$ will show that $a_1 = +1$.

If we operate with κ again, we get

$$a_1(a_1 f_1 + a_2 f_2 + \dots + a_s f_s) + a_2 f_2 + \dots + a_s f_s;$$

since, by hypothesis, $f_2, f_3, \dots, f_s$ are invariants of κ. But, since $\kappa^2 = 1$, the result of operating twice on f_1 with κ is f_1.

Hence, since $a_1 = 1$,

$$(f_1 + a_2 f_2 + \dots + a_s f_s) + a_2 f_2 + \dots + a_s f_s \equiv f_1;$$

so that $$a_1 = 1, \quad a_2 = a_3 = \dots = a_s = 0.$$

The result is now at once established by induction. Similarly for the substitution L of § 10.

12. It may now be proved that if A is orthogonal in § 10, P and Q may be taken orthogonal (and therefore symmetric, since they are of order 2).*

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* Proved by Frobenius, *Berliner Sitzungsberichte* (1911), p. 11, in a different manner. He does not obtain the invariant-factors of P and Q.

Consider a simple case.

Take an orthogonal substitution on $x_1, x_2, \ldots, x_{4s}$ with invariant-factors $(\lambda-a)^s$, $(\lambda-a)^s$, $(\lambda-a^{-1})^s$, $(\lambda-a^{-1})^s$, and the quadratic invariant

$$x_1^2+x_2^2+\ldots+x_{4s}^2.$$

It may be transformed into V,

$$\left\{\begin{array}{l} x_1' = ax_1 \;\; +x_2, \;\ldots,\; x_{s-1}' = ax_{s-1} \;\; +x_s,\; x_s' = ax_s\,; \\ \qquad\qquad\qquad\qquad \xi_1' = a\xi_1 \;\; +\xi_2, \;\ldots,\; \xi_{s-1}' = a\xi_{s-1} \;\; +\xi_s,\; \xi_s' = a\xi_s\,; \\ y_1' = a^{-1}y_1+y_2, \;\ldots,\; y_{s-1}' = a^{-1}y_{s-1}+y_s,\; y_s' = a^{-1}y_s\,; \\ \qquad\qquad\qquad\qquad \eta_1 = a^{-1}\eta_1+\eta_2, \;\ldots,\; \eta_{s-1}' = a^{-1}\eta_{s-1}+\eta_s,\; \eta_s' = a^{-1}\eta_s, \end{array}\right.$$

suppose that $x_1^2+x_2^2+\ldots+x_{4s}^2$ becomes

$$\begin{aligned} I \equiv \;& \{c_{11}f_1(x, y)+c_{11}'f_2(x, y)+c_{11}''f_3(x, y)+\ldots\} \\ &+\{c_{12}f_1(x, \eta)+c_{12}'f_2(x, \eta)+c_{12}''f_3(x, \eta)+\ldots\} \\ &+\{c_{21}f_1(\xi, y)+c_{21}'f_2(\xi, y)+c_{21}''f_3(\xi, y)+\ldots\} \\ &+\{c_{22}f_1(\xi, \eta)+c_{22}'f_2(\xi, \eta)+c_{22}''f_3(\xi, \eta)+\ldots\}, \end{aligned}$$

where
$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0.$$

Now it is possible to change the variables so that the canonical substitution V remains unaltered, but I becomes the *symmetric* invariant

$$\begin{aligned} &\{C_{11}f_1(x, y)+C_{11}'f_2(x, y)+C_{11}''f_3(x, y)+\ldots\}+\{C_{12}f_1(x, \eta)+\ldots\} \\ &+\{C_{21}f_1(\xi, y)+\ldots\}+\{C_{22}f_1(\xi, \eta)+\ldots\}, \end{aligned}$$

where
$$C_{12} = C_{21}, \quad C_{12}' = C_{21}', \quad C_{12}'' = C_{21}'', \quad \ldots.$$

By § 11 such a symmetric invariant would be an invariant of F,

$$\left.\begin{array}{ll} x_1' = & -a^2y_1+a^3y_2-\quad a^4y_3+\quad a^5y_4-\ldots \\ x_2' = & \qquad\qquad a^4y_2-{}^2C_1a^5y_3+{}^3C_1a^6y_4-\ldots \\ x_3' = & \qquad\qquad\qquad\qquad -a^6y_3+{}^3C_2a^7y_4-\ldots \\ \;\vdots & \qquad\qquad\qquad\qquad\qquad \vdots \end{array}\right\},$$

$$\left.\begin{array}{ll} y_1' = & -a^{-2}x_1+a^{-3}x_2-\quad a^{-4}x_3+\quad a^{-5}x_4-\ldots \\ y_2' = & \qquad\qquad a^{-4}x_2-{}^2C_1a^{-5}x_3+{}^3C_1a^{-6}x_4-\ldots \\ y_3' = & \qquad\qquad\qquad\qquad -a^{-6}x_3+{}^3C_2a^{-7}x_4-\ldots \\ \;\vdots & \qquad\qquad\qquad\qquad\qquad \vdots \end{array}\right\},$$

$$\left.\begin{aligned}
\xi_1' &= -a^2\eta_1+a^3\eta_2-\quad a^4\eta_3+\quad a^5\eta_4-\dots \\
\xi_2' &= \qquad\qquad a^4\eta_2-{}^2C_1a^5\eta_3+{}^3C_1a^6\eta_4-\dots \\
\xi_3' &= \qquad\qquad\qquad -a^6\eta_3+{}^3C_2a^7\eta_4-\dots \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\eta_1' &= -a^{-2}\xi_1+a^{-3}\xi_2-\quad a^{-4}\xi_3+\quad a^{-5}\xi_4-\dots \\
\eta_2' &= \qquad\qquad a^{-4}\xi_2-{}^2C_1a^{-5}\xi_3+{}^3C_1a^{-6}\xi_4-\dots \\
\eta_3' &= \qquad\qquad\qquad -a^{-6}\xi_3+{}^3C_2a^{-7}\xi_4-\dots \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}\right\}.$$

Hence when we transformed V back again into an orthogonal substitution, F would become orthogonal also; which is what we require.

To change the invariant I into a symmetric form, replace

$$\left\{\begin{aligned}
&x_1 \text{ by } t_{11}x_1+t_{21}\xi_1+s_{11}x_2+s_{21}\xi_2+r_{11}x_3+r_{21}\xi_3+\dots, \\
&\qquad\qquad \xi_1 \text{ by } t_{12}x_1+t_{22}\xi_1+s_{12}x_2+s_{22}\xi_2+r_{12}x_3+r_{22}\xi_3+\dots, \\
&x_2 \text{ by } t_{11}x_2+t_{21}\xi_2+s_{11}x_3+s_{21}\xi_3+r_{11}x_4+r_{21}\xi_4+\dots, \\
&\qquad\qquad \xi_2 \text{ by } t_{12}x_2+t_{22}\xi_2+s_{12}x_3+s_{22}\xi_3+r_{12}x_4+r_{22}\xi_4+\dots, \\
&x_3 \text{ by } t_{11}x_3+t_{21}\xi_3+s_{11}x_4+s_{21}\xi_4+r_{11}x_5+r_{21}\xi_5+\dots, \\
&\qquad\qquad \xi_3 \text{ by } t_{12}x_3+t_{22}\xi_3+s_{12}x_4+s_{22}\xi_4+r_{12}x_5+r_{22}\xi_5+\dots, \\
&\qquad \dots \qquad \dots.
\end{aligned}\right.$$

This will not alter V, for the substitution

$$\left\{\begin{aligned}
&x_1' = t_{11}x_1+t_{21}\xi_1+s_{11}x_2+s_{21}\xi_2+\dots, \quad \xi_1' = t_{12}x_1+t_{22}\xi_1+s_{12}x_2+s_{22}\xi_2+\dots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_1' = y_1, \quad \eta_1' = \eta_1, \\
&\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}\right.$$

is permutable with V.*

The invariant I must become another invariant of V: by consideration of the coefficients of

$$y_sx_1,\, y_sx_2,\, \dots,\, y_sx_s; \quad y_s\xi_1,\, \dots,\, y_s\xi_s; \quad \eta_sx_1,\, \dots,\, \eta_sx_s; \quad \eta_s\xi_1,\, \dots,\, \eta_s\xi_s,$$

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* See Hensel, *Crelle*, 127 (1904), p. 158, or Hilton, *Messenger of Math.*, 41 (1911), p. 110, for a discussion of substitutions permutable with a canonical substitution.

it is not difficult to see that it becomes, in fact,

$$\begin{aligned}&\{\tau_{11}f_1(x,\ y)+(\tau_{11}+a\sigma_{11})f_2(x,\ y)+(\tau''_{11}+a\sigma'_{11}+a^2\rho_{11})f_3(x,\ y)+\ldots\}\\ &+\{\tau_{12}f_1(x,\ \eta)+(\tau'_{12}+a\sigma_{12})f_2(x,\ \eta)+\ldots\}\\ &+\{\tau_{21}f_1(\xi,\ y)+(\tau'_{21}+a\sigma_{21})f_2(\xi,\ y)+\ldots\}\\ &+\{\tau_{22}f_2(\xi,\ \eta)+(\tau'_{22}+a\sigma_{22})f_2(\xi,\ \eta)+\ldots\},\end{aligned}$$

where, if C, C', C'', ..., T, S, ... are the substitutions with matrices

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}, \quad \begin{vmatrix} c'_{11} & c'_{12} \\ c'_{21} & c'_{22} \end{vmatrix}, \quad \begin{vmatrix} c''_{11} & c''_{12} \\ c''_{21} & c''_{22} \end{vmatrix}, \quad \ldots, \quad \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}, \quad \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}, \quad \ldots,$$

then $\begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix}, \quad \begin{vmatrix} \tau'_{11} & \tau'_{12} \\ \tau'_{21} & \tau'_{22} \end{vmatrix}, \quad \begin{vmatrix} \tau'_{11} & \tau''_{12} \\ \tau''_{21} & \tau''_{22} \end{vmatrix}, \quad \ldots, \quad \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix}, \quad \ldots,$

are the matrices of the substitutions CT, $C'T$, $C''T$, ..., CS, ...

Now choose the substitutions T, S, R, ..., so that in turn

$$CT, \quad aCS+C'T, \quad a^2CR+aC'S+C''T, \quad \ldots$$

are symmetric;* and the required change of I into a symmetric invariant is completed. The above process may be generalized.

13. The process of § 3 may be used to find invariants of the form $\Sigma e_{ij}\bar{y}_i x_j$ for the substitution at the beginning of § 3, including the Hermitian invariants in which $e_{ij} = \bar{e}_{ji}$.

A result similar to that at the beginning of § 7 may be established for substitutions with the invariant

$$x_1\bar{x}_1+x_2\bar{x}_2+\ldots+x_k\bar{x}_k-x_{k+1}\bar{x}_{k+1}-\ldots-x_m\bar{x}_m.$$

Results similar to those of § 10 may be established for a substitution transformable into the inverse of its conjugate.

The details of these investigations must be left over for another paper.

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* This is evidently always possible in an infinite number of ways; we may take, for instance, $T = C^{-1}$, ... In fact, we can choose T, S, R, ..., so that

$$C_{ij} = C'_{ij} = C''_{ij} = \ldots = 0 \quad (i \neq j).$$

A METHOD OF ESTABLISHING THE 27-LINE CONFIGURATION ON A CUBIC SURFACE

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1. The present communication gives a method whereby, when a certain homographic correspondence has been established, the configuration and chief fundamental properties of the 27 lines on a cubic surface become almost intuitive. The method was suggested by, and is the extension of, a method published in the *Quarterly Journal*, No. 167, 1911, in which I gave an elementary proof of Salmon's theorem in connexion with the cross-ratio of the four tangents to a cubic curve drawn from a point on itself. Thereafter I used the method to prove the chief focal and bitangent properties of circular cubics and bicircular quartics in two papers published in the *Proceedings of the Edinburgh Mathematical Society*, Session 1910–1911. The present paper exhibits many of the properties of the 27 lines on the cubic surface as extensions of corresponding properties in the plane cubic.

2. If we assume that the existence of one line on the cubic surface has been established, it is easy to see that any plane containing this line cuts the surface besides in a conic and that, for *five* positions of this plane, the conic degenerates into two straight lines. We shall hereafter refer to these five planes as the five tangent planes drawn from this line to the surface.

Let AB be a line on the surface, BC and CA two other lines on the surface meeting AB. Let DAB be one (other than CAB) of the five planes through AB containing degenerate conics of section, and let D be the intersection of the two lines on the surface lying in this latter plane. The cubic surface then assumes the form

$$ax^2t+by^2t+cz^2t+2dt^2z+2fyzt+2gzxt+2hxyt+2kxyz = 0 \tag{1}$$

if we take $ABCD$ as the tetrahedron of reference.

Let $t = \lambda x$ be a plane passing through the line BC and cutting the surface again in the conic S. Then the equation to the cone DS is

$$ax^2+by^2+cz^2+2\left(f+\frac{k}{\lambda}\right)yz+2(g+d\lambda)zx+2hxy = 0. \tag{2}$$

Similarly, if the plane $t = \mu y$ through CA cut the surface again in the conic S', the equation to the cone DS' is

$$ax^2+by^2+cz^2+2(f+d\mu)yz+2\left(g+\frac{k}{\mu}\right)zx+2hxy = 0. \tag{3}$$

Thus the two cones DS and DS' will be identical if

$$\lambda\mu = \frac{k}{d}. \tag{4}$$

Thus, if a plane be drawn through BC and if its conic of section S be joined to D, the second degree cone DS cuts the cubic surface in a curve of the sixth degree consisting of S, the two lines on the surface through D and another conic S′ whose plane passes through CA.

A one-to-one algebraic correspondence exists between the above two planes containing S and S′ respectively in virtue of (4).

Any one ray of the cone DS (other than the lines on the surface through D) is sufficient to identify the plane containing S' and thus to find corresponding members of the homography.

3. If S breaks up into two lines, it is plain that S' will also consist of two straight lines, and thus the four planes through BC containing degenerate conics correspond to the four planes through CA containing degenerate conics. Furthermore, the line joining the point of intersection of the two lines composing S to D plainly passes through the meeting point of the two lines composing S', and a similar property holds whatever point D we choose of the four points of contact of the planes containing degenerate conics which can be drawn through AB. We thus have the following theorems:—

The three groups of four tangent planes (*other than the plane containing the three given lines on the surface*) *that can be drawn from three coplanar lines on the surface have equal cross-ratios.*

The points of contact constitute the vertices of three "desmic tetrahedra."

If a and b be two coplanar lines on the surface, the cross-ratio of the four planes joining a to the points of contact of the four tangent planes from b is equal to the cross-ratio of the four planes joining b to the points of contact of the four tangent planes from a.

For, let α, β be the points of contact of representative tangent planes from a, b respectively. Then the line $\alpha\beta$ passes through one or other of the points of contact of tangent planes from c, where c is the line on the surface coplanar with a and b. Hence, by Art. 2, $a\beta$ and $b\alpha$ are corresponding planes according to *one* of the four homographic systems defined by the points of contact of the four tangent planes from c. The theorem is then evident.

4. The geometrical configuration of the 27 lines, and in particular the intersection diagram of Mr. G. T. Bennett, given in the *Proc. London Math. Soc.*, Series 2, Vol. 9, Part 5, can now be quite easily verified. For example, in terms of the usual notation let a_1, b_2, c_{12} be three coplanar lines of the system, and let a_2, b_1 be two other coplanar lines of the system meeting c_{12}. Consider a_3, a_4, a_5, a_6. The planes b_1a_3, b_1a_4, b_1a_5, b_1a_6 meet the surface again in c_{13}, c_{14}, c_{15}, c_{16} which are thus determined. The lines meeting a_1, complementary to c_{13}, c_{14}, c_{15}, c_{16}, are respectively b_3, b_4, b_5, b_6, which joined to a_2 give c_{23}, c_{24}, c_{25}, c_{26} respectively. Thus all the lines meeting a_1 and b_2 are arranged as far as a_2 and b_1 are concerned. Similarly, we can arrange all the lines meeting a_1 and b_2 with reference to another pair of coplanar lines meeting c_{12}.

It is found, also, that the correspondences of the special planes which give line-pairs occur with double exchanges of these line-pairs. We proceed to tabulate these exchanges. Let the tangent planes from a_1 be α_1, α_2, α_3, α_4, and let their corresponding planes with respect to one of the points of contact of the four tangent planes from b_2 be β_1, β_2, β_3, β_4 as explained in Art. 2. Denoting the four homographies thus obtained by I, II, III, IV respectively, we obtain the following sets of corresponding planes:—

$$[\alpha_1\alpha_2\alpha_3\alpha_4] \equiv [\beta_1\beta_2\beta_3\beta_4], \qquad \text{(I)}$$

$$[\alpha_1\alpha_2\alpha_3\alpha_4] \equiv [\beta_2\beta_1\beta_4\beta_3], \qquad \text{(II}$$

$$[\alpha_1\alpha_2\alpha_3\alpha_4] \equiv [\beta_3\beta_4\beta_1\beta_2], \qquad \text{(III)}$$

$$[\alpha_1\alpha_2\alpha_3\alpha_4] \equiv [\beta_4\beta_3\beta_2\beta_1]. \qquad \text{(IV)}$$

Let α_1 contain the lines b_3, c_{13},
α_2 ,, ,, b_4, c_{14},
α_3 ,, ,, b_5, c_{15},
α_4 ,, ,, b_6, c_{16},
β_1 ,, ,, a_3, c_{23},
β_2 ,, ,, a_4, c_{24},
β_3 ,, ,, a_5, c_{25},
β_4 ,, ,, a_6, c_{26}.

Next, let ab denote the plane containing the lines a, b, and let us rewrite the above correspondences (I), (II), (III), (IV) in terms of the lines lying in the various planes, with the proviso that lines in the same vertical column are coplanar. We thus obtain the following:—

I.

$b_3\ c_{13}(\alpha_1)$	$b_4\ c_{14}(\alpha_2)$	$b_5\ c_{15}(\alpha_3)$	$b_6\ c_{16}(\alpha_4)$
$c_{23}a_3\ (\beta_1)$	$c_{24}a_4\ (\beta_2)$	$c_{25}\ a_5\ (\beta_3)$	$c_{26}a_6\ (\beta_4)$
$a_2\ b_1$	$a_2\ b_1$	$a_2\ b_1$	$a_2\ b_1$.

II.

$b_3\ c_{13}(\alpha_1)$	$b_4\ c_{14}(\alpha_2)$	$c_{15}\ b_5(\alpha_3)$	$c_{16}\ b_6(\alpha_4)$
$a_4\ c_{24}(\beta_2)$	$a_3\ c_{23}(\beta_1)$	$c_{26}a_6(\beta_4)$	$c_{25}a_5(\beta_3)$
$c_{34}c_{56}$	$c_{34}c_{56}$	$c_{34}\ c_{56}$	$c_{34}\ c_{56}$.

III.

$b_3\ c_{13}(\alpha_1)$	$c_{14}\ b_4(\alpha_2)$	$b_5\ c_{15}(\alpha_3)$	$c_{16}\ b_6(\alpha_4)$
$a_5\ c_{25}(\beta_3)$	$c_{26}a_6(\beta_4)$	$a_3\ c_{23}(\beta_1)$	$c_{24}a_4(\beta_2)$
$c_{35}c_{46}$	$c_{35}\ c_{46}$	$c_{35}c_{46}$	$c_{35}\ c_{46}$.

IV.

$b_3\ c_{13}(\alpha_1)$	$c_{14}\ b_4(\alpha_2)$	$c_{15}\ b_5(\alpha_3)$	$b_6\ c_{16}(\alpha_4)$
$a_6\ c_{26}(\beta_4)$	$c_{25}a_5(\beta_3)$	$c_{24}a_4(\beta_2)$	$a_3\ c_{23}(\beta_1)$
$c_{36}c_{45}$	$c_{36}\ c_{45}$	$c_{36}\ c_{45}$	$c_{36}\ c_{45}$.

To explain the above table, we note that the order of the lines b_3, c_{13} in the plane α_1 is kept unaltered throughout. Then, when we make interchanges of the "β" planes (*e.g.*, in passing from I to II, β_1 and β_2, β_3 and β_4), we interchange the order of the lines in each of α_3 and α_4, and also in each of β_1 and β_2. Thus we obtain II from I. Again, in passing from II to III, we interchange β_1 and β_4, β_2 and β_3. We therefore interchange the lines in α_2 and α_3, β_1 and β_4, and so on.

5. Some of the other theorems described in Mr. Bennett's paper also follow at once. Thus with the relation $\lambda\mu = \text{const.}$, zero and infinity are a pair of corresponding values, and hence the plane BCA through BC corresponds to CAD through CA, and BCD through BC corresponds to CAB through CA. Interpreting this result in the notation of the double-six, we obtain

$$a_1[b_2c_{12},\ b_3c_{13},\ b_4c_{14},\ b_5c_{15},\ b_6c_{16}] \equiv b_2[a_2b_1,\ c_{23}a_3,\ c_{24}a_4,\ c_{25}a_5,\ c_{26}a_6],$$

where b_2c_{12} denotes the intersection of the lines b_2 and c_{12}, etc. In this result is contained Dixon's theorem, viz.,

$$a_1[b_3b_4b_5b_6] = b_2[a_3a_4a_5a_6],$$

where a_1b_3 denotes the plane containing a_1 and b_3, etc. The various perspectives are also evident, *e.g.*, $a_1[b_3b_4b_5b_6]$ joined respectively to $b_2[c_{23}c_{24}c_{25}c_{26}]$ meeting in a_2c_{12}, where a_1b_3 denotes the intersection of a_1 and b_3, etc.

SOME RESULTS CONCERNING THE BEHAVIOUR AT INFINITY OF A REAL AND CONTINUOUS SOLUTION OF AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

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I.

1. The results obtained in this paper have reference to the algebraic differential equation

$$(1) \qquad f(x, y, y') \equiv \Sigma A x^m y^n y'^p = 0,$$

where m, n, p are positive integers. I suppose that this equation possesses a solution

$$(2) \qquad y = y(x),$$

which is real and possesses a continuous derivative for $x > x_0$.[*] The problem is to specify as completely as possible the various ways in which y may behave as $x \rightarrow \infty$.

This problem was first attacked by Borel, in his *Mémoire sur les Séries Divergentes*.[†] Borel proved that the equation (1) cannot have a solution y, such that

$$y > e^{e^x} = e_2(x)$$

for values of x surpassing all limit. He proved further that

$$(3) \qquad f(x, y, y', y'') = 0$$

cannot have a solution y such that

$$y > e_3(x)$$

for values of x surpassing all limit;[‡] and there is no doubt of the truth of

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[*] *I.e.*, for all values of x from some value onwards ("Orders of Infinity," *Camb. Math. Tracts*, No. 12, p. 6). We assume the existence of such a solution: it is not part of the problem to consider conditions for its existence.

[†] *Annales de l'École Normale*, t. 16, pp. 26 *et seq.*

[‡] The proof is not complete, but its general lines are clearly indicated.

the corresponding general theorem, though, so far as I am aware, no strict proof has ever been given.

Borel also devoted a section of his memoir to the subject of *oscillating* solutions, but without obtaining any very definite results.

2. In a short memoir published in 1899, Lindelöf[*] returned to the questions raised by Borel, and proved the following much more precise result:

If the equation (1) *is of degree m in x, then there is a constant C, such that*

$$y < e^{Cx^{m+1}} \tag{4}$$

for $x > x_0$.

Further, he proved that[†] either

$$|y| < e^{x^{\delta}},\text{[‡]} \tag{5}$$

or

$$e^{x^{\rho-\delta}} < |y| < e^{x^{\rho+\delta}} \quad (\rho > 0), \tag{6}$$

for $x > x_0$.

The solutions of the first class may oscillate, but those of the second are ultimately monotonic, together with all their derivatives.

3. The subject of the increase (*croissance*) of solutions of the equation (1) has also been considered by Boutroux.[§]

Boutroux confines himself to the equation

$$y' = P(x, y)/Q(x, y), \tag{7}$$

where P and Q are polynomials; but he considers the whole subject from the point of view of the theory of functions of a complex variable. The distinction between the two classes of solutions (5) and (6) of course appears again, in a more precise form—there are solutions whose increase is less than that of some power of $|x|$, and solutions which, in certain angles, behave like exponentials.

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[*] *Bulletin de la Société Mathématique de France*, t. 17, p. 205

[†] Some of these results are contained in an additional note which is in part due to Borel.

[‡] The notation is that explained in my tract cited above and my paper "Properties of Logarithmico-Exponential Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 54.

[§] *Leçons sur les fonctions définies par les équations différentielles du premier ordre*, Paris, Gauthier-Villars, 1908.

4. In this paper I consider first the equation (7); but, like Borel and Lindelöf, I consider it exclusively from the point of view of the real variable. I am thus able to obtain results very much more precise than those stated in the preceding sections. I show that *all* solutions of (7) are ultimately monotonic, and specify their possible modes of increase by simple asymptotic formulæ. I also show that substantially the same results hold for the equation

$$y'^{\mu} = P(x, y)/Q(x, y), \tag{8}$$

where μ is *odd.*

I then return to the general equation (1). I find asymptotic formulæ, more precise than Lindelöf's, for the solutions which behave at infinity like exponentials, and I prove that any oscillating solution is ultimately less in absolute value than a power of x—in symbols,

$$y = O(x^{\Delta}).$$

In particular I show that, in the case of the equation (8), with μ *even*, every oscillating solution remains finite, *i.e.*,

$$y = O(1).$$

Finally, I discuss certain particular types of oscillating solutions.

Much of the argument is capable of extension, and results still more accurate may be obtained without the intervention of any fresh difficulty of principle. But, after a certain point, the work becomes too tedious to be justified by the interest of the results.

It would, however, be exceedingly interesting to see how far the methods used in the paper will go in proving the analogous results immediately suggested for equations of order higher than the first. Here I do not go beyond the first order, but I hope to return to the subject at a later opportunity.

II.

5. Let us consider the equation

$$y' = P(x, y)/Q(x, y). \tag{7}$$

I shall prove first that it is impossible that y' should vanish for a series of values of x whose limit is infinity, except of course in the trivial case in which (7) has a solution $y = \text{const.}$ In other words, *every solution is ultimately monotonic.*

Suppose the contrary. Then the curves $y = y(x)$, $P = 0$ intersect at points corresponding to an infinity of values of x surpassing all limit.

But $P = 0$ consists of a finite number of branches, and so $y = y(x)$ must intersect at least one of these infinitely often.

Now the branches of $P = 0$, which extend to infinity in the direction of the axis of x consist of (i) a finite number of straight lines

$$y = c_s \quad (\gamma_s),$$

(ii) a finite number of branches

$$y = \phi_t(x) \quad (\delta_t),$$

along which y ultimately increases or decreases steadily.

In the first place, *$y = y(x)$ cannot cut any δ_t in an infinity of points.* Suppose, for example, that y ultimately increases along δ_t, and let P, Q be two successive points of intersection. Then $y = y(x)$ crosses δ_t at P and Q, in each case from above to below (Fig. 1), and a glance at the figure is enough to show that this is impossible.*

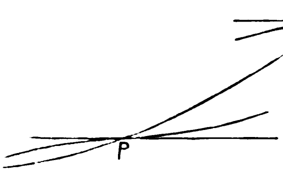

FIG. 1.

We have now to consider the possible intersections of $y = y(x)$ and γ_s. These fall under the four types represented in Fig. 2.

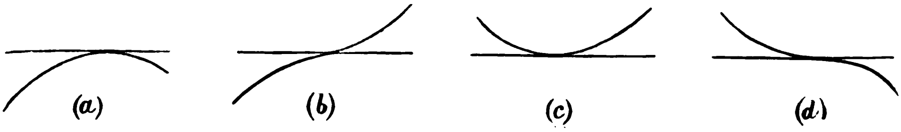

FIG. 2.

Of these we can at once rule out (a) and (c), since at such points y' would change its sign, and P would not. For a similar reason we can rule out (b) and (d), *unless the factor $y - c_s$ occurs an even number of times in P.* If this is so, and an intersection of (*e.g.*) type (b) occurs, it can occur *once only*, so far as γ_s is concerned ; for when y has once passed above γ_s it can plainly only come back to γ_s after an intersection of type (a) with some other γ_s.

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* We can suppose x large enough to ensure that P and Q cannot vanish simultaneously. Then it is easy to see that y is regular for a value of x which makes $P = 0$, and hence that there cannot be an infinity of intersections for values of x in the neighbourhood of any finite value. Hence there must be *successive* intersections. We need not elaborate this kind of point in future.

Hence y is ultimately monotonic.

6. We can go further and say that every derivative $y^{(r)}$ is ultimately monotonic. For, by differentiation and substitution, we find

$$y^{(r)} = P_r(x, y)/Q_r(x, y),$$

where P_r, Q_r are polynomials, and in fact $Q_r = Q^{2r+1}$. Our assertion will therefore follow as a corollary from the following general theorem :—

THEOREM.—*Any rational function*

$$H(x, y) = K(x, y)/L(x, y)$$

is ultimately monotonic along the curve $y = y(x)$*—unless* $L = 0$ *is a solution of the equation* (7).

This theorem I shall now proceed to prove.

7. We have

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + R\frac{\partial H}{\partial y} = T = \frac{U}{W}, \tag{8}$$

where U and W are polynomials, and d/dx implies differentiation along the curve (2). If dH/dx is not ultimately of constant sign on the curve (2), it must vanish or become infinite infinitely often on (2). In the first case (2) must have an infinity of intersections with at least one of the finite number of branches of

$$U = 0. \tag{9}$$

Now this branch may, for sufficiently large values of x, be represented in the form

$$y = A_0x^{\alpha_0} + A_1x^{\alpha_1} + \dots, \tag{9'}$$

a convergent series of (not generally integral) descending powers of x. If $\delta/\delta x$ refers to differentiation along (9),

$$\frac{\delta y}{\delta x} = A_0\alpha_0x^{\alpha_1-1} + A_1\alpha_1x^{\alpha_1-1} + \dots. \tag{10}$$

Again, along (9), $R(x, y)$ is an algebraic function of x, which may, for sufficiently large values of x, be expressed in the form

$$R = B_0x^{\beta_0} + B_1x^{\beta_1} + \dots, \tag{11}$$

another series of descending powers. And, unless the series (10), (11) are identical, we shall have, at all points of (9) from some definite point onwards,

$$\frac{\delta y}{\delta x} > R \quad \text{or} \quad \frac{\delta y}{\delta x} < R.$$

From this it follows that, at the points of intersection, (2) always crosses (9) from one and the same side to the other and the same side; which is plainly impossible.

On the other hand, if the series (10) and (11) are identical, we have

$$\frac{\delta y}{\delta x} = R,$$

and $U = 0$ is a solution of (7). In other words, H is constant along (2).

There remains only the possibility that

$$\frac{dH}{dx} = \left(L\frac{dK}{dx} - K\frac{dL}{dx}\right)\Big/L^2$$

should become infinite infinitely often, as we describe (2). This cannot be true owing to K or L or

$$\frac{dK}{dx} = \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\frac{dy}{dx}$$

or dL/dx becoming infinite, and so can only occur if L vanishes infinitely often. But then we can show as above that $L = 0$ is a solution of the equation (7).

Thus the proof of the theorem is completed.

COROLLARY.—*Any rational function*

$$H(x, y, y')$$

is ultimately monotonic, unless its denominator vanishes identically in virtue of (7).

The same is true of $H(x, y, y', y'', \ldots)$.

8. We can obtain much more accurate information concerning the increase of the solutions of

$$(7) \qquad Qy' = P.$$

The ratio of any two terms is of one of the forms

$$Ax^m y^n, \quad Ax^m y^n y';$$

and is consequently ultimately monotonic, and so, between any two terms X_i, X_j, there subsists one of the relations

$$X_i \succ X_j, \qquad X_i \asymp X_j, \qquad X_i \prec X_j.$$

It follows that there must be one pair of terms at any rate such that

$$X_i \asymp X_j.$$

If these two terms come from the same side of (7), we obtain at once

$$y \sim Ax^s, \tag{12}$$

where s is *rational*. If they come from opposite sides, we obtain a relation of the form

$$y^m y' \sim Ax^n. \tag{13}$$

Here four cases present themselves. If $m \neq -1$, $n \neq -1$, we obtain a relation of the type (12). If $m \neq -1$, $n = -1$, we obtain a relation

$$y \sim A(\log x)^{1/p}, \tag{14}$$

where p is an integer. If $m = -1$, $n \neq -1$, we obtain a relation

$$\log y \sim Ax^p,$$

$$y = e^{Ax^p(1+\epsilon)}. \tag{15}$$

Here p may be supposed a positive integer, as if p is negative $y \sim 1$.* Finally, if $m = -1$, $n = -1$, we obtain

$$\log y \sim A \log x,$$

$$y = x^{A+\epsilon}. \tag{16}$$

9. The relations (15), (16) are less precise than (12) and (14). We shall now proceed to examine them more closely.

Let us consider first the exponential solutions (15). We have

$$y' = \frac{P_0 y^r + P_1 y^{r-1} + \dots}{Q_0 y^s + Q_1 y^{s-1} + \dots},$$

where $P_0, \dots, Q_0, \dots$ are polynomials in x. It is clear that $s = r-1$, and that, for sufficiently large values of x, we have

$$y' = R_0 y + R_1 + O(x^{-\Delta}),$$

$$y'/y = R_0 + O(x^{-\Delta}),$$

---

* p is clearly at most equal to $r+1$, where r is the degree of (1) in x—this, of course, agrees with Lindelöf's result quoted in § 2.

where R_0, R_1 are rational functions of x. Hence

$$y'/y = \Pi(x) + \frac{A}{x} + O\left(\frac{1}{x^2}\right),$$

where Π is a polynomial and A a constant (not necessarily rational). Hence, integrating, we deduce

(17) $$y \sim Ax^{a}e^{\Pi(x)}.$$ [*]

It is clear that this form includes the form (12).

10. We have now to consider the last case of § 8, which is rather more difficult. There are two terms

(18) $$\lambda x^s y^t y', \quad \mu x^{s-1} y^{t+1}$$

of equal order :[†] obviously we may suppose that no other term is of greater order. We may go further, and suppose that no other term is of equal order, since the contrary assumption leads at once to a relation of the type (12). We have also

$$y = x^{A+\epsilon}, \quad A = \mu/\lambda.$$

If follows from the theorem of §§ 6, 7 that, if X_i is any third term in the differential equation, the quotient

$$(\lambda x^s y^t y' - \mu x^{s-1} y^{t+1}) / X_i$$

tends to a limit as $x \to \infty$. In other words, the difference of the two principal terms is definitely of order greater than, equal to, or less than that of any third term. We can now distinguish two possibilities.

(*a*) *There is a third term whose order is equal to that of the difference of the principal terms.*

In this case we have a relation of one of the forms

(19) $$\lambda x^s y^t y' - \mu x^{s-1} y^{t+1} \sim Mx^{\sigma} y^{\tau},$$

(20) $$\lambda x^s y^t y' - \mu x^{s-1} y^{t+1} \sim Mx^{\sigma} y^{\tau} y'.$$

First, suppose (19) holds. Putting

$$y = x^A u = x^{\mu/\lambda} u,$$

---

[*] Not, of course, with the *same* A : cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 54.

[†] We say, of course, that X_i is of order greater than, equal to, or less than that of X_j, according as $X \succ X_j$, $X_i \asymp X_j$, or $X_i \prec X$.

and substituting, we obtain

$$u^{t-\tau}u' \sim Nx^{\sigma-s+(\tau-t-1)A}.$$

But, as $u = x^{\epsilon}$, this is only possible if

$$\sigma-s+(\tau-t-1)A = -1,$$

which shows that A is *rational*, its denominator being $\tau-t-1$. Also, integrating,

$$u^{t-\tau+1} \sim N\log x.$$

Thus

$$y \sim A\,(x^{p}\log x)^{1/q}, \tag{21}$$

where p and q are integers.

Next, suppose (20) holds. Making the same substitution, we obtain

$$\{\lambda x^{s+(t+1)A}u^{t}-Mx^{\sigma+(\tau+1)A}u^{\tau}\}\,u' \sim MAx^{\sigma-1+(\tau+1)A}u^{\tau+1}.$$

But $$x^{s}y^{t} \succ x^{\sigma}y^{\tau}, \quad x^{s+tA}u^{t} \succ x^{\sigma+\tau A}u^{\tau},$$

and so $$u^{t-\tau-1}u' \sim Nx^{\sigma-s-1+(\tau-t)A};$$

and the argument may now be completed as before.

(*b*) *There is no third term whose order is equal to that of the difference of the principal terms.*

Let us denote the principal terms by X_1, $\bar{X}_1$. Then there must be at least one term X_2, such that

$$X_2 \succ X_1-\bar{X}_1;$$

and therefore another term $\bar{X}_2$, such that

$$X_2 \sim \bar{X}_2:$$

and we may suppose, as in the case of X_1, $\bar{X}_1$, that these terms come from opposite sides of the equation. We may also suppose that X_2, $\bar{X}_2$ are of higher order than any other terms other than X_1, $\bar{X}_1$. Further, we may suppose them to be of the form

$$\lambda_2 x^{s_2}y^{t_2}y', \qquad \mu_2 x^{s_2-1}y^{t_2+1},\text{*}$$

where $$\mu_2/\lambda_2 = \mu/\lambda = A.$$

Putting $y = x^{A}u$, we obtain

$$X_1-\bar{X}_1 = x^{s+(t+1)A}u^{t}u', \qquad X_2-\bar{X}_2 = x^{s_2+(t_2+1)A}u^{t_2}u';$$

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* If they were not thus related, the increase of y could be determined at once as in §§ 8, 9.

from which it at once follows that

$$X_1 - \bar{X}_1 \succ X_2 - \bar{X}_2.$$

But in this case there must be a fifth term X_3, whose order is greater than that of $X_1 - \bar{X}_1$, and a sixth term $\bar{X}_3$, such that

$$X_3 \sim \bar{X}_3,$$

and we can prove that

$$X_1 \succ X_2 \succ X_3, \qquad X_1 - \bar{X}_1 \succ X_2 - \bar{X}_2 \succ X_3 - \bar{X}_3.$$

And as this argument may be repeated indefinitely, and the number of terms is finite, we must find sooner or later that the supposition (*b*) leads either to the conclusion we desire or to a contradiction.

11. We have thus proved the following theorem :—

Any solution y[*] *of the equation*

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

is ultimately monotonic, together with all its derivatives, and satisfies one or other of the relations

$$y \sim Ax^a e^{\Pi(x)}, \qquad y \sim A(x^p \log x)^{1/q},$$

where $\Pi(x)$ *is a polynomial, and* p, q *are integers.*

These rates of increase are naturally included among the standard asymptotic forms for logarithmico-exponential functions of order 1,[†] of which they are quite special cases.

12. It is natural to attempt to extend our results to the more general equation

$$y'^{\mu} = P(x, y)/Q(x, y). \tag{8}$$

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[*] We are, of course, confining ourselves to continuous solutions : see § 1.

Examples.—The solution of $x^2 y' = (x+1)^2 y$ is

$$y = Ax^2 e^{x-(1/x)} \sim Ax^2 e^x;$$

the solution of $2x(x+1)yy' = xy^2 + (x+1)^2$ is

$$y = \sqrt{\{(x+1)(\log x + A)\}} \sim \sqrt{(x \log x)}.$$

[†] *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 76.

If μ is odd this offers no new difficulties: all our arguments apply, with appropriate modification of detail.*

But if μ is even our results are obviously no longer true. Thus

$$y'^2 = 1-y^2$$

possesses the oscillating solution

$$y = \sin x.$$

I shall now proceed to consider the general equation (1), and the particular equation (8), with μ even, with the especial idea of discovering to what limitations the existence of oscillating solutions is subject.

III.

13. I return now to the general equation (1). We can distinguish various possibilities.

(*a*) It may be possible to find a positive ρ such that

$$y > e^{x^\rho}, \tag{22}$$

for an infinity of values of x surpassing all limit. In this case Lindelöf has shown that this inequality holds for *all* sufficiently large values of x, and that y and all its derivatives are ultimately monotonic.†

In this case any rational function

$$H(x, y, y')$$

is ultimately monotonic. For, if we eliminate y' between

$$H = H(x, y, y'), \quad f(x, y, y') = 0,$$

---

[*] The standard forms of increase are

$$y \sim Ax^a e^{Bx^{p/\mu}+Cx^{(p/\mu)-1}+\dots}, \quad y \sim A(x^p lx)^{1/q}, \quad y \sim A(lx)^{\mu/q}.$$

In the first of these x^a can occur only if p/μ is integral. The form

$$y \sim A(x^p lx)^{\mu/q},$$

can only occur if $\mu = 1$ or $p = 0$.

[†] Lindelöf (*l.c.*) shows that if $y = y(x)$ cuts $y = e^{x^\rho}$ at points whose abscissæ surpass all limit, we can find values of x surpassing all limit for which

$$y' = \rho x^{\rho-1} y, \quad y > e^{x^\rho}.$$

Substituting in (1), we obtain $f_1(x, x^{\rho-1}, y) = 0,$

where f_1 is a polynomial; and it is impossible that f_1 should vanish for an infinity of pairs of values

$$(\xi_i, \eta_i), \quad \eta_i > e^{\xi_i^\rho},$$

unless it vanishes identically.

we obtain an algebraic relation

$$F(x,\ y,\ H) = 0\ ;$$

and so the points at which $\dfrac{dH}{dx} = 0$

lie on an algebraic curve, which plainly contradicts (22).

We can now argue as in § 8. The equation (1) must contain two terms of equal order, and so we deduce

$$A_1 x^{m_1} y^{n_1} y'^{p_1} \sim A_2 x^{m_2} y^{n_2} y'^{p_2},$$

$$y^{\mu} y' \sim A x^{\nu}.$$

Here μ and ν are rational, and μ must plainly be -1. Hence

$$y = e^{Ax^{s}(1+\epsilon)}. \tag{23}$$

It is clear that s can be at most greater by unity than the degree of (1) in x.*

(*b*) It may be possible to find a number K such that

$$y = O(x^{K}).$$

14. It is obvious that (*a*) and (*b*) do not exhaust the *a priori* possibilities. It is our object now to prove that no other case is really possible.

If we are not in Case (*b*), it is possible, however large be Δ, to find values of x such that

$$y > x^{\Delta}.$$

We can therefore choose an increasing sequence (Δ_{ν}), whose limit is infinity, and a corresponding sequence (x_{ν}), such that

$$y(x) > x^{\Delta_{\nu}} \quad (x = x_{\nu}).$$

We shall now construct a curve

$$y = x^{\Delta(x)} = e^{\Delta(x)\log x} = e^{\phi(x)}, \tag{24}$$

passing through the points $(x_{\nu},\ x_{\nu}^{\Delta_{\nu}})$ and satisfying certain conditions.

In the first place, we can suppose $\Delta'(x)$, and *a fortiori* $\phi'(x)$, positive and continuous. And we may suppose $\phi \prec x^{\delta}$, and *a fortiori* $\Delta \prec x^{\delta}$, since otherwise we should find ourselves again in Case (*a*).

---

* We can treat similarly the case in which y is ultimately negative.

Further, since we are at liberty to suppose the increase of the sequence (Δ_ν) as slow as we like, we may suppose that

$$x^{1-\delta}\,\Delta'(x) \to 0$$

for any positive δ.*

Now
$$\phi'(x) = \Delta'(x)\log x + \frac{\Delta(x)}{x}.$$

Hence

$$x\phi' \to \infty, \qquad x^{1-\delta}\phi' \to 0. \tag{25}$$

15. We have

$$y(x) > e^{\phi(x)}, \tag{26}$$

for an infinity of values of x surpassing all limit. We shall now show, by a modification of Lindelöf's argument, that this inequality must hold for all sufficiently large values of x.

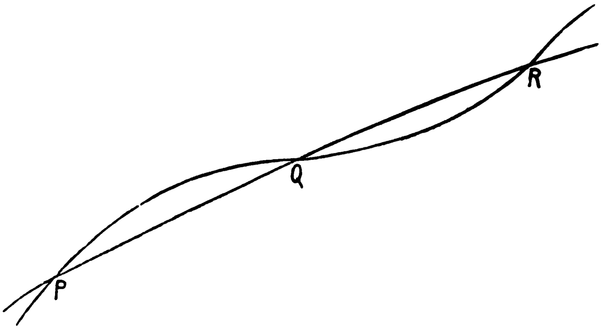

Fig. 3.

If this is not so the curves (2) and (24) must intersect in an infinity of points such as P, Q, R.†

At P (Fig. 3), we have

$$y = e^{\phi}, \qquad y' \geqslant \phi' e^{\phi} = \phi' y,$$

and at Q we have
$$y = e^{\phi}, \qquad y' \leqslant \phi' e^{\phi} = \phi' y.$$

As $y' - \phi' y$ is continuous, there must be a point between P and Q where

$$y' = \phi' y, \qquad y \geqslant e^{\phi},$$

---

* A supposition equivalent, in ordinary cases, to $\Delta(x) \prec x^{\epsilon}$.

† The argument is not affected if some of these points are points of *contact*.

and so there must be values of x surpassing all limit for which these relations hold. For these values of x,

$$f(x, y, \phi' y) = \Sigma A x^m y^{n+p} \phi'^p = 0.$$

All the terms in this equation (except those for which $n = 0$, $p = 0$) are large compared with any power of x,* and it is clear that, for any value of x for which the equation holds, there must be two terms such that

$$H x^m y^{n+p} \phi'^p < x^{m'} y^{n'+p'} \phi'^{p'} < K x^m y^{n+p} \phi'^p,$$

where H and K are numbers depending only on the form of the equation. Further, it is clear that

$$n+p = n'+p',$$

and so we have

$$H_1 < x^{m'-m} \phi'^{p'-p} < K_1,$$

say. But this plainly contradicts the relations (25), unless $m = m'$, $p = p'$, which is impossible.

16. The inequality (26) therefore holds for all sufficiently large values of x. But we can now prove, as in § 13, that any rational function $H(x, y, y')$ is ultimately monotonic, and thus arrive at the equation (23). We have thus proved the following theorem :†—

If y is any solution of the equation

$$f(x, y, y') = 0,$$

we have either

$$y = O(x^{\Delta})$$

or

$$y = e^{Ax^s(1+\epsilon)},$$

where s is rational. All solutions of the latter class are monotonic, together with all their derivatives.

IV.

17. I shall now resume the consideration of the special equation

$$(8) \qquad y'^{\mu} = P/Q,$$

where μ is *even*—if μ is odd, we have already seen that there can be no oscillating solutions.

---

* Since $y \geqslant x^{\Delta(x)}$ and $\Delta(x) \to \infty$.

† It is hardly necessary to point out again that we are considering only continuous solutions.

We have seen that any oscillating solution of the general equation (1)* must satisfy

$$y = O(x^K).$$

When the equation has the special form (8) we can go much further, and assert that any oscillating solution satisfies

$$y = O(1),$$

i.e., oscillates *finitely*.

In fact if, as in § 5, we denote by γ_s, δ_t the branches of $P = 0$ which stretch to infinity in the direction of the axis of x, we can still show, by the argument used there, that $y = y(x)$ cannot cut any δ_t infinitely often. It follows that y cannot (for sufficiently large values of x) increase beyond the greatest of the numbers c_s. For if it did so it would necessarily continue to increase until $y = y(x)$ met one of the branches δ_t. Hence y can oscillate at most finitely.

We can go further, and assert that, along any branch which does not remain finite, $H(x, y)$, any rational function of x and y—and so also any $H(x, y, y')$—is ultimately monotonic. For

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + R^{1/\mu}\frac{\partial H}{\partial y}.$$

Let

$$S = \left(\frac{\partial H}{\partial x}\right)^{\mu} - R\left(\frac{\partial H}{\partial y}\right)^{\mu},$$

and suppose, if possible, that $y = y(x)$ meets a branch of $S = 0$ infinitely often.

Along such a branch we have, as in § 7,

$$y = A_0 x^{a_0} + A_1 x^{a_1} + \dots,$$

$$\frac{\delta y}{\delta x} = A_0 a_0 x^{a_0 - 1} + A_1 a_1 x^{a_1 - 1} + \dots . \dagger \tag{26}$$

Also, along this branch $R = B_0 x^{\beta_0} + B_1 x^{\beta_1} + \dots,$

$$\frac{dy}{dx} = R^{1/\mu} = \pm (C_0 x^{\gamma_0} + C_1 x^{\gamma_1} + \dots), \tag{27}$$

as $R^{1/\mu}$ has two real values, equal and opposite.

We can now prove without difficulty that the assumption of an infinity of intersections leads to a contradiction. Let P, Q, R, ... be successive

---

[*] Such as $y = x \sin x$, which is a solution of

$$(xy' - y)^2 = x^2 (x^2 - y^2).$$

[†] As in § 7, $\delta y / \delta x$ refers to $S = 0$, and dy/dx to $y = y(x)$.

intersections. These must correspond alternately to the two signs in (27). For, if, *e.g.*, P and Q corresponded to the same sign, $y = y(x)$ would cross $S = 0$ in the same sense at P and Q (Fig. 4*a*), which is manifestly impossible. On the contrary hypothesis (Fig. 4*b*) it is clear that we could find a system of values x_ν, tending to infinity, and such that

$$y(x_\nu) \rightarrow \infty, \quad y'(x_\nu) = 0;$$

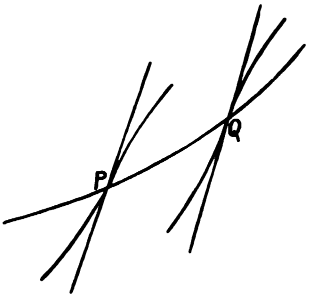

FIG. 4*a*.

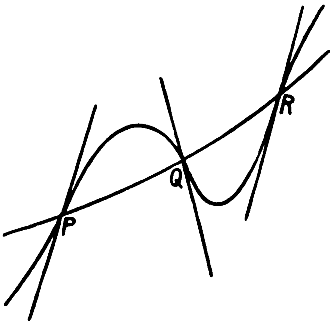

FIG. 4*b*.

and this possibility has already been excluded. Thus it has been shown that, unless y remains finite, $H(x, y)$, $H(x, y, y')$, ... are ultimately monotonic.

We can now show, as in §§ 8, 13, that *any solution of* (8), *which does not remain finite, is determined asymptotically by one or other of the formulæ*

$$y = e^{Ax^{s}(1+\epsilon)}, \ \dots$$

obtained in §§ 11, 12.

18. I shall conclude this paper by considering a few cases in which it is possible to obtain more precise information concerning the oscillating solutions.

First, let us suppose, in the equation (8), that P has no factors

$$(y-c_s)^{k_s},$$

in which k_s is even. Then $y = y(x)$ cannot *cross* a line $y = c_s$, since this would involve a change of sign on the part of P. Thus y remains continually between two adjacent lines $y = c_s$, attaining in succession maxima on the upper line and minima on the lower. In Borel's terminology, *the oscillation of* y *is of a simple and regular sinusoidal type.*

Suppose, in particular, that $\mu = 2$. Then it can be proved that, if y attains the value c_s, $y - c_s$ can occur in P as a simple factor only. For, if

$y = c_s$ for $x = \xi$, we have, near $x = \xi$,

$$y - c_s = A(x-\xi)^p + \dots,$$

$$y'^2 = B(x-\xi)^{2p-2} + \dots,$$

and $2p-2 = pk_s$, which is only possible if $p = 2$, $k_s = 1$.*

We may suppose, without loss of generality, that the lines between which y oscillates are $y-1 = 0$ and $y+1 = 0$. We have then

$$y'^2 = (1-y^2)\, S(x, y),$$

where $S > 0$. Further,

$$S = \frac{P_0 x^m + P_1 x^{m-1} + \dots}{Q_0 x^n + Q_1 x^{n-1} + \dots},$$

where $P_0, P_1, \dots, Q_0, \dots$ are polynomials in y. Suppose, to avoid complications of detail, that $P_0 > 0$, $Q_0 > 0$ for $-1 \leqslant y \leqslant 1$.† Then

$$S = R_0 x^{m-n} + O(x^{m-n-1}),$$

where R_0 is a rational function of y. Putting $y = \sin\theta$, we obtain

$$\theta'^2 = R_0(\sin\theta)\, x^{\frac{1}{2}(m-n)}(1+\epsilon),$$

or

$$\int \frac{d\theta}{\sqrt{R_0}} \sim Ax^s,$$

say. This involves a relation of the type

$$\theta \sim Bx^s.‡$$

Thus y behaves, to put it roughly, like

$$\sin(Bx^s).§$$

19. When $\mu > 2$, we can, of course, obtain more complicated types of oscillating solutions.

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* Consider, for example, the equation

$$y'^2 = (1-y^2)^2 x.$$

We find as the general solution $y = \pm \tanh(\tfrac{2}{3}x^{\frac{3}{2}} + C)$,
and y never attains the values ± 1. This is an example of a finite non-oscillating solution.

† P_0/Q_0 cannot change its sign, but P_0 or Q_0 might vanish, when we should have to take account of the other terms.

‡ If $\theta = 2\nu\pi + \phi$, where $0 < \phi < 2\pi$,

$$\int \frac{d\theta}{\sqrt{R_0}} = \nu \int_0^{2\pi} \frac{d\theta}{\sqrt{R_0}} + O(1).$$

§ A simple example of such a solution is provided by the trochoidal curve

$$x^m = \theta - a\cos\theta, \quad y = \sin\theta \quad (|a| < 1),$$

which satisfies the equation $y'^2 = \dfrac{m^2 x^{2m-2}(1-y^2)}{(1+ay)^2}$.

It is easily verified that $y = \sin^3 x$

satisfies $$y'^6 + 27y^2y'^4 + 243y^4y'^2 = 729y^4(1-y^2).$$

This suggests that the equation

$$y'^6 = Ay^4(1-y^2)$$

has an oscillating solution of the type shown in Fig. 5, and it is easy to verify that this is the case. If we take the more general equation

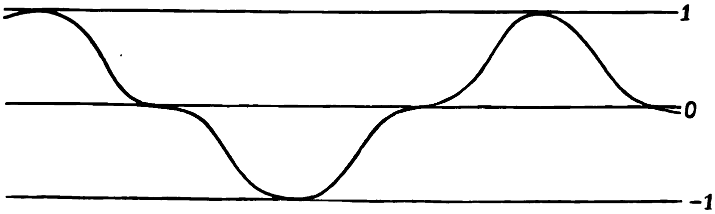

FIG. 5.

$$y'^{\mu} = Ay^a(1-y^2)^b, \quad (a,\ b > 0),$$

we find, as the conditions for the existence of continuous oscillating solutions

$$\frac{a}{\mu} < 1, \quad \frac{b}{\mu} < 1, \quad \frac{a}{\mu} = \frac{2p}{2q+1}, \quad 1 - \frac{2b}{\mu} = \frac{2r}{2s+1},$$

where p, q, r, s are integers. These cannot be satisfied if μ is odd (as is of course to be expected). For even values of μ we find, as possible cases, $\mu = 2$, $a = 0$, $b = 1$; $\mu = 6$, $a = 0$ *or* 4, $b = 1$, 3, *or* 5; and so on. The case mentioned above corresponds to $\mu = 6$, $a = 4$, $b = 1$.

[It is only since writing this paper that I have become acquainted with an important series of memoirs by Kneser and Horn, which deal with a variety of questions concerning the asymptotic behaviour of functions defined by differential equations. These memoirs are for the most part developments of the work of Poincaré on linear differential equations. The point of view adopted is very different from that of Borel and Lindelöf, and so far as I know none of the preceding results are contained in any of them. But the bibliographical indications of §§ 1–4 would be incomplete if I did not refer to them, and I accordingly add the following references, without professing that the list is complete:—

A. Kneser.—*Math. Annalen*, 42, p. 409. *Crelle's Journal*, 116, p. 178; 117, p. 72; 120, p. 267.

J. Horn.—*Math. Annalen*, 49, p. 453; 50, p. 525; 51, p. 346 and p. 360; 52, p. 271 and p. 340. *Crelle's Journal*, 116, p. 265; 117, p. 104 and p. 254; 118, p. 257; 119, p. 196 and p. 267; 120, p. 1.

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SUMMATION OF THE COEFFICIENTS OF SOME TERMINATING HYPERGEOMETRIC SERIES

By W. F. SHEPPARD.

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1. We adopt the notation*

$$a^{[r]} \equiv a(a+1)\dots(a+r-1), \qquad a^{(r)} \equiv a(a-1)\dots(a-r+1),$$
$$a_{[r]} \equiv a^{[r]}/r!, \qquad a_{(r)} \equiv a^{(r)}/r!,$$

so that

$$\left.\begin{aligned} a^{[r]} &= (a+r-1)^{(r)}, & a^{(r)} &= (a-r+1)^{[r]} \\ (-a)^{[r]} &= (-)^r a^{(r)} = (-)^r(a-r+1)^{[r]}, & & \\ & & (-a)^{(r)} &= (-)^r a^{[r]} = (-)^r(a+r-1)^{(r)} \\ a^{[r+s]} &= a^{[r]}(a+r)^{[s]}, & a^{(r+s)} &= a^{(r)}(a-r)^{(s)} \\ a^{[r]}/a^{[s]} &= (a+s)^{[r-s]}, & a^{(r)}/a^{(s)} &= (a-s)^{(r-s)}, \\ (a+r)^{[s]}/a^{[s]} &= (a+s)^{[r]}/a^{[r]}, & (a-r)^{(s)}/a^{(s)} &= (a-s)^{(r)}/a^{(r)} \end{aligned}\right\}. \tag{1}$$

Also we write

$$F\{a, \beta; \rho, \theta; x\} \equiv 1 + \frac{a.\beta}{\rho.\theta}x + \frac{a(a+1).\beta(\beta+1)}{\rho(\rho+1).\theta(\theta+1)}x^2 + \dots,$$

$$F\{a, \beta, \gamma; \rho, \theta, \phi; x\} \equiv 1 + \frac{a.\beta.\gamma}{\rho.\theta.\phi}x + \frac{a(a+1).\beta(\beta+1).\gamma(\gamma+1)}{\rho(\rho+1).\theta(\theta+1).\phi(\phi+1)}x^2 + \dots,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots;$$

and we denote the sum of the first $r+1$ terms of the series $F\{\dots\}$ by $F_r\{\dots\}$.

The general term of the generalised series, taking $x = 1$, is

$$y_r \equiv \frac{a^{[r]}\beta^{[r]}\gamma^{[r]}\dots(m \text{ factorials})}{\rho^{[r]}\theta^{[r]}\phi^{[r]}\dots(m \text{ factorials})}.$$

The properties of any particular series depend to a considerable extent on the value of

$$D \equiv \{\rho+\theta+\phi+\dots\} - \{a+\beta+\gamma+\dots\}.$$

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* Cf. *Mathematical Gazette*, Vol. iv, p. 328-330.

The value of D can usually be more conveniently obtained from the general term, written in the form

$$y_r = \frac{[\ldots(a+r-1)][\ldots(\beta+r-1)][\ldots(\gamma+r-1)]\ldots}{[\ldots(\rho+r-1)][\ldots(\theta+r-1)]\,[\ldots(\phi+r-1)]\ldots},$$

where $[\ldots(a+r-1)]$ denotes an ascending factorial ending with $a+r-1$; for the addition of factors at the beginning of one of the factorials in each term does not alter the mutual ratios of the terms, but only multiplies or divides the series as a whole by these factors. We have then

$$D = \{(\rho+r-1)+(\theta+r-1)+(\phi+r-1)+\ldots\} \\ -\{(a+r-1)+(\beta+r-1)+(\gamma+r-1)+\ldots\}.$$

We are only concerned with the cases in which $a = -n$, where n is a positive integer, so that the series terminates with the $(n+1)$th term; it being assumed that the denominator is not zero in any of the first $n+1$ terms.

If, in addition, $\rho = 1$, so that $\rho^{[r]} = r!$, then $F\{a, \beta;\, \rho, \theta;\, x\}$ becomes

$$F\{-n, \beta;\; 1, \theta;\; x\} = 1 - n_{(1)}\frac{\beta}{\theta}\,x + n_{(2)}\frac{\beta(\beta+1)}{\theta(\theta+1)}\,x^2 - \ldots,$$

and the sum of the coefficients in the complete series of $n+1$ terms is

$$F_n\{-n, \beta;\; 1, \theta;\; 1\} = \sum_{r=0}^{r=n}(-)^r n_{(r)}\beta^{[r]}/\theta^{[r]};$$

and similarly

$$F_n\{-n, \beta, \gamma;\; 1, \theta, \phi;\; 1\} = \sum_{r=0}^{r=n}(-)^r n_{(r)}\{\beta^{[r]}\gamma^{[r]}\}/\{\theta^{[r]}\phi^{[r]}\},$$

$$F_n\{-n, \beta, \gamma, \delta;\; 1, \theta, \phi, \psi;\; 1\} = \sum_{r=0}^{r=n}(-)^r n_{(r)}\{\beta^{[r]}\gamma^{[r]}\delta^{[r]}\}/\{\theta^{[r]}\phi^{[r]}\psi^{[r]}\}.$$

For this class of cases we have

$$D-2 = \{\theta+\phi+\psi+\ldots\} - \{\beta+\gamma+\delta+\ldots\} + n - 1.$$

Four-Factorial Series.

2. For the ordinary series we know that

$$F_n\{-n, \beta;\; 1, \theta;\; 1\} = (\theta-\beta)^{[n]}/\theta^{[n]}. \tag{2}$$

We can prove this in either of two ways.

(i) The simplest method, for this particular class of cases, is by induction for $n = 1, 2, 3, \ldots$. Suppose that (2) is true for a particular value of n. Then

$$\begin{aligned}
F_{n+1}\{-n-1, \beta;\ 1, \theta;\ 1\} \\
&= \sum_{r=0}^{r=n+1} (-)^r (n+1)_{(r)} \beta^{[r]}/\theta^{[r]} \\
&= 1+\sum_{r=1}^{r=n} (-)^r \{n_{(r)}+n_{(r-1)}\} \beta^{[r]}/\theta^{[r]}+(-)^{n+1}\beta^{[n+1]}/\theta^{[n+1]} \\
&= \sum_{r=0}^{r=n} (-)^r n_{(r)} \beta^{[r]}/\theta^{[r]}-(\beta/\theta) \sum_{r=0}^{r=n} (-)^r n_{(r)} (\beta+1)^{[r]}/(\theta+1)^{[r]} \\
&= (\theta-\beta)^{[n]}/\theta^{[n]}-(\beta/\theta)(\theta-\beta)^{[n]}/(\theta+1)^{[n]} \\
&= \{(\theta+n)-\beta\}\,(\theta-\beta)^{[n]}/\theta^{[n+1]} \\
&= (\theta-\beta)^{[n+1]}/\theta^{[n+1]}.
\end{aligned}$$

But $\qquad F_1\{-1, \beta;\ 1, \theta;\ 1\} = 1-\beta/\theta = (\theta-\beta)/\theta;$

hence the induction holds.

(ii) An alternative method has been given by M. J. M. Hill.* He shows by induction for $s = 1, 2, 3, \ldots$, that

$$F_s\{a, \beta;\ 1, \theta;\ 1\}-\frac{\theta-a-\beta-1}{\theta} F_{s-1}\{a+1, \beta+1;\ 1, \theta+1;\ 1\}$$
$$= \frac{(a+1)^{[s]}(\beta+1)^{[s]}}{1^{[s]}\,\theta^{[s]}}; \tag{3}$$

and therefore, putting $a = -n$, $s = n$,

$$F_n\{-n, \beta;\ 1, \theta;\ 1\} = \frac{\theta-\beta+n-1}{\theta} F_{n-1}\{-(n-1), \beta+1;\ 1, \theta+1;\ 1\}. \tag{4}$$

Replacing n by $n-1, n-2, \ldots, 1$, and aggregating the results, we obtain (2).

For the case of $D = 2$, so that $\theta-a-\beta-1 = 0$, (3) gives

$$F_s\{a, \beta;\ 1, \theta;\ 1\} = \{(a+1)^{[s]}(\beta+1)^{[s]}\}/\{s!\ \theta^{[s]}\}.$$

3. It follows from (2) that $F_n\{-n, \beta;\ 1, \theta;\ 1\}$ is zero if any one of the factors of $(\theta-\beta)^{[n]}$ is zero. Suppose, therefore, that

$$\theta-\beta+r = 0,$$

giving $\qquad D-2 = (n-1)-r,$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 5, p. 335.

where r has any of the values 0, 1, 2, ..., $n-1$. Then

$$0 = F_n\{-n,\ \theta+r;\ 1,\ \theta;\ 1\} = \sum_{s=0}^{s=n} (-)^s n_{(s)} (\theta+r)^{[s]}/\theta^{[s]}$$

$$= \sum_{s=0}^{s=n} (-)^s n_{(s)} (\theta+s)^{[r]}/\theta^{[r]},$$

by (1); and therefore

$$\sum_{s=0}^{s=n} (-)^s n_{(s)} (\theta+s)^{[r]} = 0. \tag{5}$$

But, if $f_r(\theta)$ is any polynomial of degree r in θ, we can express $f_r(\theta)$ in the form

$$f_r(\theta) = a_r\theta^{[r]}+a_{r-1}\theta^{[r-1]}+\ldots+a_1\theta+a_0;$$

and therefore, replacing r in (5) by $r-1$, $r-2$, ..., 0, it follows that

$$\sum_{s=0}^{s=n} (-)^s n_{(s)} f_r(\theta+s) = 0 \quad (r = 0,\ 1,\ 2,\ \ldots,\ n-1). \tag{6}$$

This is only another way of stating the more familiar theorem that

$$\sum_{s=0}^{s=n} (-)^s n_{(s)} s^r = 0 \quad (r = 0,\ 1,\ 2,\ \ldots,\ n-1). \tag{7}$$

4. Now let us see whether we can sum the more general four-factorial series

$$F\{\alpha,\ \beta;\ \rho,\ \theta;\ 1\} \equiv 1+\frac{\alpha.\beta}{\rho.\theta}+\ldots+\frac{\alpha^{[s]}\beta^{[s]}}{\rho^{[s]}\theta^{[s]}}+\ldots,$$

for the case of $\alpha = -n$. Adapting Hill's method, quoted above, let us write

$$f(s) \equiv F_s\{\alpha,\ \beta;\ \rho,\ \theta;\ 1\}$$
$$-\frac{(\rho+p\alpha+p'\beta+q)(\theta+p\beta+p'\alpha+q)}{\rho\theta} F_{s-1}\{\alpha+1,\ \beta+1;\ \rho+1,\ \theta+1;\ 1\},$$

where p, p', and q are constants whose values are to be chosen. Then

$$\rho\theta.f(1) = \rho\theta+\alpha\beta-(\rho+p\alpha+p'\beta+q)(\theta+p\beta+p'\alpha+q).$$

Replacing ρ by $D+\alpha+\beta-\theta$, we find that, in order that $f(1)$ may contain $\alpha+1$ and $\beta+1$ as factors, we must have

$$p' = p, \quad q = -1, \quad D = 2,$$

and either $$p = 0 \quad \text{or} \quad p = -1;$$

but the two cases are really identical, since $D = 2$ gives

$$\rho-\alpha-\beta-1 = -(\theta-1), \qquad \theta-\alpha-\beta-1 = -(\rho-1).$$

Taking $p = 0$, we can show by induction that, for this value of D,

$$F_s\{\alpha, \beta;\ \rho, \theta;\ 1\} - \frac{(\rho-1)(\theta-1)}{\rho\theta} F_{s-1}\{\alpha+1, \beta+1;\ \rho+1, \theta+1;\ 1\}$$
$$= \frac{(\alpha+1)^{[s]}(\beta+1)^{[s]}}{\rho^{[s]}\theta^{[s]}} \quad (\rho+\theta = \alpha+\beta+2). \tag{8}$$

This is true whatever the value of α may be. Writing $\alpha = -n$, $s = n$, and replacing β by $\rho+\theta-\alpha-2$, we obtain

$$F_n\{-n, \rho+\theta+n-2;\ \rho, \theta;\ 1\}$$
$$= \frac{(\rho-1)(\theta-1)}{\rho\theta} F_{n-1}\{-n+1, \rho+\theta+n-1;\ \rho+1, \theta+1;\ 1\}. \tag{9}$$

Replacing n by $n-1, n-2, \ldots, 1$, and aggregating the results, we have, finally,

$$F_n\{-n, \rho+\theta+n-2;\ \rho, \theta;\ 1\} = \frac{(\rho-1)^{[n]}(\theta-1)^{[n]}}{\rho^{[n]}\theta^{[n]}}$$
$$= \frac{(\rho-1)(\theta-1)}{(\rho+n-1)(\theta+n-1)}. \tag{10}$$

Six-Factorial Series.

5. Next consider the six-factorial series

$$F\{\alpha, \beta, \gamma;\ 1, \theta, \phi;\ 1\} \equiv 1+\frac{\alpha.\beta.\gamma}{1.\theta.\phi}+\frac{\alpha(\alpha+1).\beta(\beta+1).\gamma(\gamma+1)}{2!\,\theta(\theta+1).\phi(\phi+1)}+\ldots,$$

of which the series considered in § 4 is a particular case. It will be found, as in § 4, that Hill's method can be applied if $D = 2$, *i.e.*, if

$$\gamma = \theta+\phi-\alpha-\beta-1.$$

We have

$$1+\frac{\alpha\beta(\theta+\phi-\alpha-\beta-1)}{\theta\phi}-\frac{(\theta-\alpha-\beta-1)(\phi-\alpha-\beta-1)}{\theta\phi}$$
$$= \frac{(\alpha+1)(\beta+1)(\theta+\phi-\alpha-\beta-1)}{\theta\phi};$$

and thence, by induction for $s = 1, 2, 3, \ldots$, we can show that

$$F_s\{\alpha, \beta, \theta+\phi-\alpha-\beta-1;\ 1, \theta, \phi;\ 1\} - \frac{(\theta-\alpha-\beta-1)(\phi-\alpha-\beta-1)}{\theta\phi}$$
$$\times F_{s-1}\{\alpha+1, \beta+1, \theta+\phi-\alpha-\beta-1;\ 1, \theta+1, \phi+1;\ 1\}$$
$$= \frac{(\alpha+1)^{[s]}(\beta+1)^{[s]}(\theta+\phi-\alpha-\beta-1)^{[s]}}{s!\,\theta^{[s]}\phi^{[s]}}. \tag{11}$$

This is true whatever the value of α may be. Putting $\alpha = -n$, $s = n$, we obtain

$$F_n\{-n, \beta, \theta+\phi-\beta+n-1;\ 1, \theta, \phi;\ 1\}$$
$$= \frac{\theta-\beta+n-1}{\theta}\,\frac{\phi-\beta+n-1}{\phi}$$
$$\times F_{n-1}\{-n+1, \beta+1, \theta+\phi-\beta+n-1;\ 1, \theta+1, \phi+1;\ 1\}. \tag{12}$$

Replacing n by $n-1$, $n-2$, ..., 1, and aggregating the results, we have

$$F_n\{-n, \beta, \theta+\phi-\beta+n-1;\ 1, \theta, \phi;\ 1\}$$
$$= \frac{(\theta-\beta+n-1)^{(n)}}{\theta^{[n]}}\,\frac{(\phi-\beta+n-1)^{(n}}{\phi^{[n]}} = \frac{(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}}{\theta^{[n]}\phi^{[n]}}, \tag{13}$$

We ought also to have, by symmetry,

$$F_n\{-n, \beta, \theta+\phi-\beta+n-1;\ 1, \theta, \phi;\ 1\} = \frac{(\theta-\gamma)^{[n]}(\phi-\gamma)^{[n]}}{\theta^{[n]}\phi^{[n]}}, \tag{13a}$$

where $\gamma \equiv \theta+\phi-\beta+n-1$. This is easily verified; for

$$\theta-\gamma = -(\phi-\beta+n-1), \qquad \phi-\gamma = -(\theta-\beta+n-1),$$
$$(\theta-\gamma)^{[n]} = (-)^n(\phi-\beta+n-1)^{(n)} = (-)^n(\phi-\beta)^{[n]},$$
$$(\phi-\gamma)^{[n]} = (-)^n(\theta-\beta+n-1)^{(n)} = (-)^n(\theta-\beta)^{[n]}.$$

6. By putting $\beta = 1$ in (13), we obtain (10). The following are two other special cases.

(i) Let $\phi = \theta$. Then

$$F_n\{-n, \beta, 2\theta-\beta+n-1;\ 1, \theta, \theta;\ 1\} = \left\{\frac{(\theta-\beta)^{[n]}}{\theta^{[n]}}\right\}^2$$
$$= [F_n\{-n, \beta;\ 1, \theta;\ 1\}]^2. \tag{14}$$

(ii) Let one of the two numbers β and γ in the numerator of the second term be equal to the difference of the two θ and ϕ in the denominator; *e.g.*, let $\theta+\phi-\beta+n-1=\phi-\theta$, so that $\beta=2\theta+n-1$. Then

$$F_n\{-n,\ 2\theta+n-1,\ \phi-\theta;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=\frac{2\theta-\phi}{\phi}F_{n-1}\{-n+1,\ 2\theta+n,\ \phi-\theta;\ 1,\ \theta+1,\ \phi+1;\ 1\}$$

$$=\ldots$$

$$=\frac{(2\theta-\phi)^{[n]}}{\phi^{[n]}}. \qquad (15)$$

7. Having obtained an expression for the sum of the coefficients in the six-factorial series for the case of $D=2$, we can obtain an expression for their sum in the case where D is any positive integer greater than 2. For, if $D=2+k$, where k is a positive integer, and if y_s is the $(s-1)$th term of the series (for $x=1$), we can replace y_s by $y_s\Psi(s)$, where $\Psi(s)$ is the sum of $k+1$ terms, each consisting of k factors so chosen as to extend the factorials in the numerator of y_s or reduce the factorials in the denominator, the coefficients of the terms being chosen so as to make $\Psi(s)=1$; and we shall thus replace the series by the sum of $k+1$ series, for each of which $D=2$. This can usually be done in several different ways; but there will be three typical forms, obtained by altering (i) two factorials in the numerator, (ii) two factorials in the denominator, and (iii) one factorial in the numerator and one in the denominator, respectively.

Suppose, for instance, that $D=3$. Then we can use any one of the three relations

$$1=\{(\beta+s)-(\gamma+s)\}/(\beta-\gamma)$$

$$=\{(\theta+s-1)-(\phi+s-1)\}/(\theta-\phi)$$

$$=\{(\beta+s)-(\phi+s-1)\}/(\beta-\phi+1).$$

If we use the first, we obtain

$$\frac{\beta^{[s]}\gamma^{[s]}}{\theta^{[s]}\phi^{[s]}}=\frac{\beta}{\beta-\gamma}\,\frac{(\beta+1)^{[s]}\gamma^{[s]}}{\theta^{[s]}\phi^{[s]}}-\frac{\gamma}{\beta-\gamma}\,\frac{\beta^{[s]}(\gamma+1)^{[s]}}{\theta^{[s]}\phi^{[s]}};$$

this gives the general formula

$$(\beta-\gamma)F_s\{a,\ \beta,\ \gamma;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=\beta F_s\{a,\ \beta+1,\ \gamma;\ 1,\ \theta,\ \phi;\ 1\}-\gamma F_s\{a,\ \beta,\ \gamma+1;\ 1,\ \theta,\ \phi;\ 1\};$$

and thence it will be found that

$$F_n\{-n,\ \beta,\ \theta+\phi-\beta+n-2;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=\frac{(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}+n\beta(\theta-\beta)^{[n-1]}(\phi-\beta)^{[n-1]}}{\theta^{[n]}\phi^{[n]}}$$

$$=\frac{(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}}{\theta^{[n]}\phi^{[n]}}\left\{1+\frac{n\beta}{(\theta-\beta+n-1)(\phi-\beta+n-1)}\right\}. \quad (16)$$

Similarly for $D=4$ it may be shown that

$$F_n\{-n,\ \beta,\ \theta+\phi-\beta+n-3;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=\frac{(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}}{\theta^{[n]}\phi^{[n]}}\left\{1+\frac{2n\beta}{(\theta-\beta+n-1)(\phi-\beta+n-1)}\right.$$

$$\left.+\frac{n(n-1)\beta(\beta+1)}{(\theta-\beta+n-1)(\theta-\beta+n-2)(\phi-\beta+n-1)(\phi-\beta+n-2)}\right\}. \quad (17)$$

8. These last results suggest that the formula for the case of $D=2+k$, where k is a positive integer, is

$$F_n\{-n,\ \beta,\ \theta+\phi-\beta+n-k-1;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=\frac{(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}}{\theta^{[n]}\phi^{[n]}}F_k\{-n,\beta,-k;1,-\theta+\beta-n+1,-\phi+\beta-n+1;1\}.$$

This, however, is only a particular case of a more general formula; for we can replace $F_k\{\ldots\}$ on the right-hand side by $F_n\{\ldots\}$, and we then have a formula which, if it is true for all positive integral values of k, must be true universally. Replacing $\theta+\phi-\beta+n-k-1$ by γ, the formula becomes

$$\theta^{[n]}\phi^{[n]}F_n\{-n,\ \beta,\ \gamma;\ 1,\ \theta,\ \phi;\ 1\}$$

$$=(\theta-\beta)^{[n]}(\phi-\beta)^{[n]}F_n\{-n,\ \beta,\ \beta+\gamma-\theta-\phi-n+1;$$

$$1,\ -\theta+\beta-n+1,\ -\phi+\beta-n+1;\ 1\} \quad (18)$$

$$=(-\theta+\beta-n+1)^{[n]}(-\phi+\beta-n+1)^{[n]}F_n\{-n,\ \beta,\ \beta+\gamma-\theta-\phi-n+1;$$

$$1,\ -\theta+\beta-n+1,\ -\phi+\beta-n+1;\ 1\} \quad (18a)$$

$$=\theta'^{[n]}\phi'^{[n]}F_n\{-n,\ \beta,\ \gamma';\ 1,\ \theta',\ \phi';\ 1\}, \quad (18b)$$

where

$$\left.\begin{aligned}\gamma'&=\beta+\gamma-\theta-\phi-n+1, & \gamma&=\beta+\gamma'-\theta'-\phi'-n+1\\ \theta'&=-\phi+\beta-n+1, & \theta&=-\phi'+\beta-n+1\\ \phi'&=-\theta+\beta-n+1, & \phi&=-\theta'+\beta-n+1\end{aligned}\right\}. \quad (19)$$

To establish this, let us write

$$\Phi(n, \beta, \gamma, \theta, \phi) \equiv \theta^{[n]}\phi^{[n]}F_n\{-n, \beta, \gamma;\ 1, \theta, \phi;\ 1\}$$
$$= \sum_{r=0}^{r=n}(-)^r n_{(r)}\beta^{[r]}\gamma^{[r]}(\theta+r)^{[n-r]}(\phi+r)^{[n-r]},$$

$$\Psi(n, \beta, \gamma, \theta, \phi) \equiv (\theta-\beta)^{[n]}(\phi-\beta)^{[n]}F_n\{-n, \beta, \beta+\gamma-\theta-\phi-n+1;$$
$$1, -\theta+\beta-n+1, -\phi+\beta-n+1;\ 1\}$$
$$= \sum_{r=0}^{r=n}(-)^r n_{(r)}\beta^{[r]}(\beta+\gamma-\theta-\phi-n+1)^{[r]}$$
$$\times(\theta-\beta)^{[n-r]}(\phi-\beta)^{[n-r]},$$

$$f_n(\gamma) \equiv \Phi(n, \beta, \gamma, \theta, \phi)-\Psi(n, \beta, \gamma, \theta, \phi);$$

then we wish to show that, for any assigned values of β, θ, and ϕ, and for any assigned (positive integral) value of n, $f_n(\gamma)$ is 0 for all values of γ. Suppose this is true for values of n up to $n-1$ inclusive. Then

$$\Phi(n, \beta, \gamma+1, \theta, \phi)-\Phi(n, \beta, \gamma, \theta, \phi)$$
$$= -n\beta\Phi(n-1, \beta+1, \gamma+1, \theta+1, \phi+1),$$

$$\Psi(n, \beta, \gamma+1, \theta, \phi)-\Psi(n, \beta, \gamma, \theta, \phi)$$
$$= -n\beta\Psi(n-1, \beta+1, \gamma+1, \theta+1, \phi+1);$$

and therefore $$f_n(\gamma) = f_n(\gamma+1) = f_n(\gamma+2) = \ldots.$$

Hence $f_n(\gamma)$ is independent of γ. Now write

$$\beta+\gamma-\theta-\phi-n+1 = 0,$$

so that Ψ reduces to its first term; then

$$\gamma = \theta+\phi-\beta+n-1,$$

and we see from (13) that Φ and Ψ are equal, so that $f_n(\gamma) = 0$. Hence the step of the induction holds. But $f_n(\gamma) = 0$ for $n = 0$. Hence the theorem is true universally.

It will be noticed that, if D and D' are the values of D for the two series,

$$D = 2-\gamma', \quad D' = 2-\gamma,$$

so that the formula gives us an approximate value for the sum of the coefficients of a terminating six-factorial series for which D differs only slightly from 2.

9. The method of § 7 can be used to obtain an expression for

$$\sum_{s=0}^{s=n} y_s\chi(s),$$

where $y_s x^s$ is the $(s+1)$th term of a terminating series for which $D = 2+k$, k being a positive integer, and $\chi(s)$ is a polynomial in s of degree not exceeding k; the only difference being that the coefficients of the terms constituting $\Psi(s)$ are to be chosen so as to make $\Psi(s) = \chi(s)$.

Suppose, for instance, that we require the sum of the series

$$F_n\{-n,\ \beta,\ \gamma,\ \epsilon\ ;\ 1,\ \theta,\ \phi,\ \psi\ ;\ 1\}\ ;$$

the constants being subject to the conditions

$$\epsilon = \psi+k, \qquad (1+\theta+\phi+\psi)-(-n+\beta+\gamma+\epsilon) = 2,$$

where k is a positive integer. This is ostensibly an eight-factorial series; but, since

$$(\psi+k)^{[s]}/\psi^{[s]} = (\psi+s)^{[k]}/\psi^{[k]},$$

it is really of the form mentioned above. It will be found that the three typical formulæ are

$$F_n\{-n,\ \beta,\ \gamma,\ \psi+k\ ;\ 1,\ \theta,\ \phi,\ \psi\ ;\ 1\}$$

$$= \sum_{t=0}^{t=k} (-)^t k_{(t)}(\beta-\gamma+k-2t)$$

$$\times \frac{\beta^{[k-t]}\gamma^{[t]}(\psi-\beta)^{[t]}(\psi-\gamma)^{[k-t]}(\theta-\gamma-t)^{[n]}(\phi-\gamma-t)^{[n]}}{(\beta-\gamma-t)^{[k+1]}\psi^{[k]}\theta^{[n]}\phi^{[n]}} \tag{20}$$

$$= \sum_{t=0}^{t=k} (-)^t k_{(t)}(\phi-\theta+k-2t)$$

$$\times \frac{(\phi-\psi-k)^{[k-t]}(\theta-\psi-k)^{[t]}(\theta-\beta-k+t)^{[n]}(\phi-\beta-t)^{[n]}}{(\phi-\theta-t)^{[k+1]}\psi^{[k]}\theta^{[n-k+t]}\phi^{[n-t]}} \tag{21}$$

$$= \sum_{t=0}^{t=k} k_{(t)} \frac{\beta^{[t]}(\theta-\psi-k)^{[t]}(\psi-\beta)^{[k-t]}(\theta-\beta)^{[n-k]}(\phi-\beta-t)^{[n]}}{\psi^{[k]}\theta^{[n-k+t]}\phi^{[n]}}, \tag{22}$$

these being all subject to the condition

$$\theta+\phi = \beta+\gamma+k-n+1.$$

THE SYSTEM OF LINES OF A CUBIC SURFACE

By G. T. BENNETT.

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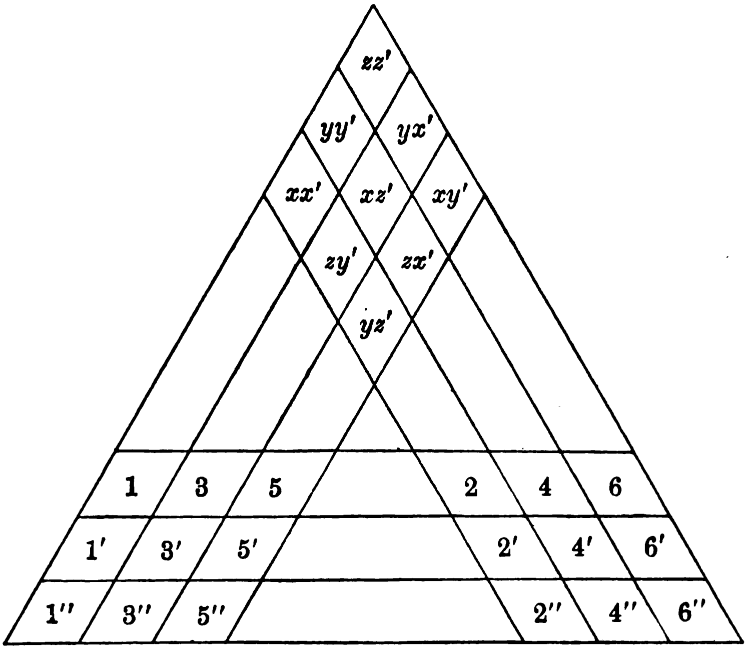
1. In a recent paper* the author gave a form of diagram by which the intersections, 135 in number, of the 27 lines of a cubic may be summarily represented. The diagram, as here repeated,



consists of nine strips arranged in triangular fashion so as to give 27 cells, in which the names of the lines are entered. A set of nine lines appear in each of three "corners" of the diagram. The rule is that intersection occurs for any two lines which are in different corners and in the same strip, or in the same corner and not in the same strip. The object of this supplementary note is to show a method of filling up the diagram by stages in such a way as to build up the configuration of lines in question *ab initio*, without any appeal to the geometry of cubics or quadrics. The property of existence of the double six occurs as an incident of the process.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 349.

The first stage consists in filling the nine cells of the top corner of the diagram with the lines in which any three planes x, y, z are cut by any three other planes x', y', z'. The lines may be regarded either as forming three triangles in planes x, y, z with the point $x'y'z'$ as common centre of perspective, or conversely. The nine lines will be regarded as fixed base lines on which the rest of the figure is to be constructed. The two identical relations among the six planes may be written

$$x+y+z \equiv x'+y'+z', \tag{i}$$

$$ax+by+cz \equiv a'x'+b'y'+c'z', \tag{ii}$$

whence the identity

$$(a+\lambda)\,x+(b+\lambda)\,y+(c+\lambda)z \equiv (a'+\lambda)\,x'+(b'+\lambda)\,y'+(c'+\lambda)z', \tag{iii}$$

true for all values of the parameter λ.

The second stage consists in supplying the first of the horizontal strips with six lines 1, 2, 3, 4, 5, 6. Such a set of lines are furnished by the following table:—

	2	4	6
1	$(a+\lambda)\,x = (a'+\lambda)\,x'$	$(b+\lambda)\,y = (b'+\lambda)\,y'$	$(c+\lambda)\,z = (c'+\lambda)\,z'$
3	$(c+\lambda)\,z = (b'+\lambda)\,y'$	$(a+\lambda)\,x = (c'+\lambda)\,z'$	$(b+\lambda)\,y = (a'+\lambda)\,x'$
5	$(b+\lambda)\,y = (c'+\lambda)\,z'$	$(c+\lambda)\,z = (a'+\lambda)\,x'$	$(a+\lambda)\,x = (b'+\lambda)\,y'$

The three planes whose equations appear in any row of the table are consistent in virtue of the identity (iii) and give a line of intersection, the name of which is put at the end of the row; and similarly for columns. Each of the lines 1, 3, 5 meets each of the lines 2, 4, 6; and each of the lines meets three of the fixed base lines. The conditions necessary to be observed in filling the strip of the diagram are all satisfied. The theorem which arises at this stage may be stated thus:—Given three triangles with a common centre of perspective, there exist a variable set of six lines, of which three meet the other three, such that each of the six lines meets three non-corresponding sides of the three triangles. (The variable lines satisfy 27 conditions with only 24 available parameters.)

The third stage consists in filling a second horizontal strip with six lines 1′, 2′, 3′, 4′, 5′, 6′ obtained by changing the parameter λ into a different value μ. The fresh set of lines are properly related to the nine base-lines of the top corner, but they must also be properly related to the lines 1, 2, 3, 4, 5, 6. The twelve necessary intersections may be summarised as making a closed hexagon of the lines 1, 3′, 5, 1′, 3, 5′ and another closed hexagon of the lines 2, 4′, 6, 2′, 4, 6′. Now consider any one of the intersections, say that of 1 and 3′. The planes

$$(a+\lambda)\,x = (a'+\lambda)\,x', \qquad (b+\lambda)\,y = (b'+\lambda)\,y', \qquad (c+\lambda)\,z = (c'+\lambda)\,z',$$

giving line 1, and the planes

$$(c+\mu)\,z = (b'+\mu)\,y', \qquad (a+\mu)\,x = (c'+\mu)\,z', \qquad (b+\mu)\,y = (a'+\mu)\,x',$$

giving line 3′, must be concurrent. Hence, by multiplication,

$$f(\lambda) = f(\mu),$$

where $$f(\lambda) = \frac{(a+\lambda)(b+\lambda)(c+\lambda)}{(a'+\lambda)(b'+\lambda)(c'+\lambda)} = k, \text{ say};$$

so that μ is one of the other two roots of the cubic equation $f(\lambda) = k$. The same condition gives rise to all the twelve intersections simultaneously. The figure at this stage may be regarded as showing a variable double-six

$$\begin{matrix} 1 & 3 & 5 & 2' & 4' & 6' \\ 1' & 3' & 5' & 2 & 4 & 6 \end{matrix}$$

(the only one in the figure), having each of its six pairs of lines meeting three non-corresponding sides of the three perspective triangles which form the base-lines.

The final stage consists in filling the third horizontal strip with lines 1″, 2″, 3″, 4″, 5″, 6″, got by taking as parameter the third root ν of the cubic equation $f(\lambda) = k$. The last set of six lines are thus properly related to both the preceding sets and also to the base-lines. The construction of the configuration is now complete. If the parameter k is supposed to remain variable, the eighteen lines of the three horizontal strips are left poristically variable. They may be regarded as three double sixes (with each line common to two) resting on the nine base-lines: or they may be regarded as two perspective sets of nine lines, of the same kind as the base-lines of the top corner, with the three corners mutually related.

The cubic surface containing the 27 lines has for its equation

$$x'y'z' = kxyz.$$

It occurs here only as a final corollary to the elementary construction made with points, planes, and lines.

2. Occasion may be taken here to supply an omission in § 11 of the paper already quoted. In exhibiting sets of points such as $b_1(a_3a_4a_5a_6)$ and $a_2(c_{23}c_{24}c_{25}c_{26})$ as being in perspective, the centre of perspective should be noted as $c_{12}b_2$. The line of collinearity of the points b_1a_3, a_2c_{23} and $c_{12}b_2$ is immediately apparent as the line of intersection of the two triple tangent planes $b_1a_2c_{12}$ and $a_3c_{23}b_2$, and similarly for the remaining pairs of rays. The whole system of such perspectives on the sides of any one triangle of lines of the cubic is thus comprehended in the figure of lines in which the plane is cut by the remaining 44 triple tangent planes: namely, each side of the triangle counted four times, and 32 lines concurrent, by sets of four, in eight points on each side.

It may be worth stating a means of generating such a plane figure. The triangle being XYZ, let an involution of points containing the pair Y and Z have A and A', B and B', C and C', D and D', for other pairs. Starting from an arbitrary point P on XY a polygon $PQRST$ may be drawn with its alternate vertices on XY and XZ and its consecutive sides passing through A, B', C, D' in order. It will be found that the position of T remains the same for all 24 permutations of the four letters A, B, C, D, the letters taken second and fourth being always accented: and furthermore that the lines forming the 24 different routes from P to T give, without deficiency or redundancy, the system of 32 lines in question.

The theorem last stated holds good still if, more generally, the lines XY, XZ are replaced by a conic; and it takes a simple metrical form if Y and Z are supposed to be the circular points at infinity: namely, that taking a pencil of four pairs of lines a and a', b and b', c and c', d and d' in symmetric involution, successive reflection of an arbitrary point in a, b', c, d', or in any associated permutation, gives a closed system of 16 image points. In this form the result is readily apparent by mere summation of angles. The figure is partially representative of the section of a cubic and its lines by a plane through one of the lines; and it may be completed by using an additional pair of reflectors e and e', so placed that the fifth and final image coincides with the original point. But the figure is mentioned only for the sake of its bearing on the more special case, and further consideration of it need not be pursued here.

Added October 26th, 1911.—

A Barycentric Construction of the 27 *Double Lines of a Class Cubic.*

The figure reciprocal to the more familiar one treated above consists of 27 lines concurrent by threes in 45 points. The triangular diagram summarising the system of intersections is again available. The properties of the figure are immediately correlative to those of the original, and the analysis dealing with it may take precisely the same form. Nevertheless much conciseness may be attained in the case of the reciprocal figure by the use of barycentric parameters. The following method of constructing the figure is an immediate interpretation, with slight modification only, of the analysis reciprocal to that which has been given.

Take two triangles XYZ, $X'Y'Z'$, and let their sides cut the common line of their planes in points A, B, C, A', B', C'. Let P be any point on the line. Let particles of mass x, y, z, at X, Y, Z be equivalent to unit particle at P: and let particles x', y', z' at X', Y', Z' be equivalent to minus-unity at P. The centroid of any set of the six particles at X, Y, Z, X', Y', Z' is then the same point as the centroid of the remainder; and further the centroids of any three pairs are collinear. Let the centroid of x at X and x' at X' be denoted by xx', and similarly for others. Consider the nine points arranged in the table:—

	2	4	6
1	xx'	yy'	zz'
3	zy'	xz'	yx'
5	yz'	zx'	xy'

They lie on lines 1, 3, 5, by rows, and on lines 2, 4, 6, by columns; and the lines 1, 3, 5 and 2, 4, 6 mutually intersect in the nine points. The set of lines 1, 2, 3, 4, 5, 6 are variable as P moves. The diagram of § 1 represents the present process if the letters in the top corner are changed into capitals; and one of the horizontal strips is at present filled.

If P moves to Q a set of lines 1′, 2′, 3′, 4′, 5′, 6′ are obtained. If, now, line 1′ meets line 3, it is one of the two lines meeting XX', YY', ZZ' and line 3. The plane through line 3 and either of these positions of 1′ cuts the sides of the hexagon $XX'YY'ZZ'$ in ratios whose product is unity. Hence the value of $xyz/x'y'z'$ is the same for Q as for P: or, as equivalent,

$$PA\,.\,PB\,.\,PC/PA'\,.\,PB'\,.\,PC' = QA\,.\,QB\,.\,QC/QA'\,.\,QB'\,.\,QC'.$$

This gives two alternative positions, say Q and R, for the point companion to P. The one symmetrical relation gives simultaneously all the twelve intersections necessary for entering the lines 1′, 2′, 3′, 4′, 5′, 6′ in the second horizontal strip. As the associated pair of points P and Q vary in position the double six

$$\begin{matrix} 1 & 3 & 5 & 2' & 4' & 6' \\ 1' & 3' & 5' & 2 & 4 & 6 \end{matrix}$$

is itself variable, and rests on the nine lines joining the vertices of the triangles XYZ and $X'Y'Z'$.

Finally, if the third point R is taken in addition, so that P, Q, R are a triad of points of the cubic involution determined by the two triads A, B, C and A', B', C', the set of lines 1″, 2″, 3″, 4″, 5″, 6″, fill the last strip properly and complete the configuration.

ERRATUM.—*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 346, line 3:—

For "from" *read* "form."

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